ORDINARY DIFFERENTIAL EQUATIONS (MATH 2030)

Lecture 1: Introduction

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Please read about recommended textbooks, homework, grading policy, exams, and the structure of the course at:

https://www.math.columbia.edu/~doan/ode.html

The same information is available on Canvas. If you have any questions, don't hesitate to ask in class, during office hours, or by email:

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REFERENCES

Boyce–DiPrima sections 1.1, 1.2, 1.3; Braun sections 1.1, 1.3, 1.5.

1 GENERAL COMMENTS

Don't hesitate to ask questions in class, feel free to interrupt me when I'm talking or send a message on zoom chat; the former is preferable as I don't see the chat while sharing the whiteboard. However, if you don't want to ask a question in front of the class, you're welcome to send me a private message by zoom chat or email.

I encourage you to attend office hours. We can discuss homework problems, review material discussed in class, and discuss any topic related to the course. Even if you don't have any particular questions, please stop by: you can benefit from hearing other people's questions and comments.

Homework must be submitted through gradescope. If you don't have an access to gradescope, please let me know. Please read about grading policy and academic integrity in the course syllabus or on the website.

The class will be attended by people with various majors and diverse math background. Please be respectful, supportive, and help your peers. It is hard to make such a course balanced: some of you might already know many of the topics we will discuss, whereas for the others they will be completely new. We can always adjust the difficulty and pace of the course. After a few weeks, I will ask you for your opinion: whether the course it too slow or too fast, too easy or too difficult etc. If you find the course moving too fast, please let me know and make sure to attend office hours. On the other hand, if the course is too slow or too easy for you, also let me know and we can find together some more interesting topics to learn about and more challenging homework problems for extra credit.

2 REVIEW OF CALCULUS

2.1 *Functions*

In your calculus class, you studied a variety of functions. In this course, by a function we mean a recipe for associating to a real number x another real number $f(x)$. For example, $f(x) = \sin x$, $f(x) = x^2$, and so on. In these examples $f(x)$ is defined for any real number x, but some functions are defined only for certain x. For example, $f(x) = \sqrt{x}$ is defined for $x \geq 0$ and $f(x) = 1/x$ is defined for $x \neq 0$.

2.2 *Derivatives*

Recall that the derivative of a function f at a number a, denoted $\mathrm{d}f/\mathrm{d}x(a)$ or $f'(a)$, is the limit (if it exists)

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

This number can be interpreted as the rate of change of f at a. For example, if $x(t)$ is the position of an object moving along the x–axis at time t, then $x'(t)$ is the speed of this object at time t. Geometrically, $f'(a)$ is the slope of the tangent line to the graph $y = f(x)$ at the point with coordinates $(a, f(a))$ on the xy–plane.

If $f'(a)$ exists for all a, we say that f is differentiable. The derivative of such a function f, denoted f' or $\mathrm{d}f/\mathrm{d}x$, is also a function. Namely, to any number a it associates the number $f'(a)$. If this function is also differentiable, we can define its derivative f''. This is the second derivative of f. Similarly, we can consider higher derivatives $f^{(3)}$, $f^{(4)}$, and so on; every next function is the derivative of the previous one (provided it exists).

For this course, it is important that you are able to compute derivatives of functions such that e^x, $\ln x$, x^α, trigonometric functions, and functions that are constructed from the above. Please review the rules for differentiating sums, products, quotients, and compositions of functions (the chain rule).

2.3 *Antiderivatives and integrals*

Given a function f an antiderivative of f is any function F such that $F' = f$. The antiderivative is not unique. If $F' = f$ and C is any constant, then the function $G(x) = F(x) + C$ is also an antiderivative: $G' = f$. In fact, if you find one antiderivative F of f, then any other antiderivative G of f is of the above for for some constant C. Antiderivatives are also called indefinite integrals and denoted $\int f(x)\mathrm{d}x$ or simply $\int f$, so for example

$$\int (x+1)\mathrm{d}x = \frac{1}{2}x^2 + x + C,$$

meaning that for any constant C the derivative of the right-hand side is the function $f(x) = x + 1$. This notation is inspired by the Fundamental Theorem of Calculus, which deals with *definite integrals*

$$\int_a^b f(x)\mathrm{d}x$$

which are defined as the area (with sign) of the region in the xy–plane bounded by the graph $y = f(x)$ and lines $x = a$, $x = b$ and the x–axis. The Fundamental Theorem of Calculus says that if $F = \int f(x)\mathrm{d}x$ is any antiderivative of f, then the definite integral is given by

$$\int_a^b f(x)\mathrm{d}x = F(b) - F(a).$$

Another formulation of this theorem is that if we define a function F by

$$F(t) = \int_{t_0}^t f(x)\mathrm{d}x$$

for any t_0, then F is an antiderivative of f, that is $F' = f$.

It is important that you know how to integrate simple functions such as e^x, $\ln x$, x^α, trigonometric functions and their inverses, rational functions (quotients of polynomials), and that you know various rules of integration (integration by parts, integration by substitution).

3 What is a differential equation?

In school, you studied algebraic equations such as

$$x^2 + 2x + 3 = 0, \quad \text{etc.}$$

Given an equation, you are interested in finding all numbers x solving it. Similarly, you can consider systems of equations for many unknown numbers $x, y, \ldots$

A differential equation is an equation for an unknown function, which involves the function and its derivatives (first, second, and higher). Coefficients of such an equations are also function. For example, we might be interested in finding all functions $x(t)$ of a variable t which satisfy

$$x'(t) + 2t^2 x(t) + \sin t = 0.$$

This equation involves the unknown function $x(t)$ and its first derivative $x'(t)$, and the coefficients are known functions of t: $2t^2$ and $\sin t$. We can also have equations which involve higher derivatives, or systems of equations involving many functions $x(t), y(t), z(t), \ldots$ and their derivatives.

Remark 1. We call a differential equation ordinary if it involves functions of one variable. Differential equations for functions of many variables $f(x, y, z, \ldots)$ (such as the ones you studied in a multivariable calculus class) involving their partial derivatives $\partial f/\partial x, \partial f/\partial y, \ldots$ are called partial differential equations. In this class, we will study only ordinary differential equations, which we will often abbreviate to ODE.

Given a differential equation such as the one above here are some typical questions we are interested in: Is there a solution to the equation? Is is unique or maybe there are many solutions? Can we find a solution (or all solutions), either by guessing or by using some special technique? In this course, we will learn a few such techniques. Perhaps even if we cannot write a formula for the solution, maybe we can say something about its behavior? For example, does the solution grow or decay, or has a limit or asymptotic behaviour as $t \to \infty$? Are there any quantities associated with the solution that are constant in t, such as energy in physical systems? Can we approximate the solution with a solution to a different, simpler equation that we understand better?

First, can we solve it. Is there a solution? Given a solution, is it the only one or maybe there are many? Can we write it down explicitly (in terms of functions we know)? If not, can we still say something about how this solutions behaves? For example, what happens when $t \to \infty$?

Example 2 (Simplest case)**.** The simplest example of a differential equation is one of the form
$$x'(t) = f(t),$$
where x is an unknown function and f is known function of t. This equation says that x is an antiderivative of f, so that any solution is given by

$$x(t) = \int f(t) \mathrm{d}t.$$

Since the indefinite integral is defined only up to adding a constant, there is an entire family of solutions obtained as follows. If $y(t)$ is any f solution, then all other solutions are of the form $x(t) = y(t) + C$ for some constant C.

Suppose we are given t_0 and x_0 and we look for a solution $x(t)$ such that $x(t_0) = x_0$. This specifies our solution uniquely. Indeed, since any solution is of the form $x(t) = y(t) + C$, by plugging $t = t_0$ we get

$$x_0 = x(t_0) = y(t_0) + C,$$

so $C = x_0 - y(t_0)$ is determined.

Most differential equations are more complicated than the above example and we cannot solve them so easily. For example, suppose that we are given $f(t)$ and we look for a function $x(t)$ such that

$$x'(t) = x(t) + f(t).$$

We cannot solve it anymore in the same way as in the above example: if we try to integrate both sides, we get $x(t) = \int x(t) \mathrm{d}t + \int f(t) \mathrm{d}t$. This doesn't get us anywhere since in order to compute the right-hand side we would have to know $x(t)$ to begin with. However, there are more sophisticated techniques for solving differential equations such as the one above and in this class we will learn some of these techniques.

4 MOTIVATION AND EXAMPLES

Differential equations are used to model how natural or social phenomena change in time. An unknown function $x(t)$ corresponds to some quantity we are interested in, which depends on time t. We want to predict the evolution of $x(t)$. While $x(t)$ is not given by an explicit formula, we know its value at some initial time t_0, and a principle, or law, governing its evolution in time. Such a law is often expressed as a differential equation. Therefore, by solving th differential equation we can predict the future of a system from its initial state together with a law governing its evolution in time.

Example 3 (Newton's laws)**.** The first differential equation ever studied in history was Newton's law of motion: $F = ma$. Here F is the force acting on a certain body of mass m, and a is the acceleration of this body. Suppose that the body moves along the x–axis and its position at time t is denoted by $x(t)$.

The acceleration is then defined as the second derivative of the position with respect to time: $a(t) = x''(t)$.

Suppose that there is no force acting on the body: $F = 0$. Thus, our differential equation is simply $x''(t) = 0$. Integrating, this gives us $x'(t) = C$ for some constant C. Integrating again, we get $x(t) = Ct + D$ for some other constant D. Thus, we recover Newton's first law: an object upon which no forces act will remain in motion with constant velocity. What is the interpretation of the constants C and D? By plugging $t = 0$ we get $x(0) = D$ and $x'(0) = C$. Therefore, C and D are, respectively, the velocity and position of the body at time $t = 0$.

We similarly solve the equation if the force is constant $F = \text{const}$. This is the case, for example, for a massive object placed in a constant graviational field. Denote for simplicity $F/m = k$. In this case, integration gives us $x'(t) = kt + C$ and $x(t) = \frac{1}{2}kt^2 + Ct + D$. We see that the motion is described by a quadratic function. Indeed, if you throw a rock it will move along a parabola, which corresponds to the fact that its vertical position is a quadratic function of time, whereas the horizontal position is a linear function of time (because the gravitational force acting in the vertical direction is constant and there is no force acting in the horizontal direction). Again by plugging $t = 0$ we interpret C and D as the initial velocity and position.

Example 4 (Harmonic oscillator)**.** In general, the force in Newton's equation $F = mx''(t)$ can depend on the position x and time t, and we cannot solve the equation simply by integrating both sides. For example, if $x(t)$ describes the position of a body attached to an elastic spring, Hooke's law of elasticity states that the force experiencing by the body is proportional to the deformation of the spring, i.e. position, if $x = 0$ corresponds to the spring being undeformed: $F = -kx(t)$ for some constant $k > 0$; we have the minus sign because the body will be pulled back by the spring. We end with a differential equation

$$x''(t) = -\frac{k}{m}x(t).$$

Can you guess a solution? For simplicity, assume $k/m = 1$. The equation is now easier

$$x''(t) = -x(t).$$

Your first guess might be the exponential function $x(t) = e^t$, but this doesn't work because of the sign. Indeed $x(t) = e^t$ satisfies the equation $x''(t) = x(t)$ and not $x''(t) = -x(t)$. But if we try

$$x(t) = \sin t \quad \text{or} \quad x(t) = \cos t,$$

we easily compute that these functions solve the equation. In fact, for any constants A, B the function

$$x(t) = A \sin t + B \cos t$$

solve the equation. We will see in future lectures that, in fact, any solution to the equation $x''(t) = -x(t)$ is of this form. How can be interpret the constants A and B? Again, by plugging $t = 0$, we see that $B = x(0)$ is the initial position. On the other hand, differentiating the formula for $x(t)$ we get

$$x'(t) = A \cos t - B \sin t$$

and plugging $t = 0$ we see that $A = x'(0)$ is the initial velocity. Note that the movement of the body is periodic because the functions $\sin t$ and $\cos t$ are periodic. This is very different from the behavior of the function $x(t) = e^t$ which solved the other equation $x''(t) = x(t)$; the exponential function is not periodic, in fact it grows very fast as $t \to \infty$.

Having guessed a solution in the special case $k/m = 1$, we can easily modify it to the general case. You can check that any function of the form

$$x(t) = A \sin \omega t + B \cos \omega t$$

for a constant ω, satisfies the equation $x''(t) = -\omega^2 x(t)$. Thus, for $\omega = \sqrt{k/m}$ we find a solution to the original problem.

The harmonic oscillator equation is extremely important, not because we care so much about springs, but because it models the behavior of many other systems, and is the first order approximation of many differential equations. In particular, suppose that the force depends only on the position, $F = F(x)$. Any function $F(x)$ has a Taylor expansion

$$F(x) \approx F_0 + F_1 x + F_2 x^2 + \dots$$

If in the initial position $x = 0$ there is no force, $F_0 = 0$, and if the first order expansion is negative, i.e. the force pulls the body back rather than pushing it forward, we get a differential equation

$$mx''(t) = -kx(t) + \text{ higher order terms.}$$

Thus, as long as $x(t)$ stays sufficiently small, the harmonic oscillator is a good approximation of the general model.

More examples in the next lecture!

ORDINARY DIFFERENTIAL EQUATIONS (MATH 2030)

Lecture 2: Separable equations

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REFERENCES

Boyce–DiPrima sections 2.2, 2.5; Braun sections 1.3, 1.4, 1.5.

1 MOTIVATION AND EXAMPLES

Last time we discussed some examples of differential equations and two basic methods of finding solutions. The first method applies to trivial differential equations of the form $x'(t) = f(t)$. In this case, we simply integrate both sides to find $x(t)$. The second method is simply trying to guess a solution, based on our knowledge of functions and their derivatives, as we did in the example of the harmonic oscillator $x''(t) = -x(t)$. Let us discuss two more examples.

Example 1 (Population models)**.** Let $p(t)$ be the population of a certain species. We are interested in how $p(t)$ evolves in time. The simplest model is to assume that the birth and death rates are constant in time. If this is the case, the rate of change of $p(t)$ is proportional to $p(t)$:

$$p'(t) = \lambda p(t) \quad \text{for a constant } \lambda > 0. \tag{1.1}$$

As in the case of the harmonic oscillator, this equation is simple enough so that we can guess a solution. Suppose first that $\lambda = 1$, so that we look for a function $p(t)$ such that $p'(t) = p(t)$. The obvious candidate is $p(t) = e^t$. In fact, $p(t) = Ce^t$ works for any constant C. This suggests the solution for a general value of λ:

$$p(t) = Ce^{\lambda t}.$$

We easily check that $p(t)$ solves (1.1). What is the meaning of the constant C? As usual, set $t = 0$, then we see that

$$p(0) = C,$$

so C is the initial population and our solution is

$$p(t) = p(0)e^{\lambda t}.$$

This model predicts that the population will grow exponentially fast in time. This simple model was proposed by the English scholar Thomas

Malthus in 1798. Based on it, Malthus predicted that such a fast growth of population will lead to a catastrophe as the growth of population will outpace agricultural production, thus restricting humanity's access to food and other resources and resulting in famine and diseases. Malthus' book was highly influential. It gave Charles Darwin the idea that with the growth of populations, animals have to compete for limited resources and only the best adapted survive. Thus, Malthus' simple model was an inspiration for Darwin's theory of natural selection.

However, the Malthussian model is too simple to be realistic. There are many other models that try to correct it. One idea is to assume that it is valid as long as the population is not too large. However, as the population grows, the Malthussian model fails to be realistic because it does not take into account that organisms compete for limited resources. One model that takes this into account is the following *logistic model* proposed by the Dutch mathematician Verhulst in 1937:

$$p'(t) = ap(t) - bp(t)^2, \tag{1.2}$$

for some constants $a, b > 0$. In this case, it's not so easy to guess a solution directly, as we did before (try it!), so we need to develops new tools for solving differential equations of this form.

Example 2 (Radioactive decay)**.** The simplest model of radioactive decay is the following. If $N(t)$ denotes the number of atoms of a given radioactive substance at time t, then rate of decay of $N(t)$ is proportional to $N(t)$, that is:

$$N'(t) = -aN(t)$$

for a constant $a > 0$. As before, we can guess a solution

$$N(t) = N(0)e^{-at}.$$

We can consider a more complicated model in which one radioactive substance A decays to another one B and then B decays further. In that case, the equation should reflect the fact that the amount of B will at the same time (a) increase to due radioactive decay of A to B, and (b) decay due to radioactive decay of B. If $r(t)$ denotes the rate of decay of A, and $N(t)$ denotes the number of atoms of B at time t, then

$$N'(t) = r(t) - aN(t),$$

where the first time corresponds to (a) and the second to (b). In some situations the half-life of A is much longer than that of B. If we are interested in how $N(t)$ changes over a period of time much shorter than the half-life of A, we can for all practical purposes assume that $r(t)$ does not change much in that period of and therefore set $r(t) = \text{const}$. This leads to the equation

$$N'(t) = r - aN(t), \tag{1.3}$$

for $r > 0$ constant. This simple model was actually used to detect art forgeries of Vermeer's paintings! You can read about in section 1.3 of Braun's book. In this lecture will learn how to solve (1.3) and similar equations.

