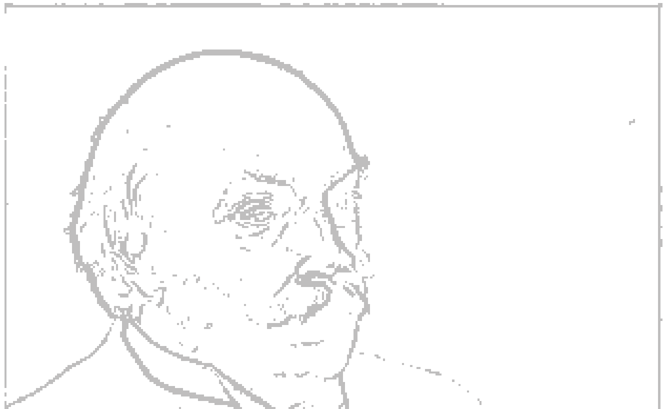
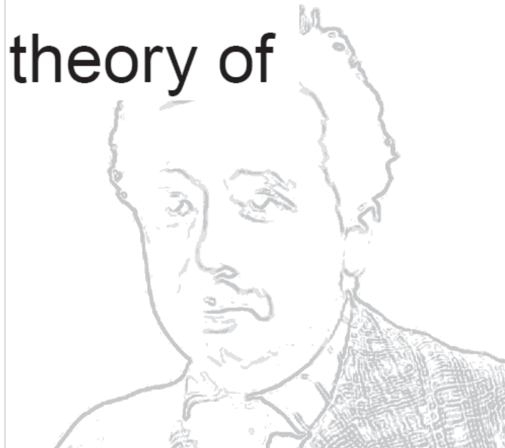
Random walk origins



Lord Rayleigh's theory of sound (1880s)

Einstein's theory of



Brownian motion (1905–08)

Louis Bachelier's

thesis on random models of stock prices (1900)

Karl Pearson's

theory of random migration (1905-06)

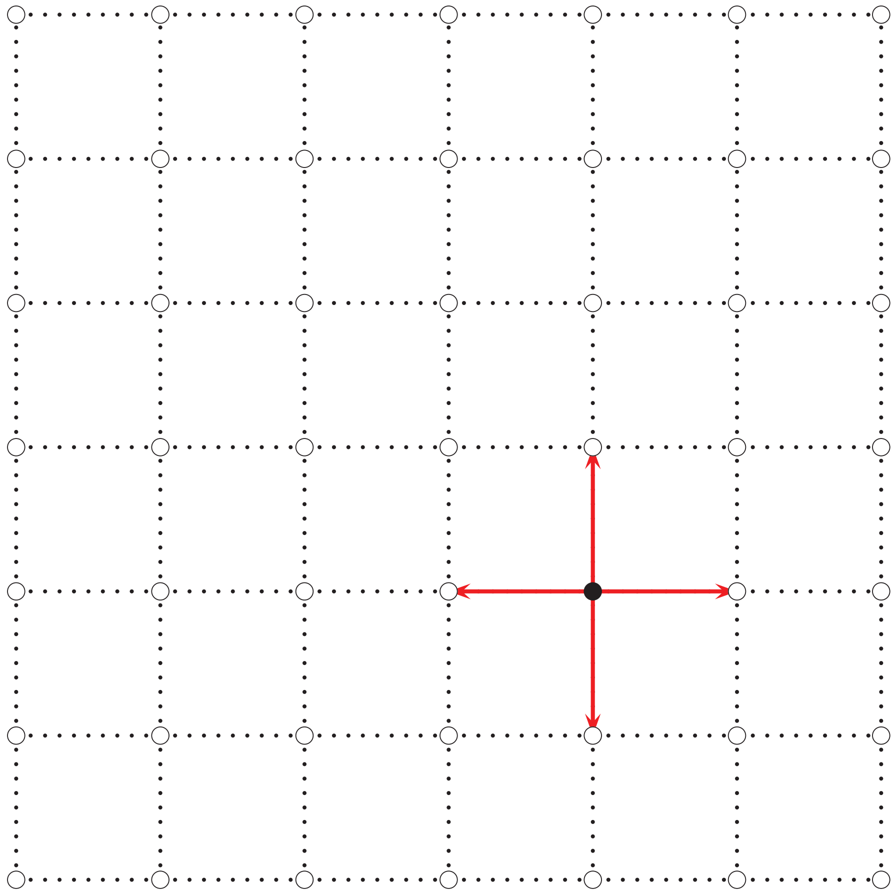
Mathematical developments

George Pólya (1887–1985).

