Cutpoints of non-homogeneous random walks

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Cutpoints

Suppose that $X = (X_n; n \in \mathbb{Z}_+)$ is a discrete-time stochastic process adapted to a filtration $(\mathcal{F}_n; n \in \mathbb{Z}_+)$ and taking values in a measurable $\mathcal{X} \subset \mathbb{R}_+$ with $\inf \mathcal{X} = 0$ and $\sup \mathcal{X} = \infty$. We permit $\mathcal{F}_0$ to be rich enough that X_0 is random.

A point x of $\mathbb{R}_+$ is a *cutpoint* for a given trajectory of a stochastic process if, roughly speaking, the process visits x and never returns to $[0, x)$ after its first entry into (x, ∞).

Motivation

Under mild conditions, cutpoints may appear only in the transient case, when trajectories escape to infinity.

The more cutpoints that a process has, the ‘more transient’ it is, in a certain sense.

A fundamental question is: does a transient process have infinitely many cutpoints, or not?

Cutpoints

Definition

(i) The point $x \in \mathbb{R}_+$ is a *cutpoint* for X if there exists $n_0 \in \mathbb{Z}_+$ such that $X_n \leq x$ for all $n \leq n_0$, $X_{n_0} = x$, and $X_n > x$ for all $n > n_0$.
(ii) The point $x \in \mathbb{R}_+$ is a *strong cutpoint* for X if there exists $n_0 \in \mathbb{Z}_+$ such that $X_n < x$ for all $n < n_0$, $X_{n_0} = x$, and $X_n > x$ for all $n > n_0$.

Number of cutpoints

Let $\mathcal{C}$ denote the set of cutpoints, and let $\mathcal{C}_s$ denote the set of strong cutpoints; the random sets $\mathcal{C}$ and $\mathcal{C}_s$ are at most countable, with $\mathcal{C}_s \subseteq \mathcal{C}$.

In this presentation we give conditions under which either (i) $\#\mathcal{C}_s = \infty$, or (ii) $\#\mathcal{C} < \infty$.

The example of a trajectory on $\mathbb{Z}_+$ which follows the sequence $(0, 0, 1, 1, 2, 2, \ldots)$ shows that it is, in principle, possible to have $\#\mathcal{C} = \infty$ and $\#\mathcal{C}_s < \infty$, but our results show that such behaviour is excluded for the models that we consider (with probability 1).

Some literature

For simple symmetric random walk (SSRW) on $\mathbb{Z}^d$, $d \geq 3$,

Erdős and Taylor (1960): Cutpoints have a positive density in the trajectory if $d \geq 5$;

Lawler (1991): Transient SSRW has infinitely many cutpoints in dimension $d \geq 4$;

James and Peres (1997): Transient SSRW has infinitely many cutpoints in dimension $d \geq 3$.

Recently, examples of transient Markov chains on $\mathbb{Z}_+$ with finitely many cutpoints were produced (e.g. by Csáki et. al (2010)): these processes are nearest-neighbour birth-and-death chains that are ‘less transient’ than SSRW on $\mathbb{Z}^3$.

Some assumptions

Bounded Increments:

(B) Suppose that there exists a constant $B < \infty$ such that, for all $n \in \mathbb{Z}_+$,

$$\mathbb{P}(|X_{n+1} - X_n| \leq B \mid \mathcal{F}_n) = 1.$$

Non-confinement condition:

(N) Suppose that $\limsup_{n\to\infty} X_n = +\infty$, a.s.

Some assumptions cont'

For $n \in \mathbb{Z}_+$, we will impose conditions on the conditional increment moments $\mathbb{E}[(X_{n+1} - X_n)^k \mid \mathcal{F}_n]$, $k = 1, 2$, that are required to hold uniformly (in n and a.s.) on $\{X_n > x\}$ for large enough x. To formulate these conditions, we suppose that we have (measurable) functions $\underline{\mu}_k, \bar{\mu}_k : \mathcal{X} \to \mathbb{R}$ such that

$$\underline{\mu}_k(X_n) \leq \mathbb{E}(\Delta_n^k \mid \mathcal{F}_n) \leq \bar{\mu}_k(X_n), \text{ a.s.}$$

for all $n \in \mathbb{Z}_+$.

Additional mild assumption:

(V) Suppose that $\liminf_{x\to\infty} \underline{\mu}_2(x) > 0$.

