Non-homogeneous random walks on a semi-infinite strip

Chak Hei Lo

Joint work with Andrew R. Wade

World Congress in Probability and Statistics
11th July, 2016

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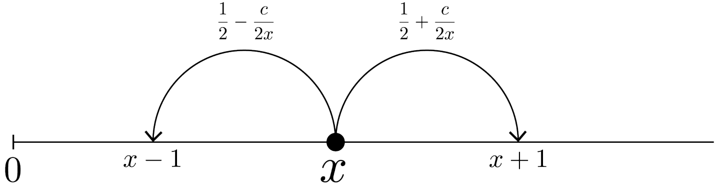
Outline

- Motivation: Lamperti's problem
- Our Model
  - Non-homogeneous RW on semi-infinite strip
  - Classification of the random walk
  - Assumptions
- Main results
  - Constant drift
  - Lamperti drift
  - Generalized Lamperti drift
- Examples: Correlated random walks

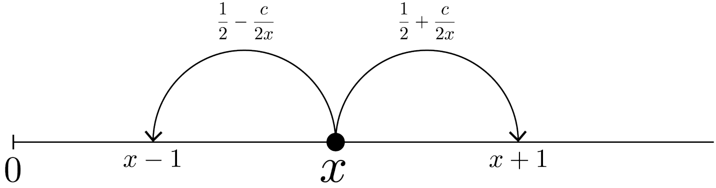
Motivation: Lamperti's problem

- Let X_n be a nearest neighbour random walk on $\mathbb{Z}_+$.



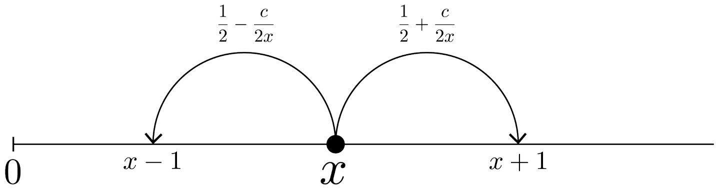
Motivation: Lamperti's problem

- Let X_n be a nearest neighbour random walk on $\mathbb{Z}_+$.
- Denote the mean drift at x by $\mu(x)$.



Motivation: Lamperti's problem

- Let X_n be a nearest neighbour random walk on $\mathbb{Z}_+$.
- Denote the mean drift at x by $\mu(x)$.
- Lamperti's problem: $\mu(x) = O(1/x)$ when $x \to \infty$.



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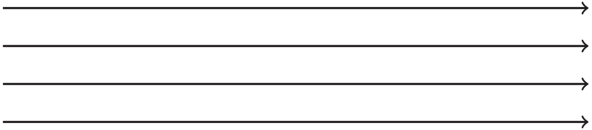
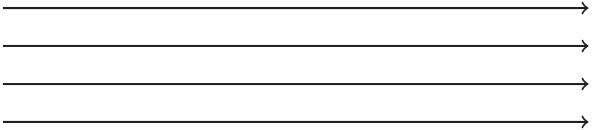
Non-homogeneous RW on semi-infinite strip

- Let S be a finite non-empty set.



Non-homogeneous RW on semi-infinite strip

- Let S be a finite non-empty set.
- Let Σ be a subset of $\mathbb{R}_+ \times S$ that is *locally finite*, i.e., $\Sigma \cap ([0, r] \times S)$ is finite for all $r \in \mathbb{R}_+$.



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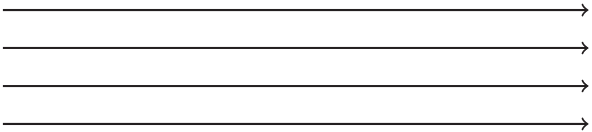

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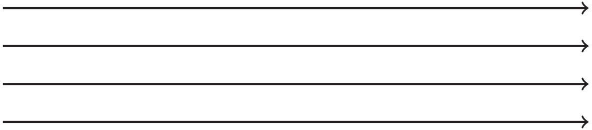
- E.g. $\Sigma = \mathbb{Z}_+ \times S$.
- Define for each $k \in S$ the *line* $\Lambda_k := \{x \in \mathbb{R}_+ : (x, k) \in \Sigma\}$.



