British Isles Graduate Workshop 2019 – Schedule

Breakfast: 6:00-10:00
Lunch: 13:00
Dinner: 18:00

	Sunday, 09.06.	Monday, 10.06.	Tuesday, 11.06.	Wednesday, 12.06.	Thursday, 13.06.	Friday, 14.06.	Saturday, 15.06.
9:30–10:30		1.1 Spinors [Matt Turner]	1.5 Harmonic analysis [Udhav Fowdar]	2.3 Spin(7)-instantons, complex ASD, Hermitian-Einstein connections [Mateo Galdeano Solans]	2.5 Weighted Sobolev spaces, Fredholm properties of the linearised operator, and estimates [Vasileios Ektor Papoulias]	3.3 Seiberg-Witten theory in dimension three [Joe Driscoll]	10:00 Check-out
11:30-12:30		1.2 Nearly Kähler six-manifolds [Jakob Stein]	1.6 Deformation theory for nearly Kähler instantons [Derek Harland]	2.4 Ingredients for the construction and approximate solutions [Peter Panagiotis Angelinos]	2.6 Construction of Spin(7)-instantons [Yuuji Tanaka]	3.4 Generalized Seiberg-Witten equations [Jacob Gross]	
12:40-12:50				Take group photo (meet in hostel lobby)			
14:00-15:00	13.00 Earliest check-in	1.3 Reductive homogeneous spaces [Yang Li]	2.1 Basics on ASD instanton moduli space [Luya Wang]		3.1 Hyperkähler structures and the ADHM construction [Christoff Krüger]	3.5 SW-equations with multiple spinors; Fueter maps and G2-instantons [Greg Parker]	
16:00-17:00	19:00 Welcome	1.4 Instantons on nearly Kähler six-manifolds [Corvin Paul]	2.2 Joyce's examples of compact Spin(7)-manifolds [Holly Mandel]		3.2 Explicit examples [Yuan Yao]	3.6 The blow-up set for the SW-equations with 2 spinors [Andriy Haydys]	

Derek Harland: Nearly Kähler instantons and their deformations

In this course we will learn what nearly Kähler 6-manifolds and their instantons are, why people care about them, and how to develop a deformation theory. Along the way we will learn some useful techinques from harmonic analysis applied to homogeneous manifolds.

1. Spinors [3, 6, 9, 13] [Matt Turner]
Definition of Clifford algebra and relation with exterior algebra.
Isomorphisms with matrix algebras, spinors.
Spin groups and their Lie algebras as subsets of Clifford algebras.
The case of dimension 6 (Clifford algebra isomorphic to 8×8 real matrices, spin group isomorphic to SU(4), stabiliser of any spinor is SU(3)…)
Spin manifolds, spin structures, spin bundles.

2. Nearly Kähler six-manifolds [1, 5, 6, 8, 14, 17] [Jakob Stein]
Definitions (via Killing spinors, via SU(3)-structures, via almost complex structures, via G_2 cones) and their equivalence.
Example: S^6 as a submanifold of the imaginary octonions.
The canonical/characteristic connection on a nearly Kähler six-manifold.
If time allows: the canonical nearly Kähler structure on a twistor space, or results of Foscolo, Haskins, Moroianu-Semmelmann, Nagy.

3. Reductive homogeneous spaces [3, 5, 12] [Yang Li]
Definition of a reductive homogeneous space G/H, the canonical bundle $G \to G/H$, tangent bundle as an associated bundle.
The canonical connection, its curvature and torsion, the Levi-Civita connection.
Symmetric spaces and 3-symmetric spaces.
The four homogeneous nearly Kähler six-manifolds, their almost complex structures and metrics.

4. Instantons on nearly Kähler six-manifolds [6, 10, 15, 17, 16] [Corvin Paul]
Definitions (via spinors, via SU(3)-structures, via G_2-cones).
Example: the canonical connection.
Instantons are Yang-Mills.
Instantons are critical points of a Chern-Simons functional.
(If time/interest allows) Yang-Mills connections on n-spheres [4], or the twistor lift of an instanton on S^4 as an example of a nearly Kähler instanton.

5. Harmonic analysis [3, 7, 11] [Udhav Fowdar]
The Peter-Weyl theorem and Frobenius reciprocity.
The Laplace operator as a Casimir and the Freudenthal formula.
Example: spectrum of the Laplacian on S^2.
Example: spectrom of the Dirac operator on S^2.

6. Deformation theory for nearly Kähler instantons [6] [Derek Harland]
I will outline deformation theory for nearly Kähler instantons, focusing on the homogeneous examples, and describe some ongoing and related work.