A sufficient condition for infinitely many strong cutpoints

Theorem 1 (L., Menshikov, Wade, 2020)

Suppose that (B), (N), *and* (V) *hold. Suppose also that*

$$\liminf_{x\to\infty}\left(2x\underline{\mu}_1(x) - \bar{\mu}_2(x)\right) > 0, \tag{1}$$

$$\limsup_{x\to\infty}\left(x\bar{\mu}_1(x)\right) < \infty.$$

Then $\mathbb{P}(\#\mathcal{C}_s = \infty) = 1$. *Moreover, if* $\mathbb{E}\, X_0 < \infty$ *then there is a constant* $c > 0$ *such that* $\mathbb{E}\,\#(\mathcal{C}_s \cap [0,x]) \geq c \log x$ *for all* x *sufficiently large.*

The hypotheses of Theorem 1 imply $X_n \to \infty$ a.s. is a result of Lamperti. By Lamperti's result, condition (1) is sufficient for transience and is equivalent to $d \geq 3$ in SSRW on $\mathbb{Z}^d$.

A sufficient condition for finitely many cutpoints

Our second result applies only to the Markov case.

Theorem 2 (L., Menshikov, Wade, 2020)

Suppose that some stronger regularity assumption on the process, (B)*, and* (V) *hold. Suppose also that there exist constants* $x_0 \in \mathbb{R}_+$ *and* $D < \infty$ *such that*

$$\mu_1(x) \geq 0 \text{ and } 2x\mu_1(x) - \mu_2(x) \leq \frac{D}{\log x}, \text{ for all } x \geq x_0. \quad (2)$$

Then $\mathbb{P}(\#\mathcal{C} < \infty) = 1$*.*

An example of transient processes with $\#\mathcal{C} < \infty$

Intuitively, we want processes that are 'less transient' than SSRW in $\mathbb{Z}^3$.

A more refined recurrence classification (see Menshikov et. al. (1995)) says that a sufficient condition for transience is, for some $\theta > 0$ and all x sufficiently large,

$$2x\mu_1(x) \geq \left(1 + \frac{1+\theta}{\log x}\right)\mu_2(x),$$

and a sufficient condition for recurrence is the reverse inequality with $\theta < 0$.

Example cont'

Example 1

If

$$\lim_{x\to\infty} \mu_2(x) = b \in (0,\infty), \text{ and } \mu_1(x) = \frac{a}{2x} + \frac{c + o(1)}{2x \log x},$$

then $a > b$ implies that there are infinitely many cutpoints by Theorem 1, and $a < b$ is recurrent (regardless of c).

The critical case has $a = b$, and then $c < b$ implies recurrence and $c > b$ implies transience.

This latter regime provides examples of processes with few cutpoints, as we show in Theorem 2.

See Csáki et. al (2010) for a sharper version in the nearest neighbour case.

Application to higher dimensions

Elliptic random walks were introduced in Georgiou et. al. (2016) and are non-homogeneous random walks with zero drift that can be transient in any dimension $d \geq 2$.

Theorem 3 (L., Menshikov, Wade, 2020)

Suppose that Ξ is a time-homogeneous transient elliptic random walk on $\Sigma \subseteq \mathbb{R}^d$. Then a.s., there are infinitely many cut annuli.

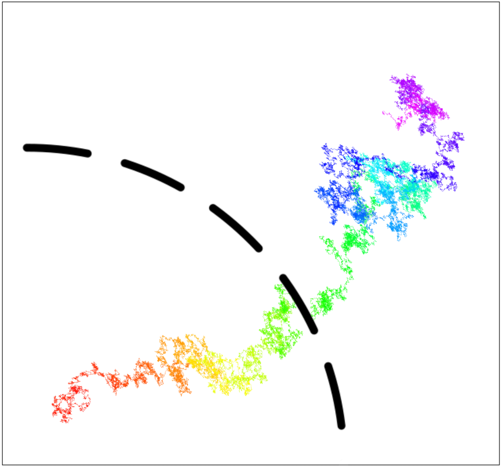
Application to higher dimensions (cont')

The following corollary is essentially due to James and Peres(1997), now follows as a special case of Theorem 3.

Corollary

Suppose we have a homogeneous random walk on $\mathbb{Z}^d$ *with bounded jumps, zero drift and finite variance. Then the random walk is transient and has infinitely many cut annuli.*

Example



A transient elliptic random walk and a cut annulus

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