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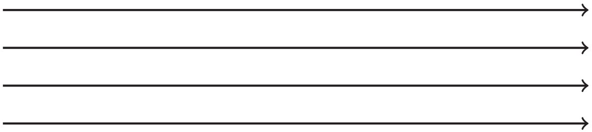
- Suppose that for each $k \in S$ the line Λ_k is unbounded.



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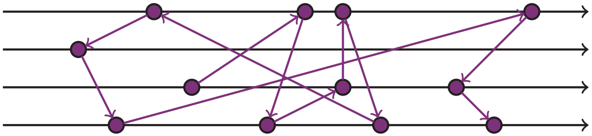
Non-homogeneous RW on semi-infinite strip

- Suppose that (X_n, η_n), $n \in \mathbb{Z}_+$, is a time-homogeneous, irreducible Markov chain on Σ, a locally finite subset of $\mathbb{R}_+ \times S$.



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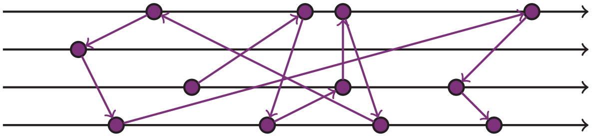
Non-homogeneous RW on semi-infinite strip

- Suppose that (X_n, η_n), $n \in \mathbb{Z}_+$, is a time-homogeneous, irreducible Markov chain on Σ, a locally finite subset of $\mathbb{R}_+ \times S$.
- Neither coordinate is assumed to be Markov.



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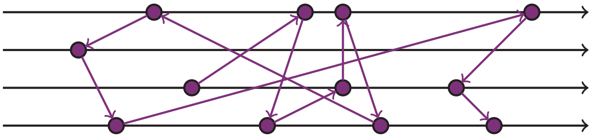
Motivating examples

We can view S as a set of internal states, influencing motion on the lines $\mathbb{R}_+$. E.g.,

- Operations research: modulated queues
  (S = states of server)



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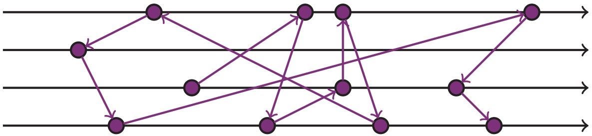
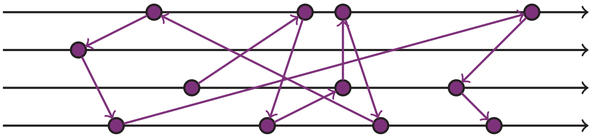


Motivating examples

We can view S as a set of internal states, influencing motion on the lines $\mathbb{R}_+$. E.g.,

- Operations research: modulated queues
  (S = states of server)
- Economics: regime-switching processes
  (S contains market information)
- Physics: physical processes with internal degrees of freedom
  (S = energy/momentum states of particle)



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Classification of the random walk

Recall (X_n, η_n) is a time-homogeneous irreducible Markov chain on the state-space $\Sigma \in \mathbb{R}_+ \times S$.

(i) If (X_n, η_n) is transient, then $X_n \to \infty$ *a.s.*

(ii) If (X_n, η_n) is recurrent, then $\mathbb{P}(X_n \in A \text{ i.o.}) = 1$ for any bounded region A.

(iii) Define $\tau = \min\{n \geq 0 : X_n \in A\}$. If (X_n, η_n) is positive-recurrent, then $\mathbb{E}[\tau] < \infty$ for any bounded region A.

(iv) If (X_n, η_n) is recurrent but not positive recurrent, then we call it null-recurrent.

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Assumptions

- Moments bound on jumps of X_n
  (B_p) $\quad \exists\ C_p < \infty$ s.t.

$$\mathbb{E}[|X_{n+1} - X_n|^p \mid (X_n, \eta_n) = (x, i)] \leq C_p$$

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$$\mathbb{E}[|X_{n+1} - X_n|^p \mid (X_n, \eta_n) = (x, i)] \leq C_p$$

- Notation for moments of the displacements in the X-coordinate

$$\mu_i(x) = \mathbb{E}[X_{n+1} - X_n \mid (X_n, \eta_n) = (x, i)]$$
$$\sigma_i(x) = \mathbb{E}[(X_{n+1} - X_n)^2 \mid (X_n, \eta_n) = (x, i)]$$

Assumptions (cont.)