2 Separable equations

We said that the simplest differential equations are ones of the form

$$x'(t) = f(t).$$

We can solve it simply by integrating both sides with respect to t. The next simplest case are *separable equations* of the form

$$x'(t) = \frac{f(t)}{g(x(t))} \quad \text{or simply} \quad x' = \frac{f(t)}{g(x)}. \tag{2.1}$$

They are called separable because the right-hand side can be separated into two terms: one depending only on t and one depending only on x. Equations (1.2) and (1.3) we just saw are examples of such equations, with $f(t) = 1$ in both cases and $g(x)$ given, respectively, by

$$g(x) = \frac{1}{ax - bx^2} \quad \text{and} \quad g(x) = \frac{1}{-ax + r}.$$

A separable equation can be solved as follows. Suppose that $x(t)$ is a solution, multiply both sides by $g(x)$ and integrate with respect to t:

$$\int g(x(t))x'(t)\mathrm{d}t = \int f(t)\mathrm{d}t. \tag{2.2}$$

If you remember integration by substitution, you will notice that the left-hand side is simply

$$\int g(x(t))x'(t)\mathrm{d}t = \int g(x)\mathrm{d}x.$$

Suppose that we can compute both integrals $\int f(t)\mathrm{d}t$ and $\int g(x)\mathrm{d}x$, i.e. find antiderivatives of f and g. Denote them by $F(t)$ and $G(x)$. By (2.2),

$$G(x(t)) = F(t). \tag{2.3}$$

We can the find $x(t)$ by solving the above equation for $x(t)$. In technical terms, we apply the inverse function of G, i.e. a function G^{-1} such that $G^{-1}(G(x)) = x$. (We assume here that there exists such a function.) Applying the inverse to both sides we find

$$x(t) = G^{-1}(F(t)).$$

Remember that an antiderivative is determined only up to a constant. Once we have chosen an antiderivative F of f, we can write other equations as

$$x(t) = G^{-1}(F(t) + C)$$

for some constant C. (Similarly, G is only determined up to a constant. However, we can choose our favorite G and incorporate the constant in the right-hand side of (2.3).)

Example 3 (Population model)**.** Suppose we couldn't guess a solution to the Malthussian model (1.1). This is an example of a separable equation so we can apply the general method. Write (1.3) as

$$\frac{p'(t)}{p(t)} = \lambda$$

and integrate both sides with respect to t:

$$\int \frac{p'(t)\mathrm{d}t}{p(t)} = \lambda t + C$$

for some constant C. Integrating by the substitution $p = p(t)$ we get

$$\int \frac{\mathrm{d}p}{p} = \lambda t + C,$$

and since $\int p^{-1}\mathrm{d}p = \ln|p|$ and we assume here that p is positive,

$$\ln p(t) = \lambda t + C.$$

Applying the exponential function to both sides and using the formulae $e^{\ln x} = x$ and $e^{x+y} = e^x e^y$, we get

$$p(t) = e^C e^{\lambda t}.$$

Since C was an arbitrary constant, we can simply rename $A = e^C$ and now A is an arbitrary positive constant, so the solution is

$$p(t) = Ae^{\lambda t}.$$

(Of course, for $A < 0$ the resulting function is also a solution but for our model only the positive solutions make sense.)

Next week we will do more examples of separable equations and learn how to solve first order linear equations.

ORDINARY DIFFERENTIAL EQUATIONS (MATH 2030)

Lecture 3: First order linear equations

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REFERENCES

Boyce–DiPrima sections 1.3, 2.1, 2.5 (problems 22–24); Braun section 1.2.

1 SEPARABLE EQUATIONS

Last time we discussed separable equations:

$$y' = \frac{f(t)}{g(y)}.$$

Remark 1. Remember our convention to write for simplicity $y, y', \ldots$ instead of $y(t), y'(t), \ldots$ where $y = y(t)$ is the unknown function in the differential equation. The known coefficients of the equation will still be denoted $a(t), b(t), \ldots$ to stress that they are functions of t rather than constants. For example, today we will consider equations of the form $y' + a(t)y + b(t) = 0$.

The algorithm for solving separable equations is:

1. multiply both sides by $g(y)$,
2. integrate both sides with respect to t,
3. integrate $\int g(y(t))y'(t)\mathrm{d}t$ by substitution $y = y(t)$,
4. solve the resulting equation for $y(t)$ (if possible).

Example 2 (Epidemics)**.** As an example, we will discuss a simple model describing the spread of a contagious disease. (We will discuss more complicated, and realistic, models later in the course once we develop more tools.) Let $x = x(t)$ and $y = y(t)$ denote the proportion of the population which at time t is, respectively, healthy and infected. We have $x + y = 1$. In this simple model, we assume that everyone in the population is susceptible to the disease. Moreover, we assume that the members of the population interact freely with each other, and that the rate with which the disease spreads is proportional to the number of contacts between healthy and infected individuals. Since this number of contacts is proportional to xy, we get the differential equation:

$$y' = \alpha xy$$

for some constant $\alpha > 0$. There are two unknown functions in this equation, x and y. However, using $x + y = 1$, we get

$$y' = \alpha y(1 - y).$$

This is a separable equation, so we solve it using our algorithm. First, we separate the variables

$$\frac{y'}{y(1-y)} = \alpha.$$

(Note that in dividing by $y(1-y)$ we assume that $y \neq 0$ and $y \neq 1$. Constant functions $y(t) = 0$ and $y(t) = 1$ are solutions but they are not interesting as they describe a population in which everybody is healthy or everybody is already infected.) Integrating with respect to t and using the substitution $y = y(t)$ yields

$$\int \frac{\mathrm{d}y}{y(1-y)} = \int \frac{y'(t)}{y(t)(1-y(t))}\mathrm{d} = \int \alpha \mathrm{d}t = \alpha t + C.$$

The left-hand side is an example of an integral of a rational function. It will be useful for this course if you remind yourself how to integrate rational functions. (Just search 'integration of rational functions' or 'integration partial fractions', or simply ask me during office hours.) The trick is to write

$$\frac{1}{y(1-y)} = \frac{1}{y} + \frac{1}{1-y}.$$

Therefore,

$$\int \frac{\mathrm{d}y}{y(1-y)} = \int \frac{\mathrm{d}y}{y} + \int \frac{\mathrm{d}y}{1-y} = \ln|y| - \ln|1-y|.$$

The minus sign in the second terms comes from integrating by substitution $u = 1 - y$, $\mathrm{d}u = -\mathrm{d}y$. Recall that y is the proportion of infected population. In particular, $0 < y < 1$ and we can drop the absolute value. (There are other solutions to the equations which don't satisfy this condition, but we discard them as unimportant for our model.) In the end, we get

$$\ln y - \ln|1-y| = \alpha t + C.$$

Applying the exponential function to both sides yields

$$\frac{y(t)}{1-y(t)} = Ae^{\alpha t}, \tag{1.1}$$

where $A = e^C$ is a positive constant. We can compute it by plugging $t = 0$:

$$A = \frac{y(0)}{1-y(0)} = \frac{y(0)}{x(0)}.$$

We can solve equation (1.1) for $y(t)$. Simple algebraic manipulations give us

$$y(t) = \frac{Ae^{\alpha t}}{1+Ae^{\alpha t}} \quad \text{and} \quad x(t) = 1 - y(t) = \frac{1}{1+Ae^{\alpha t}}.$$

It is interesting to see what this tells us about the long-time behavior of our model. We easily see that

$$\lim_{t\to\infty} y(t) = 1 \quad \text{or equivalently} \quad \lim_{t\to\infty} x(t) = 0.$$

Therefore, our simple model predicts that eventually the disease spreads through the entire population.

A differential equation involves a function, say y, and its derivatives:

$$y', y'', y^{(3)}, y^{(4)}, \ldots$$

The *order* of a differential equation is the highest number n such that the n–th derivative $y^{(n)}$ appears on the equation. Here are some examples of differential equations of order one, two, and three.

equation	order
$y'(t) + ay(t) + by^2(t) = 0$	1
$(y''(t))^2 + aty'(t) + by(t) + ct^2 = 0$	2
$y^{(3)}(t) + ty''(t) + a(1 - y'(t)^2) = 0$	3
$\ldots$	

A differential equation is *linear* if it is of the form

$$a_n(t)y^{(n)} + a_{n-1}(t)y^{(n-1)} + \ldots + a_1(t)y' + a_0(t)y + b(t) = 0 \tag{2.1}$$

for some functions $a_n, \ldots, a_0, b$. (Recall that we often drop the variable t and write y', y instead of $y'(t)$, $y(t)$.) Otherwise we call the equation *nonlinear*. For example, the following equations are linear

$$\begin{aligned} &y' - 2ty - 1 = 0, \\ &y'' + \frac{1}{t}y' - \cos t\; y + e^t = 0, \\ &y^{(4)} + e^t\; y = 0, \\ &\ldots \end{aligned}$$

On the other hand, all of the examples in the table above are non-linear. They are, in fact, equations that appear in actual scientific problems: we have already seen the first one in the logistic model of population growth. For more examples of interesting non-linear equations, see:

https://en.wikipedia.org/wiki/List_of_nonlinear_ordinary_differential_equations

We say that a linear differential equation (2.1) has *constant coefficients* if the functions $a_n, a_{n-1}, \ldots, a_0$ are all constant. However, we do not require that b is constant. Here are some examples of linear equations with constant coefficients (compare with the previous list):

$$\begin{aligned} &y' + 5y = \sin t, \\ &y'' + 10y = 0, \\ &y^{(4)} + y = e^t, \\ &\ldots \end{aligned}$$

Finally, a linear differential equation is (2.1) is *homogenous* if $b = 0$ and *non-homogenous* otherwise.

To get used to this terminology, have a look at various differential equations in the textbook or on Wikipedia and determine their order and if they are linear or non-linear. If they are linear, do they have constant coefficients? Are they homogenous?

Our next goal is to learn how to solve first order linear differential equations:

$$y' + a(t)y + b(t) = 0.$$

Let us first consider some special cases.

3.1 *Trivial case*

If $a(t) = 0$, then the equation is $y' = -b(t)$ and we can simply solve it by integrating both sides with respect to t:

$$y(t) = -\int b(t)\mathrm{d}t.$$

3.2 *Homogenous case*

Another special case is when the equation is homogenous, that is $b(t) = 0$:

$$y' + a(t)y = 0.$$

This is a separable equation, so we already know how to solve it. First, we separate terms depending on y and those depending on y:

$$\frac{y'}{y} = -a(t).$$

Now we integrate both sides with respect to t

$$\int \frac{y'(t)}{y(t)}\mathrm{d}t = -\int a(t)\mathrm{d}t + C,$$

(where we add a constant of integration C to remind ourselves that the indefinite integral is determined only up to a constant) and use integration by substitution

$$\int \frac{y'(t)}{y(t)}\mathrm{d}t = \int \frac{\mathrm{d}y}{y} = \ln|y(t)|$$

and

$$\ln|y(t)| = -\int a(t)\mathrm{d}t + C$$

Applying the exponential function to both sides, we obtain

$$|y(t)| = e^{-\int a(t)\mathrm{d}t + C} = e^C e^{-\int a(t)\mathrm{d}t}.$$

In other words, if we have chosen any specific indefinite integral $\int a(t)\mathrm{d}t$, then any solution is of the form

$$y(t) = Ae^{-\int a(t)\mathrm{d}t}$$

for some constant A. Conversely, for any constant A the above function is a solution.

Remark 3. About constants: in the previous notation, $A = \pm e^C$ where the sign appears when we drop the absolute value. Note that since C can be anything, the expression $A \pm e^C$ can be any non-zero real number. On the other hand, $y(t) = 0$ is clearly also a solution. In our method of solving the equation, we missed that solution by dividing the equation by y: in order to do that, we assume that $y \neq 0$.

3.3 *General case*

We can now solve a general first order linear equation

$$y' + a(t)y + b(t) = 0.$$

We already know how to solve the equation in the trivial case $y' = -b(t)$. The trick of solving the general case is to reduce it to the trivial case. We do it by multiplying the equation by a function $\mu = \mu(t)$ (called the *integrating factor*), which we will specify later:

$$\mu(t)y' + a(t)\mu(t)y = -\mu(t)b(t). \tag{3.1}$$

Why do we do this? To make the left-hand side look similar to the formula for the derivative of the product

$$(\mu y)' = \mu(t)y' + \mu'(t)y.$$

If only we had

$$\mu'(t) = a(t)\mu(t) \tag{3.2}$$

then using the formula for $(\mu y)'$ we could rewrite (3.1) as

$$(\mu y)' = -\mu(t)b(t).$$

But now we can compute μy simply by integrating both sides. That would give us a formula for μy:

$$\mu(t)y(t) = -\int \mu(t)b(t)\mathrm{d}t \tag{3.3}$$

and after dividing by $\mu(t)$ we would get a formula for a solution $y(t)$. Of course, if we take any function μ, it will not satisfy equation (3.2). But the point is that in our method we didn't specify what μ was. We are allowed to choose any μ, so in particular we can choose one that satisfies (3.2). How do we find such a μ? Observe that (3.2) is a homogenous, first order linear differential equation for μ and we already know how to solve such equations! Therefore, we can first find a solution μ to (3.2) and then plug it to (3.3) to find the desired solution $y(t)$.

You should not memorize the final formula for $y(t)$ but rather understand the algorithm for deriving it. This algorithm is summarized as follows:

1. Multiply the equation $y' + a(t)y + b(t) = 0$ by a function $\mu(t)$.
2. Observe that this equation is equivalent to $(\mu y)' = -\mu(t)b(t)$ if μ satisfies the homogenous equation $\mu' = a(t)\mu$.
3. Solve the homogenous equation $\mu' = a(t)\mu$ (this is a separable equation).
4. Find μy by integrating both sides of $(\mu y)' = -\mu(t)b(t)$.
5. Finally, divide by $\mu(t)$ to get a formula for y.

Next time we will discuss some examples of first order linear equations.

ORDINARY DIFFERENTIAL EQUATIONS (MATH 2030)

Lecture 4: Existence and uniqueness of solutions

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REFERENCES

Boyce–DiPrima section 2.3 (Problem 16), section 2.8; Braun section 1.10.

1 FIRST ORDER LINEAR EQUATIONS

Last time we discussed the following algorithm for solving a first order linear differential equation

$$y' + a(t)y + b(t) = 0.$$

1. Multiply the equation by a function $\mu(t)$ (the integrating factor).
2. Observe that the resulting equation

   $$\mu(t)y' + \mu(t)a(t)y = -\mu(t)b(t)$$

   is equivalent to $(\mu y)' = -\mu(t)b(t)$ if μ satisfies

   $$\mu' = a(t)\mu.$$

3. Solve the homogenous equation $\mu' = a(t)\mu$ (it is separable).
4. Find μy by integrating both sides of $(\mu y)' = -\mu(t)b(t)$.
5. Finally, divide by $\mu(t)$ to get a formula for y.

Remark 1. Observe that the equation $\mu' = a(t)\mu$ satisfied by the integrating factor **is different from** the homogenous equation obtained by setting $b(t) = 0$ in the original equation $y' + a(t)y + b(t) = 0$. They differ by a sign.

