On the centre of mass of a random walk

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Joint work with Andrew R. Wade

Near-Critical Stochastic Systems: a workshop in celebration of Mikhail Menshikov's 70th birthday
28th March, 2018

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Outline

The centre of mass of a random walk

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- Strong law of large numbers
- Central limit theorem

Main results

- Local central limit theorem
- Lattice distribution
- One dimension
- Two dimensions or more

Ideas of proofs and a conjecture

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The centre of mass of a random walk

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- Sequence of i.i.d. random variables: $X, X_1, X_2, \ldots$
- Random walk: $(S_n, n \in \mathbb{Z}_+)$ in $\mathbb{R}^d$ defined by

$$S_0 := \mathbf{0} \quad \text{and} \quad S_n := \sum_{i=1}^{n} X_i \quad (n \geq 1).$$

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$$S_0 := \mathbf{0} \quad \text{and} \quad S_n := \sum_{i=1}^{n} X_i \quad (n \geq 1).$$

- *Centre of mass process*: $(G_n, n \in \mathbb{Z}_+)$, corresponding to the random walk, defined by

$$G_0 := \mathbf{0} \quad \text{and} \quad G_n := \frac{1}{n} \sum_{i=1}^{n} S_i \quad (n \geq 1).$$

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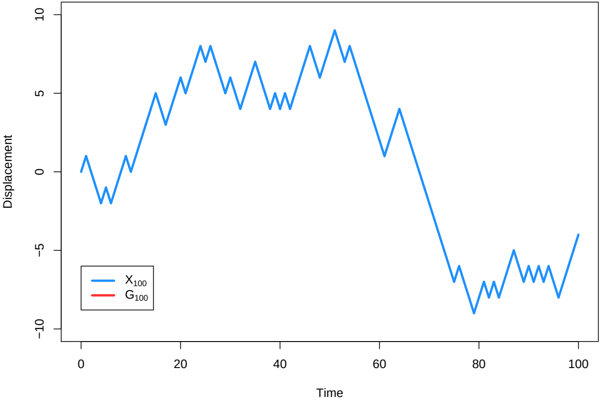
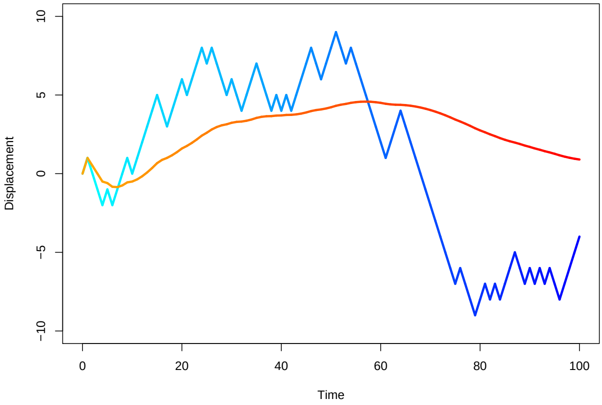
Center of mass in one dimension



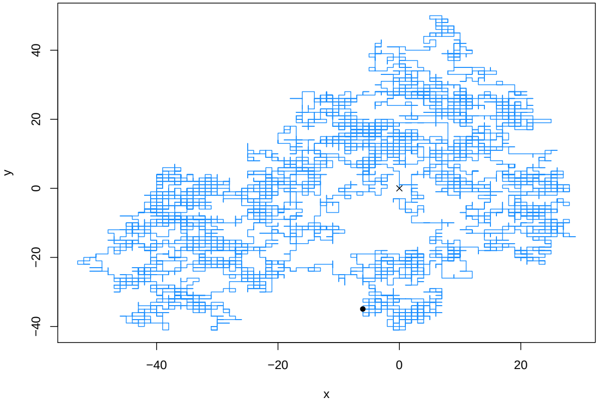
Blue: Random walk

Center of mass in one dimension



Light blue to blue: Random walk
Orange to red: Centre of mass

Center of mass and random walk in two dimensions



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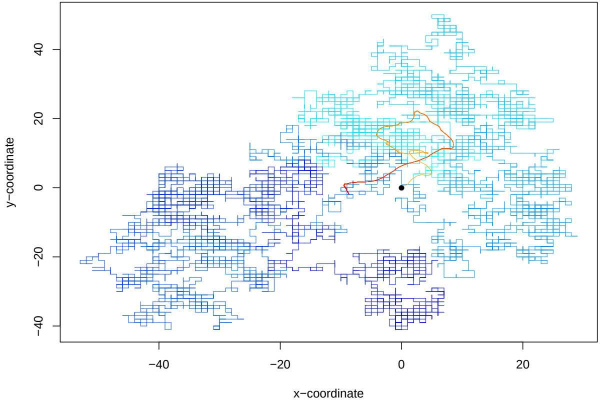
Center of mass and random walk in two dimensions