- η_n is "close to being Markov" when X_n is large

  Define
  $$q_{ij}(x) = \mathbb{P}[\eta_{n+1} = j \mid (X_n, \eta_n) = (x, i)]$$

  (Q_∞) $q_{ij} = \lim_{x\to\infty} q_{ij}(x)$ exists for all $i, j \in S$
  and (q_{ij}) is irreducible

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  Define
  $$q_{ij}(x) = \mathbb{P}[\eta_{n+1} = j \mid (X_n, \eta_n) = (x, i)]$$
  (Q$_\infty$) $\quad q_{ij} = \lim_{x\to\infty} q_{ij}(x)$ exists for all $i, j \in S$ and (q_{ij}) is irreducible
- Let $\boldsymbol{\pi}$ be the unique stationary distribution on S corresponding to (q_{ij}).

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Constant drift

- Constant-type moment condition
  $(\mathsf{M_C})$ $\quad \exists\, d_i \in \mathbb{R}$ for all $i \in S$ such that

$$\mu_i(x) = d_i + o(1).$$

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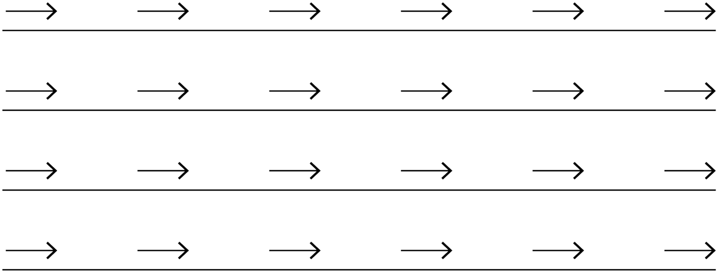
Constant drift



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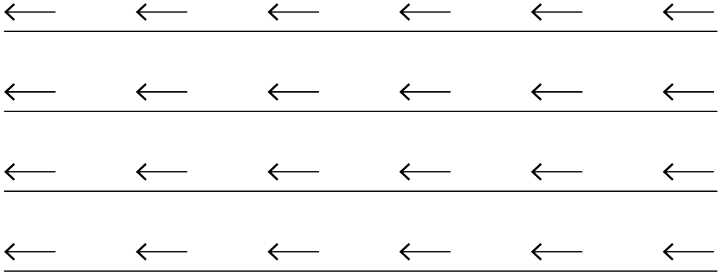

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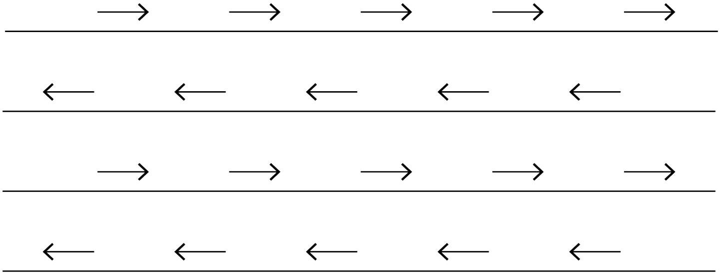

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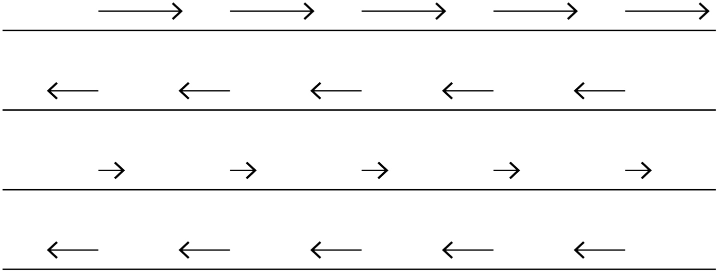

Recurrence classification for constant drift

The following theorem is from Georgiou, Wade (2014), extending slightly earlier work of Malyshev (1972), Falin (1988), and Fayolle et al (1995).