Example 2 (Newton's law of cooling)**.** Let $u = u(t)$ be the temperature of the body at time t and let $T(t)$ be the temperature of its surroundings, which can also vary in time. Newton's law of cooling is:

$$u' = -k(u - T(t)).$$

Suppose that the temperature of the surroundings oscillates periodically, for example $T(t) = T_0 + T_1 \cos \omega t$. We get a first order linear, non-homogenous equation:

$$u' + ku = k(T_0 + T_1 \cos \omega t).$$

To simplify calculations suppose that $k = T_1 = \omega = 1$ and $T_0 = 0$:

$$u' + u = \cos t.$$

According to the algorithm from the last lecture, first we solve the corresponding homogenous problem to find the integrating factor:

$$\mu' = \mu.$$

Of course, $\mu(t) = e^t$ is a solution. We multiply the original equations by μ:

$$\mu u' + \mu u = \mu \cos t,$$

which, by the definition of μ, is equivalent to

$$(\mu u)' = \mu u' + \mu' u = \mu \cos t.$$

Therefore,

$$u(t) = \frac{1}{\mu(t)} \int \mu(t) \cos t \text{ d}t = e^{-t} \int e^t \cos t \text{ d}t.$$

Integrating by parts, we compute

$$\int e^t \cos t \mathrm{d}t = \frac{1}{2} e^t (\sin t + \cos t) + C.$$

(If you don't remember how to integrate functions like $e^t \cos t$, $t^n e^t$, etc. by parts, please review this or ask me during office hours.) We can slightly rewrite the right-hand side using the fact that any expression of the form

$$a \sin t + b \cos t \quad \text{where } a^2 + b^2 = 1$$

can be written as $\sin(t + \varphi)$ for some angle φ computed from a and b. Indeed, for $\varphi = \arctan b/a$ we have

$$a = \cos \varphi \quad \text{and} \quad b = \sin \varphi,$$

and, using the formula for $\sin(t + \varphi)$,

$$a \sin t + b \cos t = \cos \varphi \sin t + \sin \varphi \cos t = \sin(t + \varphi).$$

In our case, we use $a = b = 1/\sqrt{2}$ to get $\varphi = \arctan 1 = \pi/4$ and

$$\sin t + \cos t = \sqrt{2} \left(\frac{1}{\sqrt{2}} \sin t + \frac{1}{\sqrt{2}} \cos t \right) = \sqrt{2} \sin(t + \pi/4).$$

In the end, our solution is

$$u(t) = \frac{\sqrt{2}}{2} \sin(t + \pi/4) + Ce^{-t}. \tag{1.1}$$

We see that as $t \to \infty$ the second term converges to zero and the solution is asymptotic to the periodic function

$$\frac{\sqrt{2}}{2} \sin(t + \pi/4).$$

The period is 2π, the same as that of the temperature of the surroundings $T(t) = \cos t$. However, as you can see by plotting the graphs of both functions, the sinusoidal oscillation of the above function is shifted with respect to the oscillation of T.

The above example illustrates a general principle, which we have already seen many times. When we solve a first order differential equation

$$y' = f(t,y)$$

we obtain an entire family of solutions, parametrizes by a constant C, as in (1.1). This is called the *general solution*. Typically, however, we are interested in finding a specific solution solving the *initial value problem*, that is:

$$\begin{cases} y' = f(t,y), \\ y(t_0) = y_0, \end{cases}$$

for some given numbers t_0, y_0. In all examples that we studied earlier, the initial condition determined the solution, that is: among all solutions of the general form there was exactly one satisfying the given initial value condition $y(t_0) = y_0$.

In general, given an initial value problem as the one above, we ask:

1. (Existence) Is there a solution to the initial value problem?
2. (Uniqueness) If yes, is this solution unique, i.e. is there no other function solving the same initial value problem?

These questions are of great importance, both theoretically, and in applications. If we use a differential equation to model natural phenomena, and the equation does not have any solutions satisfying our initial value problem, that means that our model is wrong. On the other hand, uniqueness is related to the question whether our model is deterministic, i.e. whether we can predict the future behavior of our model from its past. If we can't, then we can't use our model to make any predictions.

Fortunately, as we will see, for most of the initial value problems that are important in applications, and almost all that we will study in this class, the solution always exists and is unique. This is known as the existence and uniqueness theorem. Before we state the general theorem, let us consider a simple example.

Example 3 (Linear homogenous equations)**.** Let us show that an initial value problem for any linear homogenous equation

$$\begin{cases} y' + a(t)y = 0 \\ y(t_0) = y_0 \end{cases}$$

has a unique solution, as long a the function $a(t)$ is continuous. (See the next section for the discussion of continuous functions.) Suppose first that $y_0 \neq 0$ and $y \neq 0$. The constant function $y(t) = 0$ is always a solution, but it's not particularly interesting. The method of solution discussed in the last lecture showed that $y(t)$ must be of the form

$$y(t) = \pm e^{\int -a(t)\mathrm{d}t}$$

for some indefinite integral, or antiderivative, of the function $-a(t)$. If $a(t)$ is a continuous function, there always exists an antiderivative. Moreover, given any specific antiderivative, call it $A(t)$, any other is of the form $A(t) + c$ for a constant c. Therefore, if we choose any such antiderivative $A(t)$, y is given by the formula

$$y(t) = \pm e^c e^{A(t)}$$

for some constant c and some choice of the sign $\pm$. Denote $e^c = C$. This is any positive number. Since we are also allowed to choose sign $\pm$, we get that $y(t)$ must be of the form

$$y(t) = Ce^{A(t)} \tag{2.1}$$

for some number C, now of any sign, and moreover any function of this form satisfies the solution. To conclude: any solution of the equation $y' + a(t)y$ must be of the form (2.1). Moreover, any function of this form is a solution.

Now we ask whether we can find a solution satisfying the initial condition $y(t_0) = y_0$ and whether such a solution is unique. For that, assume we have a solution $y(t)$ of the form (2.1). By plugging $t = t_0$ we see that the initial condition determines the constant C:

$$y_0 = Ce^{A(t_0)};$$

therefore, $C = y_0 e^{A(t_0)}$. Therefore, the initial condition determines the constant C in (2.1). Conversely, we easily check that for this choice of C, the function $y(t)$ indeed is a solution satisfying the given initial condition.

We conclude that for a homogenous linear equation with continuous coefficient $a(t)$

1. there always exists a solution satisfying any initial value condition,
2. such a solution is unique.

It turns out that this is a general principle. Let me stress again that in our example we had to know that the coefficient $a(t)$ is not some crazy function, because we needed the integral $\int -a(t)\mathrm{d}t$ to exist. It is enough, for example, to assume that $a(t)$ is continuous. We will see that for a general first order differential equation the solution to the initial value problem always exists and is unique, as long as the coefficients of the equations are sufficiently good functions.

3 REVIEW: CONTINUOUS AND DIFFERENTIABLE FUNCTIONS

Recall the following notions from calculus. We say that a function f defined on an interval (a, b) is *continuous* if for every x_0 in this interval it satisfies

$$\lim_{x \to x_0} f(x) = f(x_0).$$

Geometrically, it means that the graph of f has no holes or jumps. For example, $f(x) = \sin x$ is continuous on the real line, whereas the function

$$f(x) = \begin{cases} -1 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0. \end{cases}$$

is continuous on the intervals $(-\infty, 0)$ and $(0, \infty)$ but not on any open interval that contains $x = 0$ because it has a jump at that point. (Draw the graph of f to see this.) We say that, moreover, that f defined on an interval (a, b) is differentiable if its derivative

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

exists for every x from the interval (a, b). Geometrically, it means that the graph of f has no cusps. For example, the function $f(x) = |x|$ is differentiable everywhere but at $x = 0$, where the derivative does not exist.

We will also use functions of many variables, $f = f(x, y, z, \dots)$. For example,

$$f(x, y) = x^2 y + y^3, \quad \text{or} \quad f(x, y, z) = x \sin y + z^3 y^2 + e^{x-y}, \quad \dots$$

For concreteness, let us focus on functions of two variables $f = f(x, y)$. As before, we say that such a function is continuous if for every (x_0, y_0) for which the function is defined, we have

$$\lim_{x \to x_0, y \to y_0} f(x, y) = f(x_0, y_0).$$

We will also consider *partial derivatives* of such functions. A partial derivative $\partial f / \partial x$ is simply obtained by looking at the expression $f(x, y)$ as a function of x only, with y fixed, and differentiating with respect to x. Similarly, $\partial f / \partial y$ is obtained by considering x fixed and differentiating with respect to y. For example, for

$$f(x, y) = x^2 y + y^3$$

we have

$$\frac{\partial f}{\partial x} = 2xy,$$
$$\frac{\partial f}{\partial y} = x^2 + 3y^2.$$

Sometimes we will denote partial derivatives by $\partial_x f$ and $\partial_y f$.

4 Existence and uniqueness theorem

We can now state the main theorem, which states that the initial value problem

$$\begin{cases} y' = f(t, y), \\ y(t_0) = y_0. \end{cases} \tag{4.1}$$

always has a unique solution provided that the function f is well-behaved around the point (t_0, y_0).

Theorem 4. *Consider the initial value problem* (4.1). *If f and $\partial_y f$ are both continuous in a neighborhood of (t_0, y_0), then there exists a unique solution $y = y(t)$ to* (4.1) *defined for t from the interval $(t_0 - \epsilon, t_0 + \epsilon)$ for some $\epsilon > 0$.*

Remark 5. An important point to notice is that the solution $y = y(t)$ is not necessarily defined for all t, even if $f(t, y)$ is defined for all t. We will see some examples in the next lecture.

Remark 6. The proof of the theorem goes roughly as follows. Define the *Picard iterations* $y_1, y_2, y_3, \dots$ by the inductive process

$$y_1(t) = y_0 + \int_{t_0}^{t} f(s, y_0) \mathrm{d}s,$$
$$y_{n+1}(t) = y_0 + \int_{t_0}^{t} f(s, y_n(s)) \mathrm{d}s.$$

The main part of the proof is to show that there exists a limit function

$$y(t) = \lim_{n\to\infty} y_n(t)$$

defined for $t \in (t_0 - \epsilon, t_0 + \epsilon)$ for some $\epsilon > 0$. The limit then satisfies the equation

$$y(t) = y_0 + \int_{t_0}^{t} f(s, y(s))\mathrm{d}s.$$

Differentiating both sides with respect to t and using the Fundamental Theorem of Calculus, we get

$$y'(t) = f(t, y(t)).$$

Therefore, the limiting function is a solution. Moreover, $y(t_0) = y_0$. This procedure can be used to solve differential equations numerically. Namely, instead of taking the limit $\lim_{n\to\infty} y_n$, simply compute numerically the integrals defining $y_1, \ldots, y_N$. For N sufficiently large, the function y_N will be a reasonable approximation to the actual solution. This algorithm can be implemented on a computer.

We won't have time to discuss the proof of the theorem or Picard iterations in detail. However, if you are interested, we can talk more about this during office hours!

In the next lecture we will discuss some examples in which the theorem applies and some for which the solution is not unique because the hypothesis of the theorem is not satisfied.

ORDINARY DIFFERENTIAL EQUATIONS (MATH 2030)

Lecture 5: Direction fields, autonomous equations

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REFERENCES

Existence and uniqueness: Boyce–DiPrima section 2.8; Braun section 1.10. Direction fields: Boyce–DiPrima section.1.1. Autonomous equations: Boyce–DiPrima section 2.5. Population models: Braun section 1.5.

1 MORE ON EXISTENCE AND UNIQUENESS

Last time we discussed the existence and uniqueness theorem, which asserts that the initial value problem

$$\begin{cases} y' = f(t,y), \\ y(t_0) = y_0, \end{cases}$$

has a unique solution $y = y(t)$ defined for t close to t_0, provided that the function f and its partial derivative $\partial f/\partial y$ are both continuous in a neighborhood of the point (t_0, y_0).

Example 1. Consider the initial value problem

$$\begin{cases} y' = t + e^y, \\ y(t_0) = y_0. \end{cases}$$

The function $f(t,y) = t + e^y$ is continuous on the entire ty–plane because it's a sum of two continuous functions $(t,y) \mapsto t$ and $(t,y) \mapsto e^y$. Similarly, its partial derivative $\partial f/\partial y = e^y$ is continuous on the entire ty–plane. We conclude from the existence and uniqueness theorem that for every choice of the initial conditions (t_0, y_0) there is a unique solution $y = y(t)$ to the above initial value problem, defined for t close to t_0.

Example 2. Consider the initial value problem

$$\begin{cases} y' = \sqrt{|y|}, \\ y(t_0) = y_0. \end{cases}$$

The function $f(t,y) = \sqrt{|y|}$ is continuous on the entire ty–plane (it does not depend on t and the function $y \mapsto \sqrt{|y|}$ is the composition of two continuous functions). Moreover, for every (t,y) such that $y > 0$ the partial derivative

$$\frac{\partial f}{\partial y} = \frac{1}{2} y^{-1/2}.$$

exists and this function is continuous on the half plane $\{y > 0\}$. Similarly, the partial derivative exists and is continuous on the half-plane $\{y < 0\}$. Therefore, by the existence and uniqueness theorem, if (t_0, y_0) are so that $y_0 \neq 0$, the initial value problem has a unique solution $y = y(t)$ defined for t close to t_0.

However, at the line $\{y = 0\}$ this partial derivative does not exist, so the hypothesis of the theorem are not satisfied for initial conditions $(t_0, y_0 = 0)$. In fact, we can see directly that in this case there are two solutions to the initial value problem

$$\begin{cases} y' = \sqrt{|y|}, \\ y(t_0) = 0. \end{cases}$$

The first solution is the constant function $y(t) = 0$. The second solution can be found by separation of variables. Suppose we look for a positive solution. Separation of variables gives us

$$y^{-1/2}y' = 1,$$

and integrating with respect to t:

$$2y^{1/2} = \int y^{-1/2}\mathrm{d}y = \int y^{-1/2}y'\mathrm{d}t = \int 1\mathrm{d}t = t + C.$$

So any function of the form

$$y(t) = \frac{1}{4}(t + C)^2$$

is a solution. A specific solution satisfying $y(t_0) = 0$ is obtained by setting $C = -t_0$. We see that there are two solutions satisfying the equation with the same initial condition. This does not contradict the existence and uniqueness theorem: we simply cannot apply the theorem in this case because the function $f(t, y) = \sqrt{|y|}$ does not have a partial derivative $\partial f/\partial y$ at points (t, y) with $y = 0$.

2 Maximal interval of existence

Earlier, I emphasized that the solution $y = y(t)$ to the initial value problem

$$\begin{cases} y' = f(t, y), \\ y(t_0) = y_0, \end{cases}$$

whose existence is guaranteed by the existence and uniqueness theorem, is defined for t sufficiently close to the initial time t_0. It it possible that the solution can be continued for t far away from t_0.

Example 3. For all t_0, y_0, the initial value problem

$$\begin{cases} y' = y, \\ y(t_0) = y_0, \end{cases} \tag{2.1}$$

has a unique solution $y(t) = y_0 e^{t-t_0}$ which is defined for all times t.

Remark 4. Given an interval (a, b) we will use the notation $t \in (a, b)$ to say that that t is in the interval (a, b). Here we allow $a = -\infty$ or $b = \infty$. We will also use the notation $\mathbb{R}$ for the real line $(-\infty, \infty)$.

However, it is not always the case that the solution exists for all $t \in \mathbb{R}$. The *maximal interval of existence* for the solution $y = y(t)$ of the initial value problem (2.1) is the largest open interval (a, b) such that $t_0 \in (a, b)$ that the solution to the initial value problem $y = y(t)$ exists for all $t \in (a, b)$. If a and b are finite, that means that the solution cannot be continued either in the past beyond a or in the future beyond b. We allow here the situation $a = -\infty$, when the solution can be continued indefinitely in the past, or $b = +\infty$, when the solution can be continued indefinitely in the future.

Example 5. Consider the separable equation

$$y' = y^2.$$

The constant function $y(t) = 0$ is a solution. All other solutions can be found by separating the variables

$$-y^{-1} = \int y^{-2} \mathrm{d}y = \int y^{-2} y' \mathrm{d}t = \int 1 \mathrm{d}t = t + C,$$

which gives us that the general solution other than $y(t) = 0$ is of the form

$$y(t) = \frac{1}{C - t}$$

for some constant C. We can now see that the maximal interval of existence of a solution $y(t)$ depends on the initial value $y_0 = y(0)$. Expressing C in terms of y_0, we find that

$$y(t) = \frac{1}{1/y_0 - t}.$$

There are two cases, $y_0 > 0$ and $y_0 > 0$. In the former, the largest open interval containing $t_0 = 0$ for which the solution is defined is $(-\infty, 1/y_0)$. We see, moreover, that the solution is asymptotic to the constant solution $y(t) = 0$ at $-\infty$:

$$\lim_{t \to -\infty} y(t) = 0$$

and blows up as we approach the other endpoint of the maximal interval of existence:

$$\lim_{t \to 1/y_0} y(t) = \infty.$$

Similarly, for $y_0 < 0$, the maximal interval of existence is $(1/y_0, \infty)$ and in that case

$$\lim_{t \to 1/y_0} y(t) = -\infty, \quad \lim_{t \to \infty} y(t) = 0.$$

3 DIRECTION FIELD, INTEGRAL CURVES

A differential equation $y' = f(t, y)$ can be interpreted graphically using the *direction field*, or a *slope field*. This field is obtained by associating with each point on the plane with coordinates (t, y) a line whose slope is $f(t, y)$.

The graph of any solution $y = y(t)$ to the differential equation is a curve in the ty–plane, whose tangent line at every point $(t, y(t))$ agrees with the line of the direction field at the same point. These curves are called the *integral curves* of the differential equation.