References

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Yuuji Tanaka: A construction of Spin(7)-instantons

Summary: Spin(7)-instantons on 8-dimensional manifolds with holonomy contained in Spin(7) are one of the higher-dimensional analogues of anti-self-dual instantons in four dimensions. The moduli spaces of them on Calabi-Yau four-folds were recently studied by Borisov-Joyce and Cao-Leung to define DT4 invariants. In this course, we look into a construction of these instantons on Joyce's second examples of compact Spin(7)-manifolds.
The structure of talks:

1. Basics on ASD instanton moduli space [Luya Wang]

   Introduce ASD instantons and describe the linearisation of ASD instanton equation, the deformation complex and so on. References are e.g. [DK90, Chapters 2 and 4];

2. Joyce's examples of compact Spin(7)-manifolds [Holly Mandel]

   Section 2 of [Tan12], sketch the construction, mention examples of the ingredients, more details are in the original paper [Joy99] by Joyce and his book [Joy00];

3. Spin(7)-instantons, complex ASD, Hermitian-Einstein connections [Mateo Galdeano Solans]

   Section 3 of [Tan12], define them, and describe the linearisations and deformation complexes for them, other references are [Kim87, Kob87, Lew98, LT95, RC98, Wal17];

4. Ingredients for the construction and approximate solutions [Peter Panagiotis Angelinos]

   Section 4 of [Tan12]; describe the ingredients for the construction, approximate solutions out of them, and the estimate;

5. Weighted Sobolev spaces, Fredholm properties of the linearised operator, and estimates [Vasileios Ektor Papoulias]

   Section 5 of [Tan12], introduce weighted Sobolev spaces, discuss Fredholm properties of the linearised operator, and sketch the proof of Prop. 5.8. References for analysis on non-compact manifolds dealt in this part are [Loc87] and [LM85];

6. Construction [Yuuji Tanaka]

References

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[Wal17] Thomas Walpuski. Spin(7)-instantons, Cayley submanifolds and Fueter sections. *Comm. Math. Phys.*, 352(1):1–36, 2017.

Andriy Haydys: G_2 instantons and the Seiberg-Witten monopoles

In this series of talks we want to learn about a conjectural relation between G_2 instantons and the Seiberg-Witten monopoles in dimension three. This relation essentially boils down to the observation that degenerations both of G_2 instantons and certain Seiberg-Witten monopoles are modeled on certain Fueter sections, which will be introduced in one of the talks. A somewhat more formal way to phrase this, is via the notion of the compactification of moduli spaces. This will be central for all talks in the series.

1. [Christoff Krger] Hyperkaehler reduction, the ADHM construction of instantons on $\mathbb{R}^4$, and the hyperKaehler structure on the moduli space of instantons on $\mathbb{R}^4$.

    - Recall the hyperKaehler quotient construction following [HKLR87, 3(D)] or [Poh]. We need only the statement and the construction of the hK structure on the quotient.
    - Describe the moduli spaces of anti-self-dual instantons on $\mathbb{R}^4$ (equivalently, on the four-sphere) both as an infinite-dimensional hK quotient and a finite-dimensional one [DK90], [Ati79].
    - Describe some explicit examples, for instance the framed moduli space of SU(n) charge 1 instantons on $\mathbb{R}^4$.

2. [Yuan Yao] The Uhlenbeck compactification of the moduli space of instantons on four-manifolds and a compactness theorem for moduli spaces of instantons in higher dimensions.
   Describe compactifications of the relevant moduli spaces following [DK90, Sect. 4.4] and [Tia00, Tia02]. This includes in particular the notions of G_2 instantons, calibration, associative submanifold of a G_2 manifold etc. We are less interested in the technique here, more on the qualitative picture.

3. [Joe Driscoll] Introduction to the Seiberg-Witten theory in dimension three. Basic constructions, compactness of the moduli space, the Seiberg-Witten invariant, equivalence to the Milnor's torsion [Lim00] (despite the title this can be partially used for 3-manifolds with $b1 > 1$), [Mar99, Ch. 6], [Sal96, Ch. 10]. One can also use [Mor96] but the constructions have to be adapted to dimension 3.

This talk should serve mainly as a basis for the next ones. We are mainly interested in the compactness property of the moduli space. The cases $b_1 < 2$ can be only briefly mentioned. Also, only the formulation of the equivalence between the Seiberg-Witten invariant and the Milnor's torsion would be enough for our purposes and this may be even done witthout going into the details of definition of the Milnor's torsion.

4. [Jacob Gross] Generalized Seiberg-Witten equations, Fueter maps (sections), the generalized Seiberg-Witten equations and the hyperkaehler reduction.

    - Decribe the notion of Fueter section (beware: these have many names, in particular 'generalized harmonic spinors', 'triholomorphic maps', 'aquaternionic maps'...), the generalized Seiberg-Witten equations following [Hay17, Sec. 2] and [Tau99].
    - Describe the relation between Fueter maps into the hK quotient and generalized Seiberg-Witten equations following [Hay17].
    - Describe G_2 instanton equations as an instance of the Seiberg-Witten equations following [Hay17, Sec. 4] and references therein.

5. [Greg Parker] A compactness theorem for the Seiberg-Witten equations with multiple spinors; Fueter maps and G_2 instantons.

    - Formulate the compactness theorem for the Seiberg-Witten equations with multiple spinors [HW14].
    - Describe the results of Walpuski [Wal17] and Walpuski-Doan [DW17] on deformations of the Fueter sections.
    - Discuss [Hay17, Sec. 5].

6. [Andriy Haydys] On a blow up set for the Seiberg-Witten equations with 2 spinors. I will discuss some properties of the blow up set for the Seiberg-Witten equations with 2 spinors.

References

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