- While walking in a Zurich park in 1914, Pólya encountered the same couple several times on his walk.
- He asked: was this, after all, so unlikely?
- Some time later Pólya published his paper on an idealized version of the problem, now known as simple random walk (SRW).

Simple random walk

A random walker on the d-dimensional integer lattice $\mathbb{Z}^d$.



X_n: position after n steps.

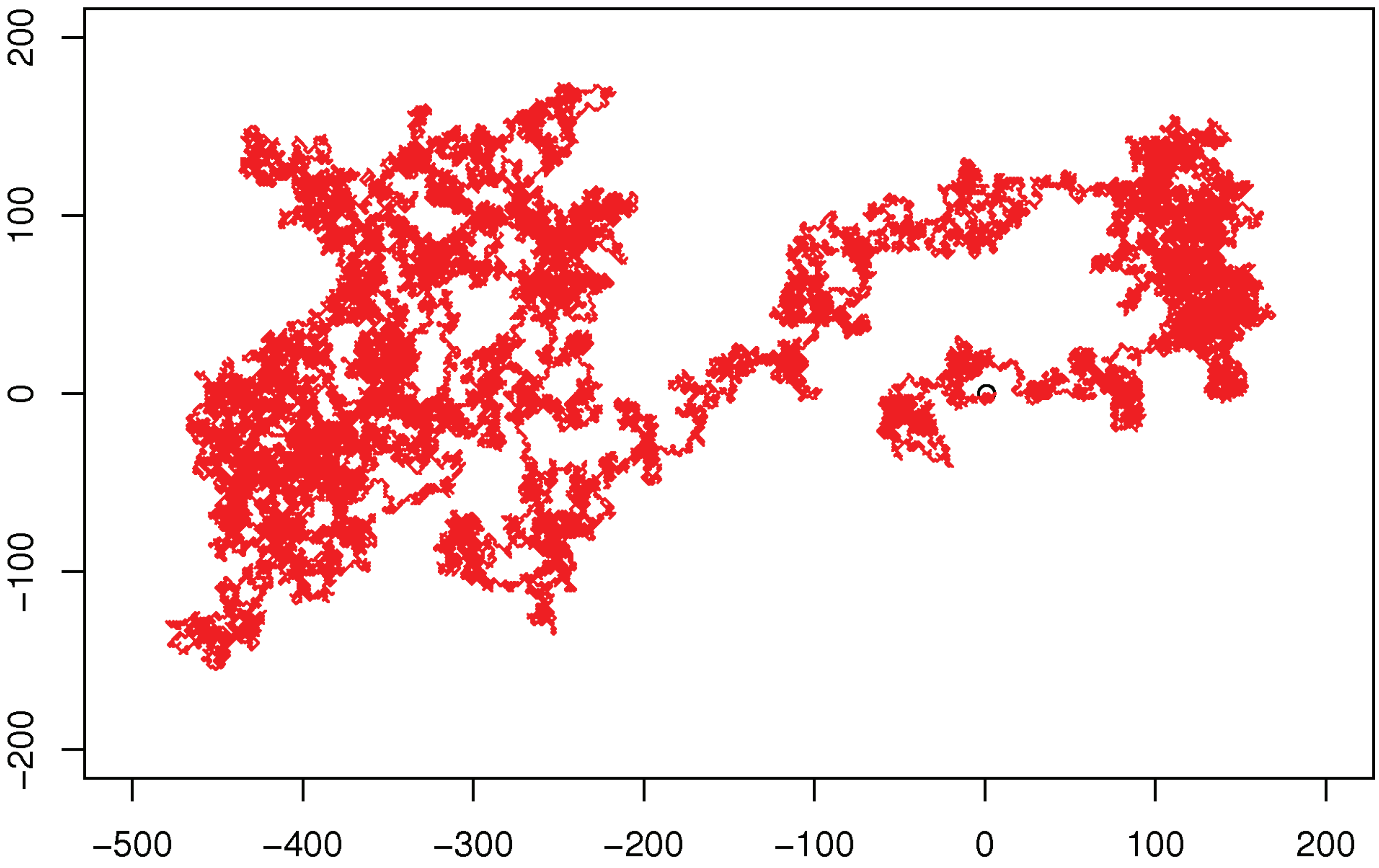
At each step, the walker jumps to one of the $2d$ neighbouring sites of the lattice, choosing uniformly at random from each.