Example 6. (Diagram with direction field and integral curves for the previous example $y' = y^2$.)

The equation $y' = y^2$ is an example of an *autonomous differential equation*, that is one of the form

$$y' = f(y).$$

For many functions f we can simply solve this equation because it is separable. However, often we can say something about the behavior of solutions without solving the equation.

For example, if y_* is such that $f(y_*)$ then the constant function $y(t) = y_0$ is a solution. Moreover, by the existence and uniqueness theorem, it is the only solution satisfying $y(t_0) = y_*$ for any t_0. Such a point y_* is called an *equilibrium* of the autonomous equation because it corresponds to a state which does not change in time.

We have seen in the previous example that some of the non-constant solutions can converge to one of these equilibrium points as t approaches one of the endpoint of the maximal interval of existence. This, in fact, is a general property of autonomous equations.

Theorem 7. *Let $y = y(t)$ be a solution of the autonomous equation $y' = f(y)$, where f is a differentiable function on the real line. If (a, b) is the maximal integral of existence of y, then either*

$$\lim_{t \to a} y(t) = \pm\infty$$

or $a = -\infty$ and

$$\lim_{t \to a} y(t) = y_*,$$

where y_* is an equilibrium, i.e. $f(y_*) = 0$.

Similarly, either

$$\lim_{t \to b} y(t) = \pm\infty$$

or $b = \infty$ and

$$\lim_{t \to b} y(t) = y_*,$$

where y_* is an equilibrium.

We will omit the proof of the theorem, but some of the ideas involved in it will be presented as a bonus problem in Homework 4. We can also discuss it during office hours.

An equilibrium y_* is called *asymptotically stable* if every solution $y(t)$ that starts near to y_* converges to y_* as $t \to \infty$. Similarly, an equilibrium y_* is *asymptotically unstable* if every solution that starts near to y_* converges to y_* as $t \to -\infty$. We can often tell whether an equilibrium is asymptotically stable, asymptotically unstable, or neither, by drawing the direction field and integral curves of the equation.

Example 8 (Logistic model of population growth)**.** The logistic equation

$$P' = \alpha P \left(1 - \frac{P}{K}\right)$$

has two equilibria $P = 0$ and $P = K$. The former is asymptotically unstable and the latter asymptotically stable.

(Diagram with direction field and integral curves)

Example 9 (Logistic model with critical threshold)**.** The equation is

$$P' = -\alpha P \left(1 - \frac{P}{T}\right)\left(1 - \frac{P}{K}\right)$$

for a constant $\alpha > 0$. We can analyze the behavior of solutions without solving the equation, by looking at the direction field.

(Diagram with direction field and integral curves.)

The equilibria are $P = 0$, $P = T$ and $P = K$. From the diagram we see that $P = 0$ is asymptotically stable, $P = T$ is asymptotically unstable, and $P = K$ is asymptotically stable. The interpretation is the following: if the initial population P_0 is below the critical treshold T, then the population becomes extinct. However, if we start with the initial population P_0 above the critical threshold, it grows and reaches the carrying capacity K as $t \to \infty$.

There is a simple criterion for determining asymptotically stable and unstable equilibria.

Theorem 10. *Let y_* be an equilibrium of the autonomous equation $y' = f(y)$.*

1. *If $f'(y_*) < 0$, then y_* is an asymptotically stable equilibrium.*
2. *If $f'(y_*) > 0$, then y_* is an asymptotically unstable equilibrium.*

Example 11.

1. For the autonomous equation $y' = y$, the equilibrium $y_* = 0$ satisfies $f'(y_*) > 0$ and is asymptotically unstable. Every other solution $y(t) = y_0 e^t$ converges to $y_* = 0$ as $t \to -\infty$.

2. For the autonomous equation $y' = -y$, the equilibrium $y_* = 0$ satisfies $f'(y_*) < 0$ and is asymptotically stable. Every other solution $y(t) = y_0 e^{-t}$ converges to $y_* = 0$ as $t \to \infty$.

Keep in mind that the theorem does not tell us anything in the case $f'(y_*) = 0$. The equilibrium could be asymptotically stable, asymptotically unstable, or neither.

Example 12. For the autonomous equation $y' = y^2$, the equilibrium $y_* = 0$ is neither asymptotically stable or unstable. For the equation $y' = y^3$, the equilibrium $y_* = 0$ is asymptotically unstable. In both cases, $f'(y_*) = 0$.

Lecture 6: Autonomous systems

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REFERENCES

Autonomous systems: Boyce–DiPrima section 9.2, Braun sections 4.3, 4.4 (you don't need to read this part now, as we will return to autonomous systems after discussing systems of linear equations).

1 AUTONOMOUS SYSTEMS OF EQUATIONS

Last time, we considered autonomous equations $y' = f(y)$. Today we will talk briefly about autonomous systems of equations. We have not talked about systems of equations yet, and we will return to this topic later in the course, but it is convenient to introduce it now to talk about exact equations.

As the name suggests, a system of differential equations consists of multiple differential equations for multiple unknown functions. We typically want to have the same number of equations and unknown functions. This is similar to the case of algebraic equations. If you want to find two unknown numbers x and y, you'd better have two equations for them, for example

$$\begin{cases} 5x + 3y = 1, \\ 3x - 7y = 0. \end{cases}$$

Only one equation $5x + 3y = 1$ would not determine x and y. Analogously, we can consider a system of n differential equations for n unknown functions $y_1 = y_1(t), \dots, y_n = y_n(t)$. We say that such a system is *autonomous* if it has the form

$$\begin{cases} y_1' = f_1(y_1, \dots, y_n), \\ y_2' = f_2(y_1, \dots, y_n), \\ \dots \\ y_n' = f_n(y_1, \dots, y_n), \end{cases}$$

for some functions $f_1, \dots, f_n$ of n variables. In other words: all of the equations are first order and the right-hand side does not involve functions depending only on t.

We will focus on autonomous systems for two unknown functions $x = x(t)$ and $y = y(t)$, that is systems of differential equations of the form:

$$\begin{cases} x' = f(x, y), \\ y' = g(x, y). \end{cases} \tag{1.1}$$

We can interpret such equations and their solutions geometrically using the direction field and integral curves as we did for first-order equations. This time the direction field associates with every point (x, y) on the plane a line whose slope is $f(x, y)/g(x, y)$. If $g(x, y) = 0$, the line is vertical. Now a solution $(x(t), y(t))$ to the autonomous system determines for every t a point on the plane. As t varies, these points form a curve. As before, we call it an integral curve. The fact that $x(t), y(t)$ satisfy the differential equation means that the integral curve is tangent to the direction field at every point.

Example 1. Consider the autonomous system

$$\begin{cases} x' = x, \\ y' = y. \end{cases}$$

The direction field at the point (x, y) is the straight line from that point to $(0, 0)$. Integral curves are straight lines passing through $(0, 0)$. In fact, we can solve each equation separately to find that solutions are

$$\begin{cases} x(t) = x_0 e^t, \\ y(t) = y_0 e^t. \end{cases}$$

Example 2. Consider the autonomous system

$$\begin{cases} x' = -y, \\ y' = x. \end{cases}$$

We see that every point (x, y) the direction field is orthogonal to the line from $(0, 0)$ to (x, y). Without solving the equation we can see that the integral curves are circles centered at $(0, 0)$.

Since the direction field of the autonomous system (1.1) depends only on the quotient $f(x, y)/g(x, y)$, it does not change when we multiply both f and g by the same, non-zero function $q = q(x, y)$. Therefore, for any such q, the autonomous system

$$\begin{cases} x' = q(x, y) f(x, y), \\ y' = q(x, y) g(x, y) \end{cases} \tag{1.2}$$

has the same direction field and integral curves as the original system (1.1).

Example 3. The integral curves of the autonomous system

$$\begin{cases} x' &= -(1 + x^2 + y^2) y, \\ y' &= (1 + x^2 + y^2) x \end{cases}$$

are the same as the integral curves of

$$\begin{cases} x' &= -y, \\ y' &= x. \end{cases}$$

Therefore, they are circles centered at $(0, 0)$, as discussed earlier.

2 CHANGE OF VARIABLES

It is important to stress that while the integral curves of the autonomous systems (1.1) and (1.2) are the same, the solutions $(x(t), y(t))$ are not the same. They differ by a *change of variable*, or *reparametrization*, as explained by the next result. Geometrically, the reparametrization means that we travel along the same curve but with different speed.

Theorem 4. *If $(x(t), y(t))$ is a solution to the autonomous system*

$$\begin{cases} x' = q(x,y)f(x,y), \\ y' = q(x,y)g(x,y), \end{cases}$$

such that $q(x(t), y(t)) \neq 0$ for all t, then there is a function $s = s(t)$, called reparametrization, *such that the functions $\tilde{x}(t) = x(s(t))$ and $\tilde{y}(t) = y(s(t))$ are a solution to the autonomous system*

$$\begin{cases} \tilde{x}' = f(\tilde{x}, \tilde{y}), \\ \tilde{y}' = g(\tilde{x}, \tilde{y}). \end{cases}$$

Proof. We need to find the reparametrization s so that $\tilde{x}$ and $\tilde{y}$ solve the second set of equations. Using the chain rule, we compute

$$\tilde{x}'(t) = x'(s(t))s'(t),$$
$$\tilde{y}'(t) = y'(s(t))s'(t).$$

Using that $(x(t), y(t))$ solve the first set of equations, we get

$$\begin{cases} \tilde{x}' = s'(t)q(\tilde{x}, \tilde{y})f(\tilde{x}, \tilde{y}), \\ \tilde{y}' = s'(t)q(\tilde{x}, \tilde{y})g(\tilde{x}, \tilde{y}). \end{cases}$$

Therefore, we want to find $s = s(t)$ such that $s'(t)q(\tilde{x}(t), \tilde{y}(t)) = 1$ or, equivalently,

$$s' = \frac{1}{q(x(s), y(s))}$$

This equation has a solution $s = s(t)$ by the uniqueness and existence theorem, provided that q and its partial derivatives are continuous. □

The proof gives us a way of finding solutions to the modified equation (1.2) by reparametrizing solutions to the original equation (1.1).

Example 5. Let us apply the above theorem to solve the autonomous system

$$\begin{cases} x' = \frac{x}{x^2+y^2}, \\ y' = \frac{y}{x^2+y^2}. \end{cases} \tag{2.1}$$

We will consider only solutions with $x^2 + y^2 \neq 0$, so that the right-hand side is well-defined. The integral curves of this system as the same as the integral curves of

$$\begin{cases} x' = x, \\ y' = y, \end{cases}$$

which are straight lines $x(t) = x_0 e^t$, $y(t) = y_0 e^t$, with $(x_0, y_0) \neq (0, 0)$. Let us find a reparametrization $s = s(t)$ such that the new functions

$$\tilde{x}(t) = x(s(t)) \quad \text{and} \quad \tilde{y}(t) = y(s(t))$$

satisfy equation (2.1). The chain rule gives us

$$\tilde{x}'(t) = x'(s(t))s'(t) = x(s(t))s'(t) = s'(t)\tilde{x}(t),$$
$$\tilde{y}'(t) = y'(s(t))s'(t) = y(s(t))s'(t) = s'(t)\tilde{y}(t),$$

so we want s to satisfy

$$s' = \frac{1}{x(s)^2 + y(s)^2} = \frac{1}{x_0^2 + y_0^2} e^{-2s}.$$

This is a separable equation. We find a solution by separating variables and integrating:

$$\frac{1}{2} e^{2s} = (x_0^2 + y_0^2)t.$$

(We don't need a constant of integration because we only need to find some solution $s = s(t)$, not all of them.) Therefore,

$$s(t) = \frac{1}{2} \ln(2(x_0^2 + y_0^2)t),$$

and the general solution to (2.1) is

$$\tilde{x}(t) = x(s(t)) = x_0 \sqrt{2(x_0^2 + y_0^2)t},$$
$$\tilde{y}(t) = y(s(t)) = y_0 \sqrt{2(x_0^2 + y_0^2)t},$$

for any $(x_0, y_0) \neq (0, 0)$.

3 EQUILIBRIA

A point (x_*, y_*) is called an equilibrium of the autonomous system (1.1) is

$$f(x_*, y_*) = 0 \quad \text{and} \quad g(x_*, y_*) = 0.$$

For such a point, the constant functions

$$x(t) = x_*, \quad y(t) = y_*$$

solve (1.1). As we did for autonomous equations for one function, we will call an equilibrium of an autonomous system

- *asymptotically stable* if any other solution $(x(t), y(t))$ which starts close to (x_*, y_*) converges to (x_*, y_*) as $t \to \infty$,
- *asymptotically unstable* if any other solution $(x(t), y(t))$ which starts close to (x_*, y_*) converges to (x_*, y_*) as $t \to -\infty$.

Example 6. We see that $(0, 0)$ is an equilibrium of the autonomous systems considered in Example 1 and Example 2. In the first example, the equilibrium is asymptotically stable. In the second example, the equilibrium is neither asymptotically stable nor asymptotically unstable.

Equilibria of two-dimensional autonomous systems are much more interesting than those of one-dimensional systems, discussed in the last lecture.

Example 7. Draw the integral curves of the autonomous systems

$$\begin{cases} x' = x, \\ y' = -y \end{cases}$$

and

$$\begin{cases} x' = \epsilon x - y, \\ y' = x + \epsilon y \end{cases}$$

to see more interesting examples of equilibria. In the first example, we have two directions: one along which solutions converge to the equilibrium, and one along which they escape from it. In the second example, the circular integral lines from Example 2 are now perturbed to trajectories which either escape from or converge to the equilibrium, depending on the sign of ϵ.

Observe that every first order equation for one function

$$y' = g(t, y)$$

can be written as an autonomous system for two functions $x = x(t)$ and $y(t)$

$$\begin{cases} x' = 1, \\ y = g(x, y). \end{cases}$$

The first equation gives us $x(t) = t$, up to adding a constant. Plugging this to the second equation gives us $y' = g(t, y)$. Therefore, every first order equation for one function naturally leads to an autonomous system for two functions. Solving one is equivalent to solving the other.

Similarly, often we can reduce an autonomous system to a first order equation for one function. Suppose we have an autonomous system

$$\begin{cases} x' = f(x, y), \\ y' = g(x, y), \end{cases}$$

and suppose that we can invert the function $y = y(t)$ to express t as a function of y, that is: $t = t(y)$. We can then consider x as a function of y as well, by $x(y) = x(t(y))$, and look for a differential equation satisfied by that function. According to the chain rule, $x = x(y)$ will satisfy the equation

$$\frac{\mathrm{d}x}{\mathrm{d}y} = \frac{\mathrm{d}x}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}y} = \frac{x'}{y'} = \frac{f(x, y)}{g(x, y)}. \tag{4.1}$$

If we can find a solution to this equation, we can solve the original equation by solving the equation for $y = y(t)$:

$$\frac{\mathrm{d}y}{\mathrm{d}t} = g(x(y), y).$$

Observe that what we are discussing here is the special case of the change of variables, or reparametrization, we already discussed earlier. Solutions of equation (4.1) are equivalent to solutions to the autonomous system

$$\begin{cases} x' = \frac{f(x,y)}{g(x,y)} \\ y' = 1, \end{cases}$$

and we already know that the integral curves of this system are the same as the integral curves of the original system. Once we solve (4.1), solving the equation for $y = y(t)$ is the same as finding a reparametrization, as we did earlier.

Example 8. Here is an alternative, but really equivalent, solution to (2.1). Let us look for solutions (x, y) such that $x = x(y)$ is a function of y. This function satisfies then the first order equation

$$\frac{\mathrm{d}x}{\mathrm{d}y} = \frac{x}{y}.$$

This is a separable equation. Separating the variables and integrating, we get

$$\ln|x| = \ln|y| + C$$

and, therefore $x(y) = \pm e^C y = Ay$ for some constant $A \neq 0$. This is the equation of a straight line passing through $(0,0)$. To find x and y as functions of t, we need to solve the equation for $y = y(t)$:

$$y' = \frac{y}{y^2(1+A^2)} = \frac{1}{(1+A^2)y}.$$

Again, this is a separable equation, with general solution (after renaming constants) of the form

$$y(t) = B\sqrt{t}, \quad B = \text{const},$$

which is the same as the solution we found earlier. Now $x(t)$ can be computed using the relation to $y(t)$, $x(t) = AB\sqrt{t}$. We see that the solution $(x(t), y(t))$ is in the end determined by two constants A and B, which can be computed from the constants x_0, y_0 we used earlier.

Lectures 7 and 8: Exact and closed autonomous systems

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REFERENCES

Boyce–DiPrima section 2.6, Braun section 1.9.