Theorem

Suppose that (B_p) holds for some $p > 1$ and conditions (Q_∞) and (M_C) hold. The following sufficient conditions apply.

- *If* $\sum_{i \in S} d_i \pi_i > 0$, *then* (X_n, η_n) *is transient.*
- *If* $\sum_{i \in S} d_i \pi_i < 0$, *then* (X_n, η_n) *is positive-recurrent.*

where π_i *is the unique stationary distribution on* S.

What about $\sum_{i \in S} d_i \pi_i = 0$?

Different drifts

(i) $\sum_{i \in S} d_i \pi_i \neq 0$, constant drift:

$$\mu_i(x) = d_i + o(1)$$

(ii) $\sum_{i \in S} d_i \pi_i = 0$ and $d_i = 0$ for all i, Lamperti drift:

$$\mu_i(x) = \frac{c_i}{x} + o(x^{-1})$$

$$\sigma_i(x) = s_i^2 + o(1)$$

(iii) $\sum_{i \in S} d_i \pi_i = 0$ and $d_i \neq 0$ for some i, generalized Lamperti drift:

$$\mu_i(x) = d_i + \frac{c_i}{x} + o(x^{-1})$$

$$\sigma_i(x) = s_i^2 + o(1)$$

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Lamperti drift

- Lamperti-type moment conditions
  $(\mathsf{M_L})$ $\quad \exists\ c_i, s_i \in \mathbb{R}$ for all $i \in S$ (at least one s_i nonzero) such that
  $$\mu_i(x) = \frac{c_i}{x} + o(x^{-1}); \quad \sigma_i(x) = s_i^2 + o(1).$$

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  $$\mu_i(x) = \frac{c_i}{x} + o(x^{-1}); \quad \sigma_i(x) = s_i^2 + o(1).$$
- When S is a singleton, this reduces to the classical Lamperti problem on $\mathbb{R}_+$.

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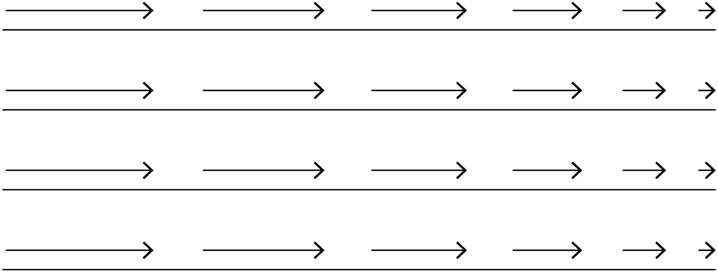
Lamperti drift



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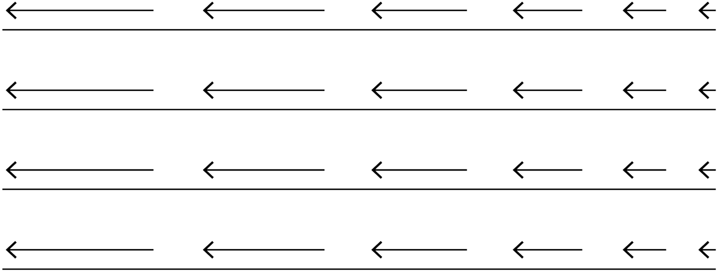

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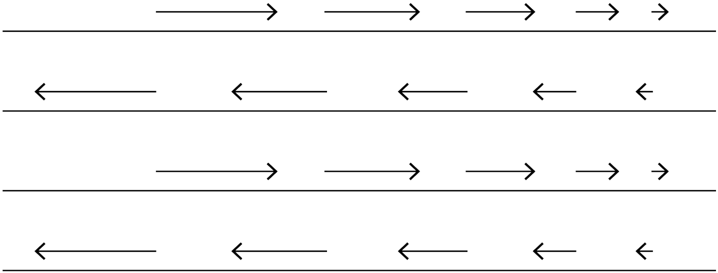

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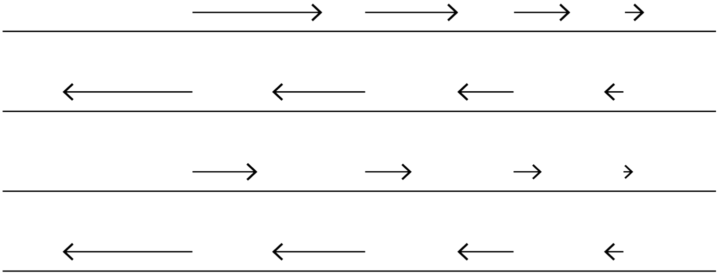

Recurrence classification for Lamperti drift

Theorem (Georgiou, Wade, 2014)

Suppose that (B_p) holds for some $p > 2$ and conditions (Q_∞) and (M_L) hold. The following sufficient conditions apply.