Pólya's question: What is the probability that the walker eventually returns to his starting point? Call it p_d.

$$p_d = \mathbb{P}[X_n = X_0 \text{ for some } n \geq 1].$$

Pólya's question

Simulation of 10^5 steps of SRW on $\mathbb{Z}^2$.



Recurrence and transience

$$p_d = \mathbb{P}[X_n = X_0 \text{ for some } n \geq 1].$$

The random walk is transient if $p_d < 1$ and recurrent if $p_d = 1$.

Theorem (Pólya)

Simple random walk on $\mathbb{Z}^d$ is

- *recurrent for $d = 1$ or $d = 2$;*
- *transient for $d \geq 3$.*

For example [McCrea & Whipple, Glasser & Zucker]:

$$p_3 = 1 - \left(\frac{\sqrt{6}}{32\pi^3} \Gamma(\tfrac{1}{24})\Gamma(\tfrac{5}{24})\Gamma(\tfrac{7}{24})\Gamma(\tfrac{11}{24}) \right)^{-1}$$

$$\approx 0.340537.$$

Recurrence and transience

Theorem (Pólya)

Simple random walk on $\mathbb{Z}^d$ is

- *recurrent for $d = 1$ or $d = 2$;*
- *transient for $d \geq 3$.*

Equivalently:

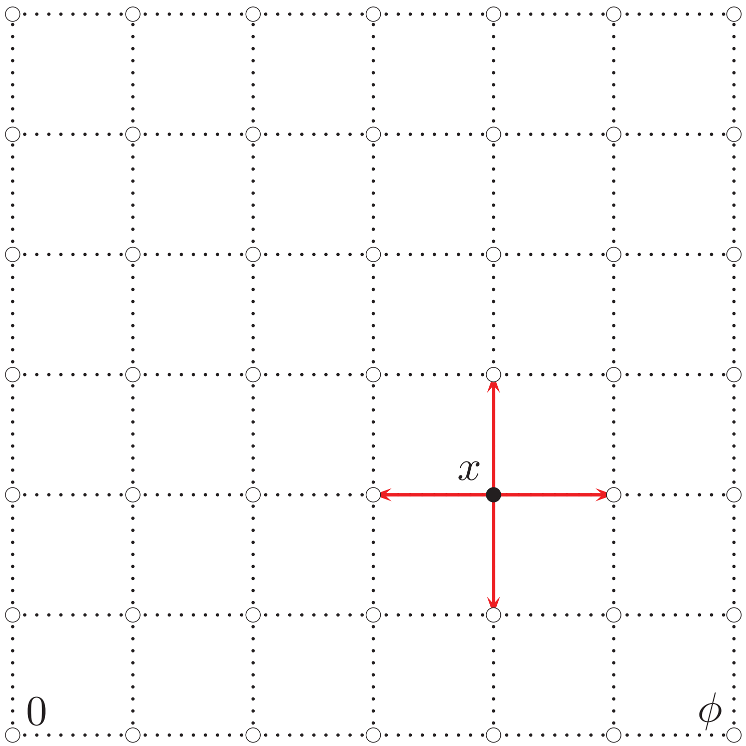
- For $d \in \{1, 2\}$, X_n visits any finite set infinitely often.
- On the other hand, if $d \geq 3$, X_n visits any finite set only finitely often.

"A drunk man will find his way home,
but a drunk bird may get lost forever."
—Shizuo Kakutani

Probabilities and potentials

Take two points in the lattice $\mathbb{Z}^d$, 0 and ϕ.

Let $p(x) = \mathbb{P}$[SRW reaches ϕ before 0 starting from x].