1 EXACT AND CLOSED AUTONOMOUS SYSTEMS

An autonomous system

$$\begin{cases} x' = f(x,y), \\ y' = g(x,y) \end{cases} \tag{1.1}$$

is called *exact* if there is a function $H = H(x,y)$ such that

$$f = -\frac{\partial H}{\partial y} \quad \text{and} \quad g = \frac{\partial H}{\partial x}.$$

Remark 1. Observe that a function H such that $f = -\partial H/\partial y$ and $g = \partial H/\partial x$ is not unique. It can always be modified by adding a constant.

While these conditions seem arbitrary at first, the point of this definition is that we can solve such differential equations by reducing them to algebraic equations.

Theorem 2. *If $(x(t), y(t))$ is a solution to the exact autonomous system*

$$\begin{cases} x' = -\frac{\partial H}{\partial y}(x,y), \\ y' = \frac{\partial H}{\partial x}(x,y), \end{cases} \tag{1.2}$$

then

$$H(x(t), y(t)) = \text{constant}.$$

Proof. This is a consequence of the chain rule for functions of two variables:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(H(x,y)\right) = \frac{\partial H}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial H}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t}.$$

Using the differential equation for (x,y), we write the right-hand side as

$$-\frac{\partial H}{\partial x}\frac{\partial H}{\partial y} + \frac{\partial H}{\partial y}\frac{\partial H}{\partial x} = 0.$$

Therefore,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(H(x,y)\right) = 0,$$

and the function $H(x(t), y(t))$ is constant. □

We conclude that the integral curves of the exact autonomous systems can be simply found by solving the algebraic equation

$$H(x, y) = \text{constant}.$$

Remark 3. If you learned classical mechanics you will immediately recognize that the exact autonomous system (1.2) is a Hamiltonian system with a Hamiltonian function H independent of time. In classical mechanics, it is well-known that the Hamiltonian function is constant along trajectories.

Example 4. The autonomous system

$$\begin{cases} x' &= -y, \\ y' &= x \end{cases}$$

is exact because for $H(x, y) = \frac{1}{2}(x^2 + y^2)$ we have

$$-\frac{\partial H}{\partial y} = -y \quad \text{and} \quad \frac{\partial H}{\partial x} = x.$$

We conclude that the integral curves are circles described by the equation

$$x^2 + y^2 = \text{constant}.$$

In this example, it was easy to see that the system is exact: we simply guessed the function H. However, in general, it is not obvious from looking at (1.1) whether it is exact and what H is. Fortunately, there is an easy criterion to determine whether an autonomous system is exact. First, recall from multivariable calculus that for any function H we have

$$\frac{\partial^2 H}{\partial x \partial y} = \frac{\partial^2 H}{\partial y \partial x}.$$

Example 5. For $H(x, y) = x^2 y^3$ we have

$$\frac{\partial^2 H}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial H}{\partial y}\right) = \frac{\partial}{\partial x}\left(3x^2 y^2\right) = 6xy^2,$$

and

$$\frac{\partial^2 H}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial H}{\partial x}\right) = \frac{\partial}{\partial y}\left(2xy^3\right) = 6xy^2.$$

In particular, if the system (1.1) is exact, that is

$$f = -\frac{\partial H}{\partial y} \quad \text{and} \quad g = \frac{\partial H}{\partial x},$$

then f an g must satisfy

$$\frac{\partial f}{\partial x} = -\frac{\partial^2 H}{\partial x \partial y} = -\frac{\partial^2 H}{\partial y \partial x} = -\frac{\partial g}{\partial y},$$

or equivalently

$$\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} = 0. \tag{1.3}$$

We call an autonomous system (1.1) for which the function f and g satisfy the above equation *closed*. We have just shown that every exact autonomous system is closed. Observe that while it might be hard to check that an autonomous system is exact, it is easy to check that it is closed: it suffices to compute the partial derivatives of f and g. Fortunately, it turns out that closed systems are exact and, moreover, we can find a formula for the function H.

Theorem 6. *Let*

$$\begin{cases} x' = f(x,y), \\ y' = g(x,y) \end{cases}$$

be an autonomous system defined on a rectangle $[x_0, y_0] \times [x_1, y_1]$ in the xy–plane. If the system is closed, i.e. (1.3) *holds, then it is exact, that is*

$$f = -\frac{\partial H}{\partial y} \quad \text{and} \quad g = \frac{\partial H}{\partial x}, \tag{1.4}$$

where H is given by the formula (1.5) *below.*

Proof. Our goal is to find H so that (1.4) holds. In particular, integrating the first equation with respect to y we see that there must be a function of one variable $\varphi = \varphi(x)$ such that

$$H(x,y) = -\int_{y_0}^{y} f(x,s)\mathrm{d}s + \varphi(x).$$

(When we integrate functions of one variable y, we get an integration constant. However, in this case, we get a different constant for every x, so the result is a function of x rather than a constant. Note that the expression $\varphi(x)$ vanishes when we take the partial derivative with respect to y.) Any such function H will satisfy $\partial_y H = -f$. We still have the freedom of choosing any function of one variable φ. We look for φ so that the second equality

$$g(x,y) = \frac{\partial H}{\partial x}(x,y)$$

holds. Expanding the right-hand side using our formula for H and differentiating with repsect to x under the integral sign, we get

$$g(x,y) = -\int_{y_0}^{y} \frac{\partial f}{\partial x}(x,s)\mathrm{d}s + \varphi'(x).$$

Now we use the fact that the equation is closed. Using (1.3) and the Fundamental Theorem of Calculus, we write the right-hand side as

$$\begin{aligned} g(x,y) &= \int_{y_0}^{y} \frac{\partial g}{\partial y}(x,s)\mathrm{d}s + \varphi'(x) \\ &= g(x,y) - g(x,y_0) + \varphi'(x). \end{aligned}$$

We see that this equation is satisfied if φ obeys the differential equation

$$\varphi'(x) = g(x,y_0).$$

We solve this equation by integrating with respect to x:

$$\varphi(x) = \int_{x_0}^{x} g(s,y_0)\mathrm{d}s.$$

In the end we get the following formula for H:

$$H(x,y) = \int_{x_0}^{x} g(s,y_0)\mathrm{d}s - \int_{y_0}^{y} f(x,s)\mathrm{d}s. \tag{1.5}$$

By construction, H satisfies (1.4). □

You should not memorize the formula for H. Rather, understand the method of finding H and apply it in specific cases.

Remark 7. Of course, we can always add a constant to H and (1.4) will still hold. Observe also that we could have started our construction using g rather than f. That is, integrating $\partial_x H = g$, we would get

$$H(x,y) = \int_{x_0}^{x} g(s,y)\mathrm{d}s + \varphi(y)$$

for some function of one variable $\varphi = \varphi(y)$. The condition $\partial_y H = -f$ then leads, in the same way as before, to the equation for φ:

$$\varphi'(y) = -f(x_0, y),$$

which we solve by integrating:

$$\varphi(y) = -\int_{y_0}^{y} f(x_0, s)\mathrm{d}s.$$

The final formula for H is then

$$H(x,y) = -\int_{y_0}^{y} f(x_0, s)\mathrm{d}s + \int_{x_0}^{x} g(s,y)\mathrm{d}s.$$

One can check directly that this formula gives the same result as (1.5).

Example 8. We will show that the following system is exact and construct H following the method from the proof of Theorem 6:

$$\begin{cases} x' = f(x,y) = 2ye^x, \\ y' = g(x,y) = -e^x y^2 + 2x. \end{cases}$$

First, let us check that this system is closed:

$$\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} = 2ye^x - 2ye^x = 0.$$

Since the system is closed, we can use the method from the proof of Theorem 6 to find a function $H = H(x,y)$ such that

$$\frac{\partial H}{\partial y} = -f(x,y) \quad \text{and} \quad \frac{\partial H}{\partial x} = g(x,y).$$

To find such H, we integrate the first of these equations with respect to y:

$$H(x,y) = -\int f(x,y)\mathrm{d}y = \varphi(y) - y^2 e^x,$$

for some function $\varphi = \varphi(x)$ depending only on x and not on y. To find φ, we use the second equation

$$\frac{\partial H}{\partial x} = g(x,y) = -y^2 e^x + 2x,$$

that is,

$$\varphi'(x) = 2x,$$

which gives us

$$\varphi(x) = x^2.$$

(We need only one solution so we can disregard the constant of integration.) We find that

$$H(x,y) = x^2 - y^2 e^x$$

and integral curves are given by the equation

$$x^2 - y^2 e^x = \text{const}.$$

Remark 9. In the proof of the theorem, it was important that the functions f and g were defined on a rectangle, for example so that we could integrate $f(x,y)$ with respect to y for every value of x. In fact, the theorem is true for autonomous systems defined on any region in the xy–plane that doesn't have any holes (such regions are called *simply-connected*). However, the theorem is false for regions with holes. For such regions, there are closed autonomous systems which are exact. For example, consider the the autonomous system

$$\begin{cases} x' = \frac{x}{x^2+y^2} \\ y' = \frac{y}{x^2+y^2} \end{cases}$$

defined on the entire plane without the point $(0,0)$. By direct calculation we check that it is exact. On the other hand, it is not exact. The integral curves of this system are the same as the curves of the system

$$\begin{cases} x' = x, \\ y' = y, \end{cases}$$

which we discussed earlier. Thus, the integral curves are straight lines starting from $(0,0)$. There is no continuous function H defined on the plane without the origin which is constant on such lines. One way to see this is to consider a point $\gamma(t)$ going around $(0,0)$ along a circle. Choose the speed of this point so that the loop closes after time $t = 1$, i.e. $\gamma(0) = \gamma(1)$. Suppose, by contradiction, that there is such a function H. The path $\gamma(t)$ is orthogonal to the lines $H = \text{const}$, so basic multivariable calculus shows that the function $H(t)$ must be increasing in t. On the other hand, we have $H(\gamma(0)) = H(\gamma(1))$, a contradiction which shows that such H cannot exist.

This is related to the theory of *differential forms* and *de Rham's theorem* in the field of mathematics called topology. If you are interested, we can talk more about this during office hours.

2 Integrating factor

Sometimes an autonomous system (1.1) is not exact but we can find a function $\mu = \mu(x,y)$ such that the modified system

$$\begin{cases} x' = \mu(x,y)f(x,y), \\ y' = \mu(x,y)g(x,y) \end{cases}$$

is exact. Such a function μ is called an *integrating factor*. Recall from the last lecture that the integral curves of the modified system as the same as those of the original system. Therefore, if we can find an integrating factor, we can solve the equation and find the integral curves even though the original autonomous system is not exact.

Example 10 (First order linear equations)**.** We already saw integrating factors in the lecture on first order equations

$$y' + a(t)y + b(t) = 0.$$

We can interpret our algorithm of solving first order equations using autonomous systems. First, as in the previous lecture, we observe that solutions to the above equation for one function can be identified with solutions to the autonomous system

$$\begin{cases} x' = f(x,y) = 1, \\ y' = g(x,y) = -a(x)y - b(x). \end{cases}$$

This system is not closed:

$$\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} = -a(x),$$

which is not zero for a general coefficient $a(x)$. So we look for a function $\mu = \mu(x, y)$ such that the modified system

$$\begin{cases} x' = \mu(x,y) f(x,y), \\ y' = \mu(x,y) g(x,y) \end{cases}$$

is exact. That means we want the following to be true:

$$\frac{\partial(\mu f)}{\partial x} + \frac{\partial(\mu g)}{\partial y} = 0.$$

Using the chain rule, we compute the left-hand side:

$$\frac{\partial(\mu f)}{\partial x} + \frac{\partial(\mu g)}{\partial y} = \frac{\partial \mu}{\partial x} + \frac{\partial \mu}{\partial y} g - \mu(x) a(x).$$

Observe that if $\mu(x, y) = \mu(x)$ depends only on x and not on y, the condition that the above sum is zero, reduces to:

$$\mu'(x) = \mu(x) a(x).$$

This is exactly the same equation for an integrating factor that we considered when discussing first order linear equations. If we find $\mu = \mu(x)$ solving this equation, we can use it as an integrating factor. The modified system is then exact and we can find an equation describing integral curves. Therefore, autonomous systems lead us to the same method of solving first order linear equations we already discussed.

However, for a general non-exact autonomous systems it is usually difficult to find an integrating factor.

ORDINARY DIFFERENTIAL EQUATIONS (MATH 2030)

Lectures 9: Second order equations

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REFERENCES

Boyce–DiPrima sections 3.1–3.2; Braun section 2.1.

1 SECOND ORDER EQUATIONS AS SYSTEMS

We will now consider second order differential equations, that is differential equations of the form

$$y'' = f(t, y, y'). \tag{1.1}$$

Here $y = y(t)$ is an unknown function of variable t and f is a function of three variables. Here are some examples of second order equations:

$$\begin{aligned} y'' &= -y, \\ y'' &= yt + at^5 \\ y'' &= -ay' - bt + ct^3 = 0, \\ y'' &= a(y')^2 + by' + cy^{-1} - t^5. \end{aligned}$$

The right-hand side can be any expression depending on t, y, and y'.

An important observation is that a second order equation can be understood as a system of two first order equations. (In general, a differential equation of order n can be written as a system of n first order equations.) Here is how to do it. Suppose that we have a solution y to (1.1). Introduce a new function $x(t) = y'(t)$. We have, using the fact that y is a solution,

$$x'(t) = y''(t) = f(t, y, y') = f(t, y, x).$$

Therefore, the pair of functions (x, y) satisfies the following system of first order differential equations

$$\begin{cases} x' = f(t, y, x), \\ y' = x. \end{cases} \tag{1.2}$$

We see that any time we have a solution y to the second order equation (1.1) we can produce a pair of functions (x, y) solving the system of first order equations (1.2). Conversely, given a pair of functions (x, y) solving (1.2), we get a solution y to (1.1). Therefore, these two points of view are completely equivalent. Depending on what we want to do, sometimes it may be easier to consider the second order equation for one function or the system of first order equations for two functions.

Example 1. The second order equation $y'' = -y$ is equivalent to the system

$$\begin{cases} x' = -y, \\ y' = x. \end{cases}$$

Recall that we have studied this system in the lecture about autonomous systems! (Important point: in general, the system (1.2) does not have to be autonomous because the right-hand side may depend on t.)

Example 2. The second order equation $y'' = a(y')^2 + by' + cy^{-1} - t^5$ is equivalent to the system

$$\begin{cases} x' = ax^2 + bx + cy^{-1} - t^5, \\ y' = x. \end{cases}$$

2 EXISTENCE AND UNIQUENESS FOR SECOND ORDER EQUATIONS

We can use the observation that second order equations are equivalent to systems of first order equations to deduce a uniqueness and existence theorem for second order equations from the theorem for first order equations.

Theorem 3. *Let t_0, y_0, y'_0 be given numbers. Consider the initial value problem*

$$\begin{cases} y'' = f(t, y, y') \\ y(t_0) = y_0, \\ y'(t_0) = y'_0. \end{cases}$$

If the functions f, $\partial f/\partial y$ and $\partial f/\partial y'$ (by this we mean the partial derivatives of f with respect to the second and third variable) are continuous in a neighborhood of the point (t_0, y_0, y'_0), then there exists a solution $y(t)$ to the initial value problem defined for t from some interval $(t_0 - \epsilon, t_0 + \epsilon)$ containing t_0. Moreover, the solution is unique in that interval.

An important point to notice is that for second order equations we have to prescribe the value of y at t_0 and the value of y' at t_0 in order to get a unique solution. This is in contrast with first order equations, for which it was enough to prescribe $y(t_0)$.