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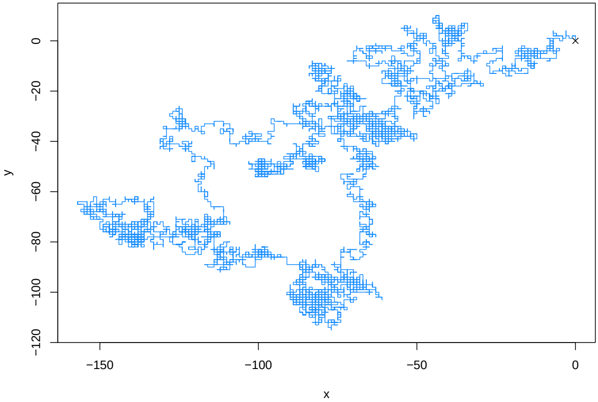
Center of mass and random walk in two dimensions (2)



Blue: Random walk

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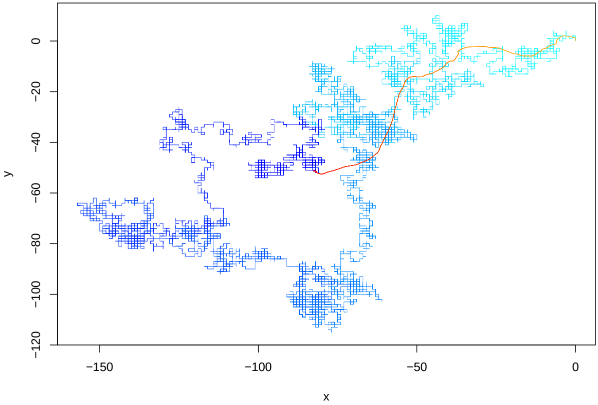
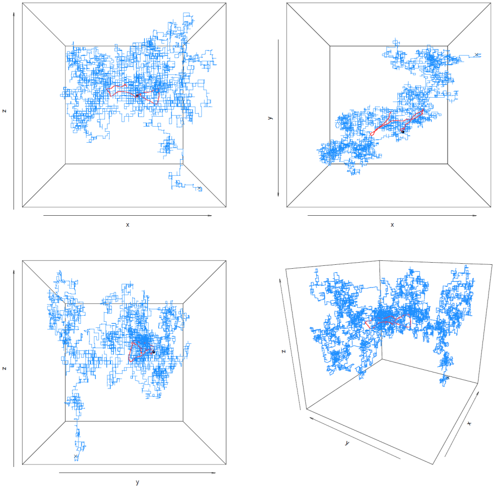


Light blue to blue: Random walk
Orange to red: Centre of mass

Center of mass and random walk in three dimensions



Blue: Random walk
Red: Centre of mass

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Motivation

- For S_n *simple symmetric random walk*, the problem of the asymptotic behaviour of G_n was posed by *P. Erdős* and solved by *K. Grill* (1988).

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- For S_n *simple symmetric random walk*, the problem of the asymptotic behaviour of G_n was posed by *P. Erdős* and solved by *K. Grill* (1988).
- G_n is an example of a *non-Markov process* of relevance for applications. E.g. if the random walk models a polymer chain, the centre of mass is of obvious physical significance.

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Notations and Assumptions

- Notation:

$$\boldsymbol{\mu} := \mathbb{E}X, \quad M := \mathbb{E}[(X - \boldsymbol{\mu})(X - \boldsymbol{\mu})^\top]$$

whenever the expectations exist; when defined, M is a symmetric d by d matrix.

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  whenever the expectations exist; when defined, M is a symmetric d by d matrix.