(i) *If $\sum_{i \in S}(2c_i - s_i^2)\pi_i > 0$, then (X_n, η_n) is transient.*

(ii) *If $|\sum_{i \in S} 2c_i\pi_i| < \sum_{i \in S} s_i^2\pi_i$, then (X_n, η_n) is null-recurrent.*

(iii) *If $\sum_{i \in S}(2c_i + s_i^2)\pi_i < 0$, then (X_n, η_n) is positive-recurrent.*

[With better error bounds in (Q_∞) and (M_L) we can also show that the boundary cases are null-recurrent.]

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Idea of proof of the theorem

- Our general analysis is based on the Lyapunov function $f_\nu : \Sigma \to \mathbb{R}$ defined for $\nu \in \mathbb{R}$ by

$$f_\nu(x, i) := x^\nu + \frac{\nu}{2} b_i x^{\nu-2}$$

where $b_i \in \mathbb{R}$.

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  where $b_i \in \mathbb{R}$.
- For appropriate choices of ν, and selecting the right b_i depending on the drift and the stationary distribution, we can show that $f_\nu(X_n, \eta_n)$ is a supermartingale and so we can apply some semi-martingale theorem. Hence we obtain the last theorem shown.

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Generalized Lamperti drift

- Generalized Lamperti type moment conditions
  Define
  $$\mu_{ij}(x) = \mathbb{E}[(X_{n+1} - X_n)\mathbf{1}\{\eta_{n+1} = j\} \mid (X_n, \eta_n) = (x, i)]$$
  ($\mathrm{M_{CL}}$) There exist $d_i \in \mathbb{R}$, $c_i \in \mathbb{R}$, $d_{ij} \in \mathbb{R}$ and $s_i^2 \in \mathbb{R}_+$, with at least one s_i^2 non-zero, such that all of the following is satisfied,

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(iv) $\sum_{i \in S} \pi_i d_i = 0$.

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  (iii) $\sigma_i(x) = s_i^2 + o(1)$,
  (iv) $\sum_{i \in S} \pi_i d_i = 0$.

- Transition probability condition

  (Q_{CL}) There exist $\gamma_{ij} \in \mathbb{R}$, such that

$$q_{ij}(x) = q_{ij} + \frac{\gamma_{ij}}{x} + o(x^{-1})$$

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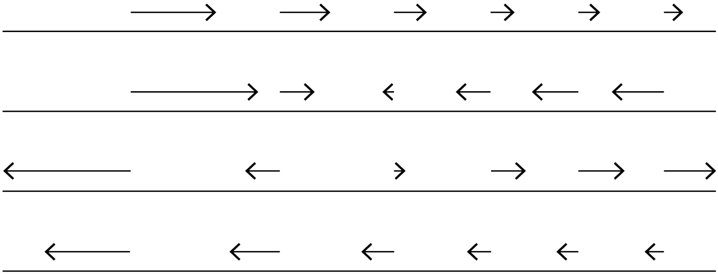
Generalized Lamperti drift



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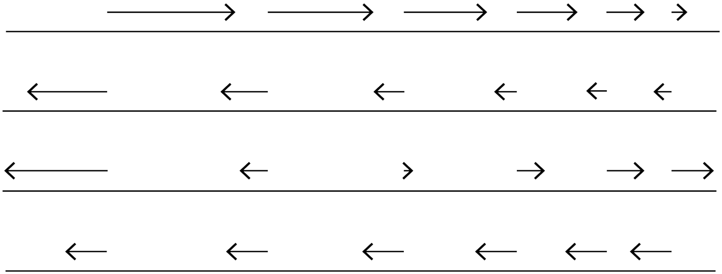

Recurrence classification for Generalized Lamperti drift

Theorem (L., Wade, 2015)

Suppose that (B_p) holds for some $p > 2$, and conditions (Q_∞), (Q_{CL}) and (M_{CL}) hold. Define a_i to be the unique solution up to translation of the system of equations $d_i + \sum_{j \in S}(a_j - a_i)q_{ij} = 0$ $\forall\ i \in S$. The following sufficient conditions apply.