Proof. From what we said in the previous section, finding a solution $y(t)$ to the second order equation is equivalent to finding a solution to the system

$$\begin{cases} x' = f(t, y, x), \\ y' = x. \end{cases} \tag{2.1}$$

We, moreover, prescribe the initial value $y(t_0) = y_0$ and $x(t_0) = y'_0$. In general, we can consider a system of second order equations

$$\begin{cases} x' = f(t, y, x) \\ y' = g(t, y, x) \end{cases}$$

with initial value $y(t_0) = y_0$ and $x(t_0) = y'_0$, for any functions f and g. The existence and uniqueness theorem for first order equations tells us that the solution exist for t close to t_0 and is unique for such t provided that f, g, and their partial derivatives with respect to x and y are continuous around (t_0, y_0, y'_0). (This is not exactly the theorem we stated in the lecture on existence and uniqueness, because we dealt with only a single equation and not a system. But the proof for systems of first order equations is exactly the same as the proof for a single first order equation.) So we see that our case is a special case of this situation, with $g(t, y, x) = x$ (which is continuous and has continuous partial derivatives). We conclude that the system (2.1) has a unique solution defined for t close to t_0. If (x, y) is a solution to the system (2.1), then y is a solution to the initial value problem for the original second order equation $y'' = f(t, y, y')$. □

Example 4. In the first lecture, we considered the harmonic oscillator equation

$$y'' + y = 0.$$

We observed that the functions $y_1(t) = \cos t$ and $y_2(t) = \sin t$ are solutions. Suppose now that we want to find a solution satisfying the initial condition

$$y(0) = y_0 \quad \text{and} y'(0) = y'_0$$

for some prescribed numbers y_0, y'_0. We can look for a solution of the form

$$y(t) = C_1 \cos t + C_2 \sin t$$

for some constants C_1 and C_2. To compute the constant, we use the initial value condition. Plugging $t = 0$, we get

$$y_0 = y(0) = C_1 \cos 0 + C_2 \sin 0 = C_1.$$

On the other hand, taking the derivative of $y(t)$, we get

$$y'(t) = -C_1 \sin t + C_2 \cos t.$$

Plugging $t = 0$ to this equation and using the initial value condition, we get

$$y'_0 = y'(0) = -C_1 \sin 0 + C_2 \cos 0 = C_2.$$

We conclude that $C_1 = y_0$ and $C_2 = y'_0$ so the solution to our initial value problem is

$$y(t) = y_0 \cos t + y'_0 \sin t.$$

Even though we only guessed this solution, the existence and uniqueness theorem tells us that this is the only solution to our initial value problem! Since y_0 and y'_0 are allowed to be anything, we conclude that any solution to the equation $y'' + y = 0$ has the form

$$C_1 \cos t + C_2 \sin t$$

for some constants C_1 and C_2.

3 LINEAR SECOND ORDER EQUATIONS

The harmonic oscillator equation is an example of a linear second order equation, that is a differential equation of the form

$$y'' + p(t)y' + q(t)y + r(t) = 0.$$

Here $p(t), q(t), r(t)$ are given functions of t. The equation is called *homogenous* if $r(t) = 0$ for all t. Otherwise we call it *non-homogenous*.

Linear equations have particularly nice properties and our goal for the next three lectures is to learn how to solve second order linear equations. Second order linear equations appear in all sort of applications, and even if you want to study non-linear equations you have to first understand linear equations as they provide easier models for more complicated equations.

For now, let us focus on homogenous equations, that is assume that $r(t) = 0$ and consider

$$y'' + p(t)y' + q(t)y = 0. \tag{3.1}$$

Recall from Homework 1 that:

1. If y is a solution to (3.1) and C is any constant, then the function Cy is a solution to (3.1).

2. If y_1 and y_2 are two solutions to (3.1) then their sum $y_1 + y_2$ is also a solution to (3.1).

3. If y_1 and y_2 are two solutions to (3.1) and C_1 and C_2 are constants, then the function
$$C_1y_1 + C_2y_2$$
is a solution to (3.1). We call functions of this form *linear combinations* of the functions y_1 and y_2.

   (The third part follows from combining the first and second part.)

Example 4 illustrates an important general principle. In order to find a general solution to a second order linear homogenous equation we need to find two independent solutions y_1 and y_2 and then any other solution will be of the form $C_1y_1 + C_2y_2$ for some constants C_1 and C_2. What do we mean by independent here? Of course, the solutions y_1 and y_2 have to be different, but that's not enough. We want them to be independent in the sense that there is no constant C such that $y_2 = Cy_1$. In the example, we had $y_1(t) = \cos t$ and $y_2(t) = \sin t$ and it's intuitive that these two functions are independent.

4 THE WRONSKIAN

In general, however, if we find to solutions y_1 and y_2 to (3.1), how do we tell that they are independent? And how do we see that any other solution is a linear combination of y_1 and y_2? The next theorem we discuss will answer these questions. However, before stating the theorem, let us introduce the following notion.

Definition 5. For two functions y_1 and y_2, the *Wronskian* $W[y_1, y_2]$ is a function defined by

$$W[y_1, y_2](t) = y_1(t)y_2'(t) - y_1'(t)y_2(t).$$

[The Wronskian is named after the Polish mathematician Jozef Hoene–Wronski (1776–1853).]

An important point: the Wronskian of two functions is itself a function, that is for every t it returns a number.

Example 6. The Wronskian of the functions $y_1(t) = e^t$ and $y_2(t) = te^{-t}$

$$W[y_1, y_2](t) = e^t(e^{-t} - te^{-t}) - e^t(te^{-t}) = -2t.$$

At the moment, the definition of the Wronskian seems unmotivated: it's not clear why we would consider such an expression. However, if you understand the proof of the theorem, you will understand how the Wronskian naturally appears. For now, let us observe the following.

Example 7. If the functions y_1 and y_2 are linearly dependent, that is $y_2 = Cy_1$ for a constant C, then their Wronskian is zero. Indeed,

$$W[y_1, y_2] = W[y_1, Cy_1] = y_1(Cy_1)' - y_1'(Cy_1) = Cy_1y_1' - Cy_1y_1' = 0.$$

We see that the Wronskian being zero or not has something to do with whether the functions are independent or not.

Without further ado, here is the main theorem of today's lecture. We will assume that the coefficients $p(t)$ and $q(t)$ in the equation are defined for all t and continuous.

Theorem 8. *Let y_1 and y_2 be two solutions to the second order homogenous equation*

$$y'' + p(t)y' + q(t)y = 0.$$

If the Wronskian $W[y_1, y_2](t)$ is non-zero for all t, then any other solution to the equation is a linear combination of y_1 and y_2, that is, any other solution has the form

$$y(t) = C_1y_1(t) + C_2y_2(t)$$

for some constants C_1 and C_2.

Proof. Choose any numbers t_0, y_0, y_0'. If we can show that any initial problem

$$\begin{cases} y'' + p(t)y' + q(t)y = 0 \\ y(t_0) = y_0, \\ y'(t_0) = y_0'. \end{cases}$$

has a solution of the form $y(t) = C_1y_1(t) + C_2y_2(t)$ for some constants C_1 and C_2, then we are done. Indeed, any solution to the equations satisfies *some* initial value problem – we can choose any t_0 and declare y_0 and y_0' to be the value of the solution and its derivative at t_0. Now the solution to the initial value problem is unique by the existence and uniqueness theorem. So if we can show that any initial value problem has a solution which is a linear combination of y_1 and y_2, we will conclude that all solution to the equation are such linear combinations.

To solve the initial value problem, given t_0, y_0, y_0', we proceed in the same way as in Example 4, that is we want to use the initial value conditions to determine the constants C_1 and C_2. Our solution should satisfy the system of algebraic equations

$$\begin{cases} y_0 = y(t_0) = C_1y_1(t_0) + C_2y_2(t_0) \\ y_0' = y'(t_0) = C_1y_1'(t_0) + C_2y_2'(t_0). \end{cases} \tag{4.1}$$

(Since we plug $t = t_0$, these are just equations for numbers and not functions!) This is a system of two algebraic equations with two unknowns C_1 and C_2. To solve it, we want to eliminate one of the unknowns, for example C_2. We do it by multiplying the first equation by $y_2'(t_0)$ and the second equation by $y_2(t_0)$ and then subtracting them:

$$y_0y_2'(t_0) - y_0'y_2(t_0) = C_1(y_1(t_0)y_2'(t_0) - y_1'(t_0)y_2(t_0)).$$

This gives us a formula for C_1:

$$C_1 = \frac{y_0 y_2'(t_0) - y_0' y_2(t_0)}{y_1(t_0)y_2'(t_0) - y_1'(t_0)y_2(t_0)},$$

and plugging this to the first equation we can also solve for C_2. The formula itself is not important, but there is one important point. We can solve the equation for C_1 and C_2 only if the formula makes sense, that is if we can divide by

$$W[y_1, y_2](t_0) = y_1(t_0)y_2'(t_0) - y_1'(t_0)y_2(t_0).$$

Observe that this number is the value of the Wronskian at t_0. So in order for our problem to have a solution we need that $W[y_1, y_2](t_0) \neq 0$. But that was our original assumption: that the Wronskian was non-zero for all values of t. We conclude that we can find C_1 and C_2 satisfying equations (4.1). For such constants, the function

$$y(t) = C_1 y_1(t) + C_2 y_2(t)$$

is a solution to the equation $y'' + p(t)y' + q(t)y = 0$ (indeed any linear combination of y_1 and y_2 is) and by construction it satisfies the initial value problem $y(t_0) = y_0$ and $y'(t_0) = y_0'$. We conclude that any initial value problem has a solution which is a linear combination of y_1 and y_2, which is what we wanted to show. □

Remark 9. This is a general fact of algebra that the system of linear algebraic equations for two unknowns x and y

$$\begin{cases} ax + by = e \\ cx + dy = f \end{cases}$$

has a unique solution if $ad - bc \neq 0$. The number $ad - bc$ is called the *determinant* of the equation. We will talk more about this when we discuss linear algebra. For now I want simply to make the point that the Wronskian appears naturally in the proof of the theorem as the determinant of the linear system (4.1).

You might remember that not every quadratic equation has two solutions. Indeed, if $p^2 - 4q = 0$, there is only one root λ of the characteristic polynomial. In this case, if $e^{\lambda t}$ is a solution. But how do we find another solution? The situation seems worse when $p^2 - 4q < 0$; in this case the characteristic polynomial has no roots so our method of finding solutions does not seem to work. We will discuss what to do in these two cases in the next lecture.

Lecture 10: Second order homogenous equations with constant coefficients

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REFERENCES

Boyce–DiPrima sections 3.1, 3.2, 3.4; Braun section 2.2.

1 ABEL'S FORMULA FOR THE WRONSKIAN

Last time we considered second order homogenous linear equations

$$y'' + p(t)y' + q(t)y = 0. \tag{1.1}$$

Given two solutions y_1 and y_2 we wanted to check if every other solution is of the form $C_1y_1 + C_2y_2$ for some constants C_1, C_2. We proved that this is the case provided that the Wronskian

$$W[y_1, y_2](t) = y_1(t)y_2'(t) - y_1'(t)y_2(t)$$

is non-zero for all t. It seems complicated to verify that the Wronskian is non-zero for all values of t. Fortunately, the following result due to the Norwegian mathematician Niels Abel (1802–1892) shows that it is enough to verify that the Wronskian is non-zero for only one value of t.

Theorem 1 (Abel's formula)**.** *If y_1 and y_2 are two solutions to equation (1.1) then their Wronskian $W = W[y_1, y_2]$ satisfies the differential equation*

$$W' + p(t)W = 0. \tag{1.2}$$

Therefore, the Wronskian is given by the formula

$$W(t) = Ce^{-\int p(t)\mathrm{d}t}.$$

In particular, either $W(t) = 0$ for all t or $W(t) \neq 0$ for all t.

Proof. First, we show that the function $W = W[y_1, y_2]$ satisfies differential equation (1.2). Using the product rule, we compute

$$\begin{aligned} W' &= (y_1y_2' - y_2'y_1)' \\ &= y_1'y_2' + y_1y_2'' - y_2''y_1 - y_2'y_1' \\ &= y_1y_2'' - y_2''y_1. \end{aligned}$$

On the other hand, since both y_1 and y_2 solve (1.1), the right-hand side is equal to

$$\begin{aligned} y_1y_2'' - y_2''y_1 &= -y_1(py_2' + qy_2) + y_2(py_1' + qy_1) \\ &= -p(y_1y_2' - y_1'y_2) = -pW. \end{aligned}$$

Therefore, $W' = -pW$ as desired. This is a separable equation. Dividing by W and integrating we get

$$\ln|W(t)| = -\int P(t)\mathrm{d}t$$

so, after exponentiating,

$$W(t) = Ce^{-\int P(t)\mathrm{d}t}$$

for some constant C. We have two possibilities: $C = 0$ or $C \neq 0$. If $C = 0$, then $W(t) = 0$ for all t. On the other hand, since e^x is non-zero for every x, if $C \neq 0$, then $W(t) \neq 0$ for all t. □

Example 2. Consider the equation $y'' + y = 0$. We verify by direct calculation that $y_1(t) = \cos t$ and $y_2(t) = \sin t$ are solutions. We compute the Wronskian

$$W[y_1, y_2](t) = \cos^2 t + \sin^2 t = 1.$$

Let's see that Abel's formula indeed works in this case. We have $p(t) = 0$ in this case so

$$Ce^{-\int p(t)\mathrm{d}t} = C,$$

and indeed the formula holds for $C = 1$. The Wronskian $W(t)$ is non-zero for all t. We conclude that every solution to $y'' + y = 0$ is of the form

$$C_1 \cos t + C_2 \sin t$$

for some constants C_1 and C_2.

Example 3. Consider the equation $y'' - y = 0$. Again, by direct calculation we check that the function $y_1(t) = e^t$ and $y_2(t) = e^{-t}$ are solutions. The Wronskian is

$$W[y_1, y_2](t) = e^t \cdot (-e^{-t}) - e^t \cdot e^{-t} = -2.$$

In this case $p(t) = 0$ as before, so the right-hand side of Abel's formula is agaqin

$$Ce^{-\int p(t)\mathrm{d}t} = C,$$

and we see that Abel's formula holds with $C = -2$. Since $W[y_1, y_2](t) \neq 0$ for all t, we conclude that every solution to $y'' - y = 0$ is of the form

$$C_1 e^t + C_2 e^{-t}$$

for some constants C_1 and C_2.

2 Characteristic polynomial; distinct roots

The last example gives us a general way of solving second order linear homogenous equations with constants coefficients, that is equations of the form

$$y'' + py' + qy = 0 \tag{2.1}$$

where now p and q are constant. Inspired by the example, we can look for solutions of the form

$$y(t) = e^{\lambda t}$$

for some constant λ. The requirement that this function satisfies equation (2.1) will give us an equation for λ which we can then solve. To see this, it is convenient to introduce the following notation. For every function y, write

$$L[y] = y'' + py' + qy.$$

Here L can be seen as a "function of functions", that is: L is an operation that takes as an input a function y of variable t and returns another function of variable t. This is similar to how regular functions work: as an input they take a number and return another number. The only difference here is that L operates on functions on numbers. This notation is useful because now saying that y is a solution to (2.1) is equivalent to saying that $L[y] = 0$. (Important point: the right-hand side has to be understood as a constant function zero, that is: the function $L[y]$ takes value zero for every t.)

In any case, we compute

$$\begin{aligned} L[e^{\lambda t}] &= (e^{\lambda t})'' + p(e^{\lambda t})' + qe^{\lambda t} \\ &= \lambda^2 e^{\lambda t} + p\lambda e^{\lambda t} + qe^{\lambda t} \\ &= (\lambda^2 + p\lambda + q)e^{\lambda t} \\ &= \chi(\lambda)e^{\lambda t}, \end{aligned} \tag{2.2}$$

where

$$\chi(\lambda) = \lambda^2 + p\lambda + q$$

is the *characteristic polynomial* of the equation. Moreover, the function $y(t) = e^{\lambda t}$ is a solution to the equation if and only if $\chi(\lambda) = 0$.

Note that this is simply a quadratic equation for λ so we know how to solve it! If p and q satisfy

$$p^2 - 4q > 0$$

then there are two solutions

$$\lambda_1 = \frac{-p + \sqrt{p^2 - 4q}}{2} \quad \text{and} \quad \lambda_2 = \frac{-p - \sqrt{p^2 - 4q}}{2}.$$

Consequently, in this case we have two solutions

$$y_1(t) = e^{\lambda_1 t} \quad \text{and} \quad y_2(t) = e^{\lambda_2 t}.$$

We can summarize this by the following theorem.

Theorem 4. *Consider a second order linear homogenous equation with constant coefficients:*

$$y'' + py' + qy = 0.$$

If $p^2 - 4q > 0$, and λ_1 and λ_2 are the two distinct roots of the characteristic polynomial $\chi(\lambda) = \lambda^2 + p\lambda + q$, then the functions

$$y_1(t) = e^{\lambda_1 t} \quad \textit{and} \quad y_2(t) = e^{\lambda_2 t}.$$

are solutions to the equation. Moreover, every solution is of the form

$$C_1 y_1 + C_2 y_2$$

for some constants C_1 and C_2.

Proof. We already verified that y_1 and y_2 are solutions (this is essentially by construction). To see that every other solution is their linear combination we need to check that the Wronskian $W = W[y_1, y_2]$ is non-zero. We compute

$$W(t) = y_1(t)y_2'(t) - y_1'(t)y_2(t) = \lambda_2 e^{\lambda_1 t}e^{\lambda_2 t} - \lambda_1 e^{\lambda_1 t}e^{\lambda_2 t} = (\lambda_1 - \lambda_2)e^{(\lambda_1+\lambda_2)t}.$$

Since $\lambda_1 \neq \lambda_2$ by assumption, and e^x is non-zero for all x, the right-hand side is non-zero. We conclude that the Wronskian is non-zero so indeed every solution to the equation is a linear combination of y_1 and y_2. □

Lecture 11: Repeated and complex roots

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REFERENCES

Boyce–DiPrima sections 3.3, 3.4; Braun section 2.2.