- Moment assumptions:
  - ($\boldsymbol{\mu}$) Suppose that $\mathbb{E}\|X\| < \infty$.
  - (M) Suppose that $\mathbb{E}[\|X\|^2] < \infty$ and M is positive-definite.

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Strong law of large numbers

From the (functional) strong law of large numbers for the random walk S_n, we get the following *strong law of large numbers* for G_n.

Proposition (L., Wade, 2017)

If (μ) *holds, then, as* $n \to \infty$,

$$n^{-1} G_n \to \frac{1}{2}\mu, \text{ a.s.}$$

Central Limit Theorem

With the help of Lindeberg–Feller theorem for triangular arrays, we have the following *central limit theorem*.

Proposition (L., Wade, 2017)

If (M) *holds, then, as* $n \to \infty$,

$$n^{-1/2}\left(G_n - \frac{n}{2}\boldsymbol{\mu}\right) \xrightarrow{d} \mathcal{N}_d(\mathbf{0}, M/3).$$

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Local central limit theorem

For our first main result, we assume that X has a lattice distribution.

(L) Suppose that X is non-degenerate. Suppose that for a constant vector $\mathbf{b} \in \mathbb{R}^d$ and a d by d matrix H with $|\det H| = h > 0$, where h is maximal, we have

$$\mathbb{P}(X \in \mathbf{b} + H\mathbb{Z}^d) = 1.$$

Also define

$$\mathcal{L}_n := \left\{ n^{-3/2} \left(\tfrac{1}{2} n(n+1)\mathbf{b} + H\mathbb{Z}^d \right) \right\}.$$

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Some examples

- Some complication on the lattice distribution. How to find the maximal span h?

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Some examples

- Some complication on the lattice distribution. How to find the maximal span h?
- This is not always immediate even for some classical random walks.

Trivial choice

- Trivial choice of lattice distribution for simple symmetric random walk, i.e. $\mathbf{b} = \mathbf{0}$ and $H = I$: Remarkably h is not maximal.

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- Which walk has this trivial choice as the right choice?
- Lazy simple symmetric random walk!
- Maybe this walk is just too lazy to bother with a complicated choice of a lattice distribution.
- How to verify that h is maximal?

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Simple symmetric random walk

Example (SSRW on $\mathbb{Z}^d$)

- Suppose that $\mathbb{P}(X = \mathbf{e}_i) = \mathbb{P}(X = -\mathbf{e}_i) = \frac{1}{2d}$ for all i.

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- Suppose that $\mathbb{P}(X = \mathbf{e}_i) = \mathbb{P}(X = -\mathbf{e}_i) = \frac{1}{2d}$ for all i.
- For SSRW the construction of H for which (L) holds is non-trivial.
- For $d = 1$, we take $b = -1$ and $h = 2$.
- In general $d \geq 2$, we take $H = (h_{ij})$ and $\mathbf{b} = (b_i)$ defined as follows.

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Simple symmetric random walk (cont.)

Example (cont.)

- If $d = 2n - 1$ for $n \geq 2, n \in \mathbb{Z}$, we take

$$b_i = -1 \quad \text{for all } i = 1, 2, \ldots, d;$$

$$h_{ij} = \begin{cases} 1 & \text{if } i - j \equiv 0 \text{ or } n \pmod{2n-1}, \\ 0 & \text{otherwise.} \end{cases}$$

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$$b_i = \begin{cases} 0 & \text{if } i = 2n, \\ -1 & \text{otherwise;} \end{cases}$$

$$h_{ij} = \begin{cases} -1 & \text{if } (i, j) = (2n, 1), \\ 1 & \text{if } j - i \equiv 0 \text{ or } 1 \pmod{2n} \text{ and } (i, j) \neq (2n, 1), \\ 0 & \text{otherwise.} \end{cases}$$

Simple symmetric random walk (cont.)

Example (cont.)

- E.g. for $d = 2$ we have

$$\mathbf{b} = \begin{pmatrix} -1 \\ 0 \end{pmatrix} \quad \text{and} \quad H = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}.$$

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- For $d = 3$, we have

$$\mathbf{b} = \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix} \quad \text{and} \quad H = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}.$$

Simple symmetric random walk (cont.)

Example (cont.)

- For $d = 4$, we have

$$\mathbf{b} = \begin{pmatrix} -1 \\ -1 \\ -1 \\ 0 \end{pmatrix} \quad \text{and} \quad H = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ -1 & 0 & 0 & 1 \end{pmatrix}.$$

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Simple symmetric random walk (cont.)