- *If $\sum_{i \in S}[2c_i - s_i^2 + 2\sum_{j \in S} a_j(\gamma_{ij} - d_{ij})]\pi_i > 0$ then (X_n, η_n) is transient.*
- *If $|\sum_{i \in S}(2c_i + 2\sum_{j \in S} a_j \gamma_{ij})\pi_i| < \sum_{i \in S}(s_i^2 + 2\sum_{j \in S} a_j d_{ij})\pi_i$ then (X_n, η_n) is null-recurrent.*
- *If $\sum_{i \in S}[2c_i + s_i^2 + 2\sum_{j \in S} a_j(\gamma_{ij} + d_{ij})]\pi_i < 0$ then (X_n, η_n) is positive-recurrent.*

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Idea of proof of the theorem

- Transform the process (X_n, η_n) with generalized Lamperti drift to a process $(\widetilde{X}_n, \eta_n) = (X_n + a_{\eta_n}, \eta_n)$

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- Find the appropriate choices of the real numbers a_i, $i \in S$, such that $(\widetilde{X}_n, \eta_n)$ has Lamperti drift i.e., the constant components of the drifts are eliminated

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- Transform the process (X_n, η_n) with generalized Lamperti drift to a process $(\widetilde{X}_n, \eta_n) = (X_n + a_{\eta_n}, \eta_n)$
- Find the appropriate choices of the real numbers a_i, $i \in S$, such that $(\widetilde{X}_n, \eta_n)$ has Lamperti drift i.e., the constant components of the drifts are eliminated
- Calculate the new increment moment estimates for the transformed process $(\widetilde{X}_n, \eta_n)$.

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Idea of proof of the theorem

- Transform the process (X_n, η_n) with generalized Lamperti drift to a process $(\widetilde{X}_n, \eta_n) = (X_n + a_{\eta_n}, \eta_n)$
- Find the appropriate choices of the real numbers a_i, $i \in S$, such that $(\widetilde{X}_n, \eta_n)$ has Lamperti drift i.e., the constant components of the drifts are eliminated
- Calculate the new increment moment estimates for the transformed process $(\widetilde{X}_n, \eta_n)$.
- Apply the results in the Lamperti drift case

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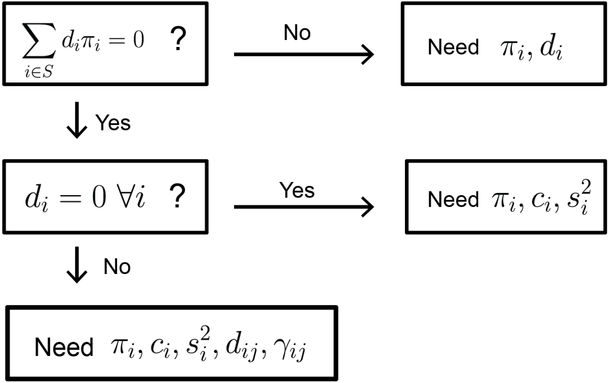
Summary of cases



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Summary of cases

$$\sum_{i\in S} d_i \pi_i = 0 \quad ?$$

No →

$$\sum_{i\in S} d_i \pi_i > 0 \qquad \text{:T}$$

$$\sum_{i\in S} d_i \pi_i < 0 \qquad \text{:PR}$$

↓ **Yes**

$$d_i = 0 \ \forall i \quad ?$$

Yes →

$$\sum_{i\in S} \left(2c_i - s_i^2\right) \pi_i > 0 \qquad \text{:T}$$

$$\left| \sum_{i\in S} 2c_i \pi_i \right| \le \sum_{i\in S} s_i^2 \pi_i \qquad \text{:NR}$$