1 DISTINCT REAL ROOTS

Last time we considered a second order homogenous equation

$$y'' + py' + qy = 0 \tag{1.1}$$

with p and q constant. We looked for solutions of the form $y(t) = e^{\lambda t}$ for λ constant, and found that a function of this form is a solution if and only if λ is a root of the *characteristic polynomial*

$$\mathrm{ch}(\lambda) = \lambda^2 + p\lambda + q.$$

This is a quadratic function of λ. There are three cases:

- $p^2 - 4q > 0$: there are two distinct roots λ_1, λ_2,
- $p^2 - 4q = 0$: there is only one root λ_1,
- $p^2 - 4q < 0$: there are no roots.

In the previous lecture we studied the first situation. We proved that in this case any solution to (1.1) is of the form

$$y(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}$$

for constants C_1, C_2.

Example 1. Consider the equation

$$y'' + y' - 2y = 0.$$

The characteristic polynomial is

$$\chi(\lambda) = \lambda^2 + \lambda - 2.$$

We see that

$$p^2 - 4q = 9 > 0$$

so there are two distinct roots. We compute that the roots are

$$\lambda_1 = -2 \quad \text{and} \quad \lambda = 1,$$

so every solution is of the form

$$y(t) = C_1 e^{-2t} + C_2 e^{t}$$

for constants C_1 and C_2.

In today's lecture we discuss what to do in the remaining two cases.

Suppose that $p^2 - 4q = 0$, so that there is only one root

$$\lambda_1 = -p/2,$$

and, according to the general principle from the last section, the function $y_1(t) = e^{\lambda_1 t}$ is a solution to the differential equation. How do we find another solution? We use the following trick. First, observe that t he characteristic polynomial in this case can be written as

$$\chi(\lambda) = \lambda^2 + p\lambda + q = \lambda^2 + p\lambda + p^2/4 = (\lambda + p/2)^2 = (\lambda - \lambda_1)^2.$$

This is just a factorization of this quadratic polynomial. From our previous calculation in the last lecture we know that for every λ

$$L[e^{\lambda t}] = \chi(\lambda)e^{\lambda t} = (\lambda - \lambda_1)^2 e^{\lambda t}.$$

(Recall that here for any function y, $L[y]$ is the function

$$L[y] = y'' + py + q$$

and y is a solution to the equation if and only if $L[y] = 0$.) Since this equation holds for all λ we can differentiate it with respect to λ. Note that we can move the differentiation inside L. The definition of L involves taking first and second derivatives with repsect to t. Recall from Multivariable Calculus that taking the derivative first with respect to t and then with respect to λ is the same as first taking the derivative with respect to λ and then with respect to t. So we have

$$\frac{\partial}{\partial \lambda} L[e^{\lambda t}] = L[\frac{\partial}{\partial \lambda} e^{\lambda t}] = L[te^{\lambda t}]$$

and the right-hand side is

$$\frac{\partial}{\partial \lambda}(\lambda - \lambda_1)^2 e^{\lambda t} = 2(\lambda - \lambda_1)e^{\lambda_t} + \lambda(\lambda - \lambda_1)^2 e^{\lambda t}.$$

Putting both things together, we get

$$L[te^{\lambda t}] = 2(\lambda - \lambda_1)e^{\lambda_t} + \lambda(\lambda - \lambda_1)^2 e^{\lambda t}.$$

Now observe that the right-hand side is zero for $\lambda = \lambda_1$. That means that the function

$$y_2(t) = te^{\lambda_2 t}$$

satisfies

$$L[y_2] = 0.$$

Therefore, we have found a second solution to the differential equation! We can summarize this in the following theorem.

Theorem 2. *Consider a second order linear homogenous equation with constant coefficients:*

$$y'' + py' + qy = 0.$$

If $p^2 - 4q = 0$, and $\lambda_1 = -p/2$ is the unique root of the characteristic polynomial, then the functions

$$y_1(t) = e^{\lambda_1 t} \quad \textit{and} \quad y_2(t) = te^{\lambda_1 t}.$$

are solutions to the equation. Moreover, every solution is of the form

$$C_1 y_1 + C_2 y_2$$

for some constants C_1 and C_2.

Proof. We already checked that y_1 and y_2 are solutions. It remains to verify, as before, that their Wronskian is non-zero. Using the product rule, we get

$$\begin{aligned} W[y_1, y_2](t) &= y_1(t)y_2'(t) - y_1'(t)y_2(t) \\ &= e^{\lambda_1 t}(e^{\lambda_1 t} + t\lambda_1 e^{\lambda_1 t}) - \lambda_1 t e^{\lambda_1 t} e^{\lambda_1 t} \\ &= e^{2\lambda_1 t}. \end{aligned}$$

The right-hand side is non-zero for all t. We conclude that the Wronskian is non-zero, and therefore any solution is a linear combination of y_1 and y_2. □

3 Review of complex numbers

When $p^2 - 4q < 0$ the equation

$$\lambda^2 + p\lambda + q = 0 \tag{3.1}$$

has no roots, so it seems that our method does not produce any solutions. However, we can still find solutions using this method if we only look for solutions λ which are *complex numbers*. In this section we briefly review the theory of complex numbers. We will apply this theory to the problem of solving linear differential equations in the next section.

The starting observation is this. Not every quadratic equation has roots. The simplest example is the equation

$$\lambda^2 = -1.$$

Indeed, a square of any number is non-negative, so there cannot be λ satisfying this equation. Imagine, however, that there exists such a solution, and call it i ('i' stands for 'imaginary' since it cannot be any standard, real number) that is: i is a number satisfying

$$i^2 = -1.$$

Other than this odd property, we declare that i can be treated like any other number: we can add it to other numbers, multiply by it, and so on. All rules of addition and multiplication hold, for example

$$(5 + i)(2 - i) = 10 + 2i - 5i - i^2 = 10 - 3i - (-1) = 9 - 3i,$$

and so on. If we only accept that there is such a number i, then it turns out that we can solve any quadratic equation! For example, if we want to find λ such that solve

$$\lambda^2 = -15$$

we can simply take $\lambda = i\sqrt{15}$ because

$$\lambda^2 = (i\sqrt{15})^2 = i^2\sqrt{15}^2 = -1 \times 15 = -15$$

and similarly for $\lambda = -i\sqrt{15}$. More generally, if we have a quadratic equation (3.1) with $p^2 - 4q < 0$, we can now take the square root of $p^2 - 4q$:

$$\sqrt{p^2 - 4q} = i\sqrt{|p^2 - 4q|}$$

and the usual formula gives us two roots

$$\lambda_1 = \frac{-p - i\sqrt{|p^2 - 4q|}}{2} \quad \text{and} \quad \lambda_2 = \frac{-p + i\sqrt{|p^2 - 4q|}}{2}. \tag{3.2}$$

We verify by direct calculation that λ_1 and λ_2 indeed are roots (3.1).

Example 3. Consider the quadratic equation

$$\lambda^2 + 4\lambda + 5 = 0.$$

We have $p^2 - 4q = 16 - 20 = -4$. The roots are given by (3.2), that is:

$$\lambda_1 = \frac{-4 - 2i}{2} = -2 - i$$
$$\lambda_2 = \frac{-4 + 2i}{2} = -2 + i.$$

To get used to these sort of calculations, let us verify that λ_1 is indeed a root. We have

$$\begin{aligned}\lambda^2 + 4\lambda + 5 &= (-2 - i)^2 + 4(-2 - i) + 5 \\ &= 4 + 4i + i^2 - 8 - 4i + 5 \\ &= (4 - 1 - 8 + 5) + (4 - 4)i = 0.\end{aligned}$$

The summary of this discussion is that we can now solve any quadratic equation. However, there is a price to pay: solutions are not ordinary, real numbers, but rather *complex numbers*, that is numbers of the form

$$z = a + ib$$

where a and b are real. That is: a complex number is a pair of two real numbers. They are called the *real* and *imaginary* part of z respectively, we write:

$$a = \text{Rez} \quad \text{and} \quad b = \text{Imz}.$$

We will now discuss some natural operations on complex numbers.

Addition and multiplication

As we said, the rules of addition and multiplication for these numbers are the same as for real numbers.

$$(a_1 + ib_1) + (a_2 + ib_2) = (a_1 + a_2) + i(b_1 + b_2)$$

and for multiplication,

$$(a_1 + ib_1)(a_2 + ib_2) = a_1 a_2 - b_1 b_2 + i(a_1 b_2 + a_2 b_1).$$

Division

We can also define division of complex numbers. In order to divide, we simply remove i from the denominator using the following trick:

$$\begin{aligned}\frac{a_1 + ib_1}{a_2 + ib_2} &= \frac{a_1 + ib_1}{a_2 + ib_2} \cdot \frac{a_2 - ib_2}{a_2 - ib_2} \\ &= \frac{(a_1 + ib_1)(a_2 - ib_2)}{a_2^2 + b_2^2}.\end{aligned}$$

Now the denominator is a real number and we can compute the numerator using our rules of multiplication. You don't need to remember this formula, just remember the trick to make the denominator into a real number. (This is similar to the way we remove a square root from the denominator, for example $1/\sqrt{2} = \sqrt{2}/2$.)

Example 4. Let us divide $1 + 2i$ by $3 - i$:

$$\frac{1 + 2i}{3 - i} = \frac{1 + 2i}{3 - i}\frac{3 + i}{3 + i} = \frac{(1 + 2i)(3 + i)}{(3 - i)(3 + i)} = \frac{1 + 7i}{9 - i^2} = \frac{1 + 7i}{10} = \frac{1}{10} + \frac{7}{10}i.$$

Conjugation and absolute value

Given a complex number $z = a + bi$ its *conjugate* is the complex number

$$\bar{z} = a - bi.$$

Note that for any z we have $\overline{(\bar{z})} = z$. The *absolute value* of z is defined by

$$|z| = \sqrt{a^2 + b^2}.$$

Observe that for any complex number z we have

$$z\bar{z} = |z|^2.$$

Indeed, we have

$$z\bar{z} = (a+bi)(a-bi) = a^2 - abi + abi - b^2 i^2 = a^2 + b^2 = |z|^2.$$

Example 5. If $z = \sqrt{2} + 4i$, then $\bar{z} = \sqrt{2} - 4i$ and

$$|z| = \sqrt{2 + 16} = \sqrt{18} = 3\sqrt{2}.$$

An important observation is that given a quadratic equation with real coefficients, its roots λ_1 and λ_2 given by the formula (3.2) are conjugate to each other, that is:

$$\lambda_2 = \bar{\lambda}_1.$$

Exponentiation

We want to make sense of the expression e^z for any complex number $z = a + bi$. We want the exponential to have the same properties that we know for real numbers that is

$$e^{z+w} = e^z e^w.$$

In particular, for a and b real we should have

$$e^{a+bi} = e^a e^{bi}.$$

Here e^a is the standard exponential function for real numbers, so we only need to make sense of e^{bi}. One way to define it is to declare that the following *Euler's formula* hold, that is, by definition,

$$e^{ib} = \cos b + i \sin b.$$

To see that this is a sensible definition, we should check that for all real numbers b_1 and b_2 we have

$$e^{ib_1} e^{ib_2} = e^{i(b_1+b_2)}.$$

The left-hand side is

$$\begin{aligned}
&(\cos b_1 + i \sin b_1)(\cos b_2 + i \sin b_2) \\
&= \cos b_1 \cos b_2 - \sin b_1 \sin b_2 + i(\sin b_1 \cos b_2 + \cos b_2 \sin b_1) \\
&= \cos(b_1 + b_2) + i \sin(b_1 + b_2),
\end{aligned}$$

where in the last line we used the formulae for sine and cosine of the sum of two angles. We see that the right-hand side is, by definition, $e^{i(b_1+b_2)}$, the our exponential of an imaginary number has the desired property. This tells us that our definition is sensible. To summarize, for an arbitrary complex number $z = a + bi$, the exponential is given by

$$e^z = e^a e^{ib} = e^a(\cos b + i \sin b).$$

Example 6. Let's compute $e^{2+i\pi/4}$. We have

$$e^{2+i\pi/2} = e^2(\cos\pi/4 + i\sin\pi/4) = \frac{e^2\sqrt{2}}{2}(1+i).$$

In the next lecture we will talk about how to use complex numbers to solve second order equations $y'' + py' + q = 0$ with $p^2 - 4q < 0$.

Lecture 12: Complex roots

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REFERENCES

Boyce–DiPrima sections 3.3; Braun section 2.2.

1 COMPLEX SOLUTIONS

We are interested in solving second order homogenous equations with constant coefficients:

$$y'' + py' + q = 0$$

for p and q constant. We proved that a function $y(t) = e^{\lambda t}$ where λ is a constant, is a solution to this equation if and only if λ is the root of the characteristic polynomial

$$\chi(\lambda) = \lambda^2 + p\lambda + q,$$

that is, if λ satisfies $\chi(\lambda) = 0$. This is a quadratic equation for λ. Last time we discussed two cases

1. If $p^2 - 4q > 0$, then there are two real roots λ_1, λ_2. The functions $y_1(t) = e^{\lambda_1 t}$ and $y_2(t) = e^{\lambda_2 t}$ are solutions and any other solution is their linear combinations.

2. If $p^2 - 4q = 0$, then there is only one, repeated real root λ_1. The function $y_1(t) = e^{\lambda_1 t}$ is a solution as before, and we proved that there is another solution $y_2(t) = te^{\lambda_1 t}$. Any other solution is a linear combination of these two,

In the remaining case $p^2 - 4q < 0$ the characteristic polynomial has no real roots. However, it still has two complex roots λ_1, λ_2, and λ_2 is conjugate to λ_1, that is:

$$\lambda_2 = \overline{\lambda_1}.$$

Recall from the last lecture that the conjugate of a complex number $z = a + bi$ is the complex number $\bar{z} = a - bi$.

Even though λ_1 and λ_2 are complex numbers, we can still define functions

$$y_1(t) = e^{\lambda_1 t} \quad \text{and} \quad y_2(t) = e^{\lambda_2 t}.$$

The difference now is that these functions are complex functions, that is: for every t the numbers $y_1(t)$ and $y_2(t)$ are complex. Nevertheless, complex functions can be differentiated in the same way as real functions (we simply differentiate separately the real and the imaginary part), and these two complex functions satisfy our differential equation.

We can write these complex solutions more explicitly using Euler's formula for e^z where $z = a + bi$ is a complex number:

$$e^{a+bi} = e^a(\cos b + i \sin b).$$

(In fact, this is how we defined the exponential of a complex number last time. There are other ways of defining it, for example, using power series.) Now if we write $\lambda_1 = a + bi$ with a and b real, then

$$y_1(t) = e^{\lambda_1 t} = e^{(a+bi)t} = e^{at}(\cos(bt) + i \sin(bt)).$$

Similarly, using that $\lambda_2 = \overline{\lambda_1} = a - bi$,

$$y_2(t) = e^{(a-bi)t} = e^{at}(\cos(-bt) + i \sin(-bt)) = e^{at}(\cos(bt) - i \sin(bt)).$$

Observe that

$$y_2(t) = \overline{y_1(t)}$$

so the second solution can be recovered from the first one by conjugation.

2 FROM COMPLEX SOLUTIONS TO REAL SOLUTIONS

In applications, we are interested in finding real solutions, since our function $y(t)$ usually has an interpretation as a real-life quantity, which should be a real number rather than a complex number. Fortunately, we can use our complex solutions to find real solutions.

We do this by taking the real and imaginary part of the complex solutions. Recall that for a complex number $z = a + bi$, we call a the real part of z and b the imaginary part of z (despite the name, both a and b are real numbers!). We write

$$a = \mathrm{Re}(z) \quad \text{and} \quad b = \mathrm{Im}(z).$$

An important observation is that for any complex number z we have

$$\mathrm{Re}(z) = \frac{z + \overline{z}}{2} \quad \text{and} \quad \mathrm{Im}(z) = \frac{z - \overline{z}}{2i},$$

so the real and imaginary part of z and linear combinations of z and $\overline{z}$ (with complex coefficients in the latter case). The second observation to make is that z is a real number if and only if $z = \overline{z}$. Indeed, if $z = \overline{z}$, then it follows from the above formula that $\mathrm{Im}(z) = 0$, that is z has no imaginary part so is a real number.

We can now prove the following result relating complex and real solutions to homogenous differential equations.