Example (cont.)

- For $d = 5$, we have

$$\mathbf{b} = \begin{pmatrix} -1 \\ -1 \\ -1 \\ -1 \\ -1 \end{pmatrix} \quad \text{and} \quad H = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{pmatrix},$$

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- Note that $h = 2$ for all such H.

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Local central limit theorem

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Local central limit theorem (cont.)

For $\mathbf{x} \in \mathbb{R}^d$, define $p_n(\mathbf{x}) := \mathbb{P}(n^{-1/2} G_n = \mathbf{x})$, and

$$n(\mathbf{x}) := \frac{\exp\{-\frac{3}{2}\mathbf{x}^\top M^{-1}\mathbf{x}\}}{(2\pi)^{d/2}\sqrt{\det(M/3)}}.$$

Theorem (L., Wade, 2017)

Suppose that (M), (L) *hold. Then we have*

$$\lim_{n\to\infty} \sup_{\mathbf{x}\in\mathcal{L}_n} \left| \frac{n^{3d/2}}{h} p_n(\mathbf{x}) - n\left(\mathbf{x} - \frac{(n+1)}{2n^{1/2}}\boldsymbol{\mu}\right)\right| = 0.$$

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One dimension: Recurrent case

- Depending on different moment assumptions, we can get very different recurrence behavour of the process. First we give a recurrence result in one dimension.

Theorem (L., Wade, 2017)

Suppose that $d = 1$ and that either of the following two conditions holds.

(i) *Suppose that $\mathbb{E}|X| \in (0, \infty)$ and $X \overset{d}{=} -X$.*

(ii) *Suppose that* (M) *holds and that $\mathbb{E}X = 0$.*

Then we have $\liminf_{n\to\infty} G_n = -\infty$, $\limsup_{n\to\infty} G_n = +\infty$ and $\liminf_{n\to\infty} |G_n| = 0$.

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One dimension: Recurrent case

- Depending on different moment assumptions, we can get very different recurrence behavour of the process. First we give a recurrence result in one dimension.
- In the case of SSRW the fact that G_n returns infinitely often to a neighbourhood of the origin is due to Grill[1988].

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One dimension: Transient case

On the other hand, if the first moment does not exist, G_n may be transient. The condition we assume is as follows.

(S) Suppose that $X \stackrel{d}{=} -X$ and X is in the domain of normal attraction of a symmetric α-stable distribution with $\alpha \in (0, 1)$.

Theorem (L., Wade, 2017)

Suppose that $d = 1$ and (L) holds, i.e., $\mathbb{P}(X \in b + h\mathbb{Z}^d) = 1$ for $b \in \mathbb{R}$ and $h > 0$. Also suppose that (S) holds. Then we have $\lim_{n\to\infty} |G_n| = \infty$.

Two dimensions or more

- The following theorem implies that $\lim_{n\to\infty} \|G_n\| = +\infty$ and moreover gives a diffusive rate of escape.

Theorem (L., Wade, 2017)

Suppose that $d \geq 2$ and that (M) and (L) hold. Also suppose that $\mu = \mathbf{0}$. Then

$$\lim_{n\to\infty} \frac{\log \|G_n\|}{\log n} = \frac{1}{2}, \ \textit{a.s.}$$

Two dimensions or more

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Idea of proof for recurrent case

Suppose $d = 1$.

- $$G_n = \sum_{i=1}^{n} \left(\frac{n - i + 1}{n} \right) X_i,$$

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- $|G_{n+1} - G_n| \to 0$ as $n \to \infty$.

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Idea of proof for transient case

Suppose $d \geq 2$. We sketch the proof of transience only.

- The idea is to use the local limit theorem to control (via Borel–Cantelli) the visits of G_n to a growing ball, along a subsequence of times suitably chosen so that the slow movement of the centre of mass controls the trajectory between the times of the subsequence as well.

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- The idea is to use the local limit theorem to control (via Borel–Cantelli) the visits of G_n to a growing ball, along a subsequence of times suitably chosen so that the slow movement of the centre of mass controls the trajectory between the times of the subsequence as well.
- Step 1: Local limit theorem implies that $\mathbb{P}(G_n \in \mathcal{B}) = O\left(n^{-\frac{d}{2}}\right)$ for a fixed ball $\mathcal{B}$.