$$\sum_{i\in S} \left(2c_i + s_i^2\right) \pi_i < 0 \qquad \text{:PR}$$

↓ **No**

$$\sum_{i\in S} \left[2c_i - s_i^2 + 2\sum_{j\in S} a_j(\gamma_{ij} - d_{ij}) \right] \pi_i > 0 \qquad \text{:T}$$

$$\left| \sum_{i\in S} \left(2c_i + 2\sum_{j\in S} a_j \gamma_{ij} \right) \pi_i \right| \le \sum_{i\in S} \left(s_i^2 + 2\sum_{j\in S} a_j d_{ij} \right) \pi_i \qquad \text{:NR}$$

$$\sum_{i\in S} \left[2c_i + s_i^2 + 2\sum_{j\in S} a_j(\gamma_{ij} + d_{ij}) \right] \pi_i < 0 \qquad \text{:PR}$$

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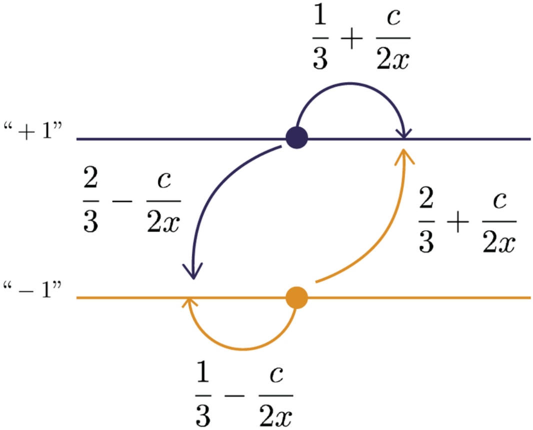
Example: One-step Correlated random walk



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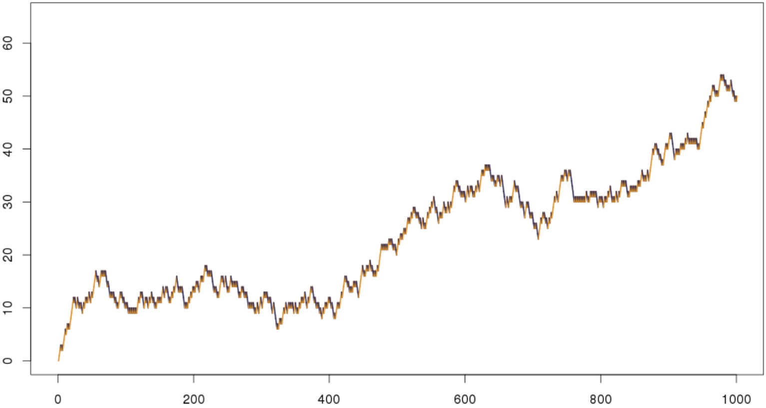
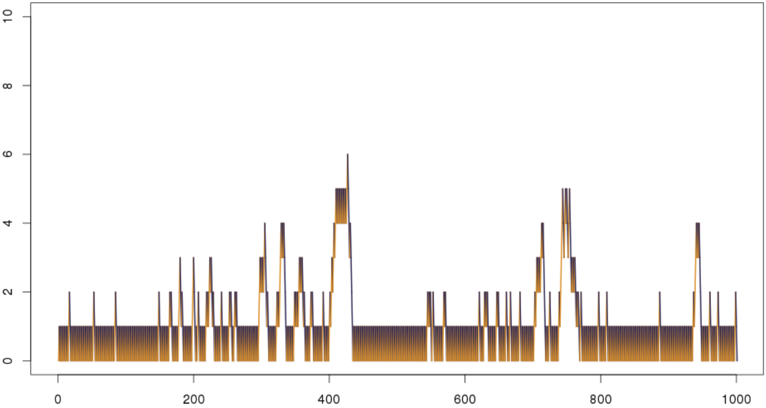
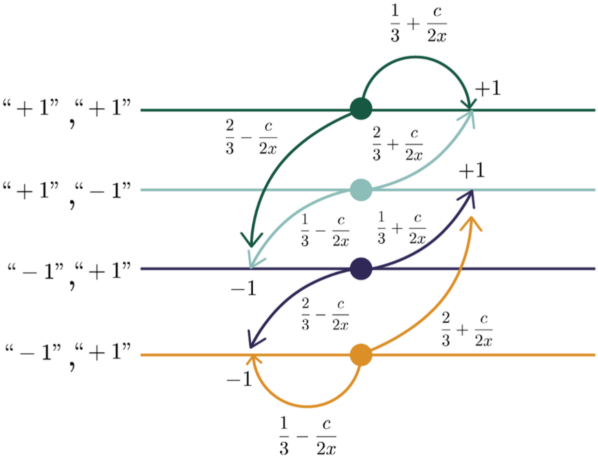
Simulation results on One-step Correlated RW