Then $p(0) = 0$ and $p(\phi) = 1$. For $x \notin \{0, \phi\}$, by conditioning on the first step of the walk, for which there are $2d$ possibilities,

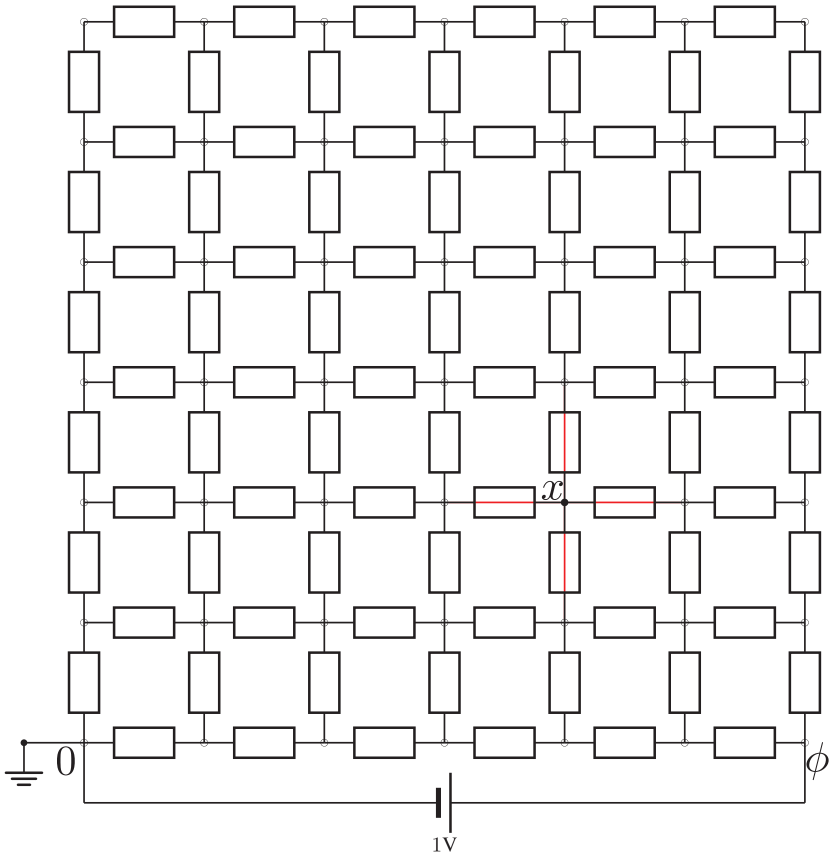
$$p(x) = \frac{1}{2d} \sum_{y \sim x} p(y),$$

where sum over $y \sim x$ means those y that are neighbours of x.

Rearranging, we get $\sum_{y \sim x} (p(y) - p(x)) = 0$.

Probabilities and potentials

There is an equivalent formulation in terms of a resistor network. In the first instance, this makes sense on a finite subgraph $A \subset \mathbb{Z}^d$.



Replace each edge of A with a 1 Ohm resistor.

Ground the point 0 and attach a 1 Volt battery across 0 and ϕ.

Let $v(x)$ be the potential at point x.

Then $v(0) = 0$ and $v(\phi) = 1$. By Kirchhoff's laws, the net flow of current at x vanishes, and the flow across any edge is given by the potential difference, so

$$\sum_{y \sim x} (v(y) - v(x)) = 0.$$

Probabilities and potentials

So both p and v solve the same boundary value problem

$$\sum_{y \sim x} (v(y) - v(x)) = 0$$

with the same boundary conditions.

The solutions are (discrete) harmonic functions.

The connections to classical potential theory run deep. For example, one can study recurrence and transience:

Theorem (Nash-Williams)

The SRW on $\mathbb{Z}^d$ is recurrent if and only if the effective resistance of the resistance network on $A \subset \mathbb{Z}^d$ tends to ∞ as $A \to \mathbb{Z}^d$.

Martingales and boundary value problems

The effectiveness of this connection to potential theory relies on certain symmetry properties of SRW. In particular, SRW is both a Markov chain and a space-homogeneous martingale (which means that the walk has zero drift).

The connection extends to a large class of processes in both discrete and continuous time.

For example, the continuous-time, continuous-space analogue of SRW is Brownian motion.

And in the continuous setting solving boundary value problems amounts to solving PDEs.

The stochastic approach provides a powerful tool for studying PDEs, and has applications in e.g.

- quantum theory;
- mathematical finance;
- etc.