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Idea of proof for transient case

- We have the following estimate on the deviations.

Lemma

Suppose that (M) *holds and that* $\mu = \mathbf{0}$. *Then, for any* $\varepsilon > 0$, *a.s. for all but finitely many* n,

$$\max_{n^2 \le m \le (n+1)^2} \| G_m - G_{n^2} \| \le n^{\varepsilon}.$$

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$$\max_{n^2 \le m \le (n+1)^2} \|G_m - G_{n^2}\| \le n^{\varepsilon}.$$

- Step 2: Slow movement of G_n implies that it suffices to control G_{n^2}.
- Step 3: $\mathbb{P}(G_{n^2} \in \mathcal{B}) \approx n^{-d}$, which is summable if $d \ge 2$.

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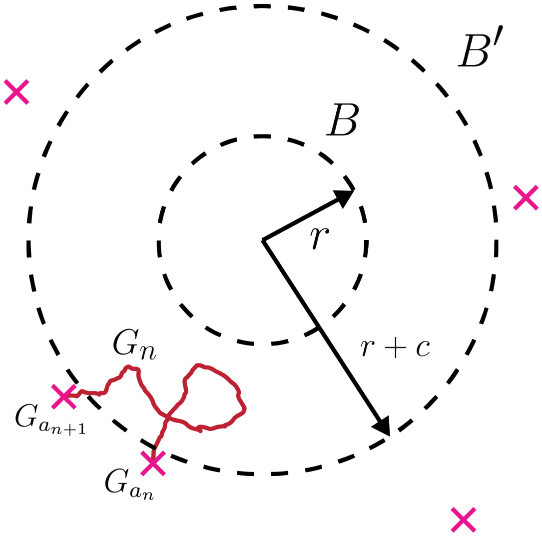
Idea of proof for transient case



Conjecture

- Obtaining necessary and sufficient conditions for recurrence and transience of G_n is an open problem.

Conjecture (L., Wade, 2017)

Suppose that $\operatorname{supp} X$ *is not conatined in a one-dimensional subspace of* $\mathbb{R}^d$*. Then*

$$\liminf_{n\to\infty} \frac{\log \|G_n\|}{\log n} \geq \frac{1}{2}, \ \textit{a.s.}$$

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Conjecture

- Obtaining necessary and sufficient conditions for recurrence and transience of G_n is an open problem.
- For $d \geq 2$, we believe that G_n is always ‘at least as transient’:

Conjecture (L., Wade, 2017)

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Back to Lattice assumption

- Denote

$$\mathcal{H} := \{H : \mathbb{P}(X \in \mathbf{b} + H\mathbb{Z}^d) = 1 \text{ for some } \mathbf{b} \in \mathbb{R}^d\}.$$

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$$\mathcal{H} := \{H : \mathbb{P}(X \in \mathbf{b} + H\mathbb{Z}^d) = 1 \text{ for some } \mathbf{b} \in \mathbb{R}^d\}.$$

- Suppose that the minimal subgroup of $\mathbb{R}^d$ associated with X is $L := H\mathbb{Z}^d$ with $h := |\det H| > 0$. Let

$$\mathcal{H}_0 := \{H \in \mathcal{H} : L = H\mathbb{Z}^d\}.$$

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$$\mathcal{H}_0 := \{H \in \mathcal{H} : L = H\mathbb{Z}^d\}.$$

- Let $K := \{|\det H| : H \in \mathcal{H}\}$.
- Denote $\varphi(\mathbf{t}) := \mathbb{E}[e^{i\mathbf{t}^\top X}]$ to be the characteristic function of X. Set $U := \{\mathbf{t} \in \mathbb{R}^d : |\varphi(\mathbf{t})| = 1\}$. Set $S_H := 2\pi(H^\top)^{-1}\mathbb{Z}^d$.

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Back to Lattice assumption

Lemma (L., Wade, 2017)

Suppose that X is non-degenerate and $H \in \mathcal{H}$. The following are equivalent.

(i) $H \in \mathcal{H}_0$.

(ii) $|\det H|$ *is the maximal element of* K.

(iii) $S_H = U$.

Acknowledgement

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