Simulation results on One-step Correlated RW



Example: Two-steps Correlated random walk



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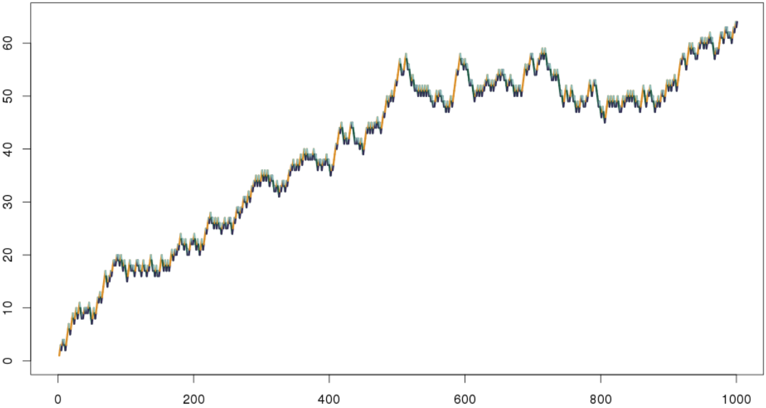
Simulation results on Two-steps Correlated RW



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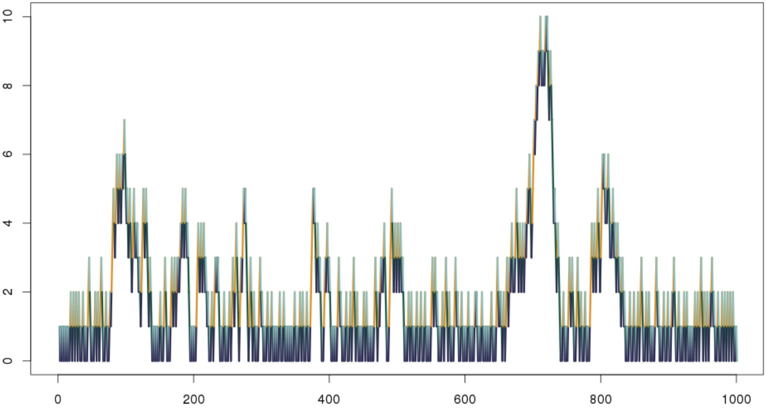
Simulation results on Two-step Correlated RW



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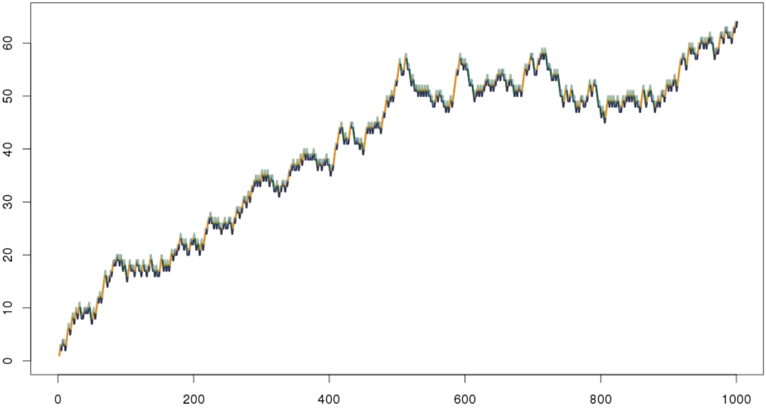
Simulation results on Two-steps Correlated RW



Existence and Non-existence of Moments results

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- We used similar techniques and ideas in our proofs.

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