Theorem 1. *Let $y(t)$ be a complex function which is a solution of the homogenous differential equation*

$$y'' + p(t)y' + q(t)y = 0$$

with $p(t)$ and $q(t)$ real functions. (They don't have to be constant.) Then the conjugate $\overline{y(t)}$ and the real and imaginary parts $\mathrm{Re}(y(t))$, $\mathrm{Im}(y(t))$ are also solutions of the same equation.

Remark 2. Note that $\overline{y(t)}$ is a complex solution whereas $\mathrm{Re}(y(t))$ and $\mathrm{Im}(y(t))$ are real solutions.

Proof. The operation of conjugating complex number has the following two properties, for any complex numbers z and w we have

$$\overline{z+w} = \overline{z} + \overline{w} \quad \text{and} \quad \overline{z \cdot w} = \overline{z} \cdot \overline{w}.$$

Using this and the fact that y is a solution to our differential equation and that p and q are real, i.e. $\overline{p} = p$ and $\overline{q} = q$, we get

$$\overline{y}'' + p(t)\overline{y}' + q(t)\overline{y} = \overline{y'' + p(t)y' + q(t)y} = \overline{0} = 0.$$

Therefore, $\overline{y}$ is a solution. Since the equation is linear, any linear combination of solutions is a solution. Since the functions

$$\text{Re}(y(t)) = \frac{y(t) + \overline{y(t)}}{2} \quad \text{and} \quad \text{Im}(y(t)) = \frac{y(t) - \overline{y(t)}}{2i},$$

are linear combinations of y and $\overline{y}$, which are solutions, we conclude that they are also solutions. □

The above theorem shows that if we take the real and imaginary part of our complex solutions $y_1(t) = e^{\lambda_1 t}$ and $y_2(t) = e^{\lambda_2 t}$, where λ_1 and λ_2 are complex roots of the characteristic polynomial, then we find real solutions to our differential equation. Observe that it suffices to use only either y_1 or y_2 because

$$y_2(t) = \overline{y_1(t)}$$

and therefore

$$\text{Re}(y_2(t)) = \text{Re}(y_1(t)) \quad \text{and} \quad \text{Im}(y_2(t)) = -\text{Im}(y_1(t)).$$

So by taking the real and imaginary part of y_2 we don't find new solutions. We can summarize this discussion by the following theorem.

Theorem 3. *Consider the second order homogenous equation*

$$y'' + py' + q = 0,$$

with p and q constant and real. Suppose that $p^2 - 4q < 0$. Let λ_1 and $\lambda_2 = \overline{\lambda_1}$ be the complex roots of the characteristic polynomial $\chi(\lambda) = \lambda^2 + p\lambda + q$. Then the following holds

1. *The complex functions*

   $$y_1(t) = e^{\lambda_1 t} \quad \textit{and} \quad y_2(t) = e^{\lambda_2 t} = \overline{y_1(t)}$$

   are complex solutions and any other complex solution has the form

   $$y(t) = C_1 y_1(t) + C_2 y_2(t)$$

   for complex constants C_1 and C_2.

2. *The real functions*

   $$\text{Re}(y_1(t)) \quad \textit{and} \quad \text{Im}(y_1(t))$$

   are real solutions and any other real solution has the form

   $$y(t) = C_1 \text{Re}(y_1(t)) + C_2 \text{Im}(y_2(t))$$

   for real constants C_1 and C_2.

Proof. We already proved that the given functions are solutions. It remains to see that any other solution is their linear combination. To do that, we simply compute the Wronskian and see that it is non-zero. This is a straightforward computation similar to the ones we did in the previous lectures. □

Example 4 (Harmonic oscillator)**.** Consider the harmonic oscillator equation $y'' + y = 0$. The characteristic polynomial is $\chi(\lambda) = \lambda^2 + 1$. Its roots are $\lambda_1 = i$ and $\lambda_2 = -i$. Therefore, the complex solutions are

$$y_1(t) = e^{it} = \cos t + i \sin t \quad \text{and} \quad y_2(t) = e^{-it} = \cos t - i \sin t.$$

The real solutions are

$$\mathrm{Re}(y_1(t)) = \cos t \quad \text{and} \mathrm{Im}(y_1(t)) = \sin t.$$

Any real solution is their linear combination:

$$y(t) = C_1 \cos t + C_2 \sin t$$

with C_1 and C_2 real constants.

Example 5. Consider the equation

$$y'' - 10y' + 29y = 0.$$

The characteristic polynomial is

$$\chi(\lambda) = \lambda^2 - 10\lambda + 29.$$

Since $p^2 - 4q = -16 < 0$, there are two complex roots. We find them to be

$$\lambda_1 = 5 - 2i \quad \text{and} \quad \lambda_2 = 5 + 2i.$$

The complex solutions are

$$y_1(t) = e^{(5-2i)t} = e^{5t}(\cos(2t) - i \sin(2t)),$$
$$y_2(t) = e^{(5+2i)t} = e^{5t}(\cos(2t) + i \sin(2t)).$$

Any real solution is a linear combination of the real and imaginary part (for example, of y_2), that is any real solution is of the form

$$y(t) = C_1 e^{5t} \cos(2t) + C_2 e^{5t} \sin 2t$$

for real constants C_1 and C_2.

3 SUMMARY

Let us summarize what we did in the last two lectures. We considered a second order homogenous equation with constant coefficients:

$$y'' + py' + qy = 0,$$

with p and q constant. The first step to solving this equation is to find roots of the characteristic polynomial of the equation

$$\chi(\lambda) = \lambda^2 + p\lambda + q.$$

We look for λ such that $\chi(\lambda) = 0$. This is a quadratic equation, so there are three cases.

Real distinct roots

If $p^2 - 4q > 0$ then there are two distinct real roots λ_1, λ_2. The functions

$$y_1(t) = e^{\lambda_1 t} \quad \text{and} \quad y_2(t) = e^{\lambda_2 t}$$

are solutions and the general solution is their linear combination:

$$y(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}$$

for any constants C_1 and C_2.

Repeated root

If $p^2 - 4q = 0$ then there is real one double root λ_1. In this case, the functions

$$y_1(t) = e^{\lambda_1 t} \quad \text{and} \quad y_2(t) = te^{\lambda_1 t}$$

are solutions and the general solution is their linear combination:

$$y(t) = C_1 e^{\lambda_1 t} + C_2 te^{\lambda_1 t}.$$

Complex roots

If $p^2 - 4q < 0$ then there are two complex roots λ_1 and λ_2 which are conjugate, that is: $\lambda_2 = \overline{\lambda_1}$. In this case, the function

$$y_1(t) = e^{\lambda_1 t} \quad \text{and} \quad y_2(t) = e^{\lambda_2 t} = e^{\overline{\lambda_1} t} = \overline{y_1(t)}$$

are complex solutions. To find real solutions we take the real and imaginary part of y_1 (or we can also use y_2, it doesn't matter since $\text{Re}(y_2) = \text{Re}(y_1)$ and $\text{Im}(y_2) = -\text{Im}(y_1)$. The general real solution is

$$y(t) = C_1 \text{Re}(y_1(t)) + C_2 \text{Im}(y_1(t)).$$

To write down the formula explicitly, we write $\lambda_1 = a + bi$ where $a = \text{Re}(\lambda_1)$ and $b = \text{Im}(\lambda_1)$. Then

$$\text{Re}(y_1(t)) = e^{at} \cos(bt),$$
$$\text{Im}(y_1(t)) = e^{at} \sin(bt)$$

so the general real solution is

$$y(t) = C_1 e^{at} \cos(bt) + C_2 e^{at} \sin(bt).$$

However, it is not practical to remember these formulae. Rather, understand the process by which we found them, that is: first we write the complex solutions in the same way as we did for real roots, then we take their real and imaginary parts.

ORDINARY DIFFERENTIAL EQUATIONS (MATH 2030)

Lecture 13: Non-homogenous equations

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REFERENCES

Boyce–DiPrima sections 3.5, 3.7, 3.8; Braun sections 2.3. 2.5, 2.6.

1 NON-HOMOGENOUS EQUATIONS

Last time we discussed an algorithm for solving second order homogenous equations with constant coefficients

$$y'' + py' + qy = 0.$$

Now we turn to non-homogenous equations. In fact, for now let us consider the general case of a second order non-homogenous equation

$$y'' + p(t)y' + q(t) = r(t)$$

where $p(t)$, $q(t)$, $r(t)$ are functions, not necessarily constant. The following theorem tells us how to find general solutions to non-homogenous equations. Recall that by a *general solution* we mean the general expression for all functions satisfying the differential equation, whereas a *particular solution* is any such function. For example, the general solution to the harmonic oscillator equation

$$y'' + y = 0$$

is

$$y(t) = C_1 \sin t + C_2 \cos t$$

for C_1 and C_2 any constants, whereas the function

$$y(t) = 5 \sin t - 11 \cos t$$

is a particular solution. When we want to find all solutions to the equation, we look for the general solution. When we solve an initial value problem, we look for a particular solution.

Theorem 1. *Consider a second order non-homogenous equation*

$$y'' + p(t)y' + q(t) = r(t). \tag{1.1}$$

If $x(t)$ is a particular solution of the non-homogenous equation (1.1)*, and if*

$$C_1 y_1(t) + C_2 y_2(t)$$

is the general solution of the corresponding homogenous equation

$$y'' + p(t)y' + q(t) = 0, \tag{1.2}$$

then the general solution of the non-homogenous equation (1.1) *is*

$$y(t) = C_1 y_1(t) + C_2 y_2(t) + x(t). \tag{1.3}$$

In other words, the theorem says that

general solution of non-homogenous = general solution of homogenous + particular solution of non-homogenous

that is: in order to find all solutions to a non-homogenous equation we need to find or guess one solution and then find all solutions to the corresponding homogenous equation.

Proof. We want to show that all solutions of the non-homogenous equation (1.1) are of the form (1.3). Suppose that $y(t)$ is any solution of (1.1). Define a function

$$z(t) = y(t) - x(t).$$

We will show that $z(t)$ is a solution of the homogenous equation (1.2). In order to do that, observe that

$$z' = y' - x' \quad \text{and} \quad z'' = y'' - x''.$$

Since both $x(t)$ and $y(t)$ satisfy (1.1), we get

$$\begin{aligned} z'' + p(t)z' + q(t)z &= y'' - x'' + p(t)(y' - x') + q(t)(y - x) \\ &= (y'' + p(t)y' + q(t)y) - (x'' + p(t)x' + q(t)x) \\ &= r(t) - r(t) = 0. \end{aligned}$$

Therefore, $z(t)$ is a solution of the homogenous equation (1.2). By assumption, all solutions of the homogenous equations are linear combinations of $y_1(t)$ and $y_2(t)$, so there exist constants C_1 and C_2 such that

$$y(t) - x(t) = z(t) = C_1 y_1(t) + C_2 y_2(t).$$

This shows that $y(t)$ is of the form (1.3), as we wanted to show. □

Example 2. Let us find the general solution to the non-homogenous equation

$$y'' + y = t.$$

First, we can guess that the function $x(t) = t$ is a particular solution. Indeed, we have $x'(t) = 1$, $x''(t) = 0$, so

$$x''(t) + x(t) = 0 + t = t.$$

On the other hand, we know that the general solution to the corresponding homogenous equation

$$y'' + y = 0$$

is

$$C_1 \sin t + C_2 \cos t.$$

Using our theorem, we conclude that the general solution to the non-homogenous equation is

$$y(t) = C_1 \sin t + C_2 \cos t + t.$$

Theorem 1 tells us that if we happen to know one solution of a non-homogenous equation, we can find the general solution by finding the general solution of the corresponding homogenous equation. But how do we find that one particular solution? There are essentially two methods. One is the method of *variation of parameters*, which works in a variety of cases. This method is described in detail in the textbook and I encourage you to read about it. We won't have time to discuss this method in class; instead, we focus on examples. Another method is guessing. The basic idea is that given a non-homogenous equation with constant coefficients

$$y'' + py' + qy = r(t) \tag{2.1}$$

we look for a solution of a particular form, depending on what the function $r(t)$ is. This is similar to how we found solutions to homogenous equations by looking at functions of the special form $y(t) = e^{\lambda t}$. Let us see how this method works in some special cases.

Polynomial right-hand side

Consider equation (2.1) with right-hand side being a polynomial, that is a function of the form

$$r(t) = r_0 + r_1 t + \cdots + r_n t^n.$$

In that case, we can look for a particular solution of (2.1) which is also a polynomial. Typically we look for a solution which is a polynomial of the same degree n, that is:

$$y(t) = a_0 + a_1 t + \ldots + a_n t^n.$$

Plugging this to (2.1) we get a bunch of linear equations for the coefficients $a_0, \ldots, a_n$, which we can solve.

Example 3. Let us find a particular solution of

$$y'' + y' + y = t^2.$$

We look for a solution of the form

$$y(t) = a_0 + a_1 t + a_2 t^2.$$

We need to find the coefficients a_0, a_1, a_2. We compute

$$\begin{aligned} y'(t) &= a_1 + 2a_2 t, \\ y''(t) &= 2a_2. \end{aligned}$$

We compute the left-hand side of the differential equation and group all the terms according to the powers of t:

$$\begin{aligned} y'' + y' + y &= 2a_2 + (a_1 + 2a_2 t) + (a_0 + a_1 t + a_2 t^2) \\ &= (2a_2 + a_1 + a_0)t^0 + (2a_2 + a_1)t^1 + a_2 t^2. \end{aligned}$$

In order for this expression to be equal to t, we need a_0, a_1, a_2 to satisfy the following equations

$$\begin{aligned} &2a_1 + a_1 + a_0 = 0, \\ &2a_2 + a_1 = 0, \\ &a_2 = 1. \end{aligned}$$

We find that $a_2 = 1, a_1 = -2, a_0 = 0$, so the function

$$y(t) = -2t + t^2$$

is a particular solution to our non-homogenous equation.

Remark 4. In some cases it happens that the resulting system of linear equations for the coefficients $a_0, \ldots, a_n$ has no solutions. In that case, we look for solution of the form

$$y(t) = t(a_0 + a_1 t + \ldots + a_n t^n + a_n t_n)$$

that is, a polynomial of degree one higher than the degree of $r(t)$. (This is similar to how we found solutions to linear homogenous equations in the case of repeated root. For a detailed discussion, see Boyce–DiPrima section 3.5 and Braun section 2.5.)

Exponential right-hand side

Now consider the case

$$r(t) = (r_0 + r_1 t + r_n t^n)e^{\alpha t}$$

for some constants $a_0, \ldots, a_n$ and α. That is: the right-hand side is a polynomial times the exponential function. This is similar to the polynomial case (and, in fact, can be reduced to that one, see the textbook). In that case, we look for a solution of the same form:

$$y(t) = (a_0 + a_1 t + a_n t^n)e^{\alpha t}.$$

As before, by plugging this to the differential equation we get a system of equations for the coefficients $a_0, \ldots, a_n$, which we then solve.

Example 5. Let us find a particular solution of

$$y'' - 2y = te^t.$$

We look for a solution of the form

$$y(t) = (a_0 + a_1 t)e^t.$$

We compute, using the product rule and factoring out e^t,

$$\begin{aligned} y'(t) &= (a_0 + a_1 + a_1 t)e^t, \\ y''(t) &= (a_0 + 2a_1 + a_1 t)e^t. \end{aligned}$$

Therefore, the left-hand side of the differential equation is

$$\begin{aligned} y'' - 2y &= \{(a_0 + 2a_1 + a_1 t) - 2(a_0 + a_1 t)\} e^t \\ &= (-a_0 + 2a_1 - a_1 t)e^t. \end{aligned}$$

We want this to be equal to te^t. By comparing coefficients next to the powers of t we see that we must have

$$\begin{aligned} -a_0 + 2a_1 &= 0, \\ -a_1 &= 1. \end{aligned}$$

Therefore, $a_0 = -2$, $a_1 = -1$ and the function

$$y(t) = -2e^t - te^t$$

is a particular solution.

Remark 6. As in the polynomial case, sometimes it happens that the resulting equations for the coefficients $a_0, \dots, a_n$ have no solutions. This happens, for example, if the right-hand side is a polynomial times $e^{\alpha t}$, where α is the root of the characteristic polynomial

$$\chi(\lambda) = \lambda^2 + p\lambda + q.$$

In that case, we look for a particular solution of the form

$$y(t) = t(a_0 + \dots + a_n t^n)e^{\alpha t}$$

or

$$y(t) = t^2(a_0 + \dots + a_n t^n)e^{\alpha t}.$$

For a detailed discussion, see Boyce–DiPrima section 3.5 and Braun section 2.5. In general, you don't have to memorize all these cases. What is important that you know that if the right-hand side $r(t)$ of the non-homogenous equation is of a special type, we can often find a particular solution by guessing the general form of the function and then finding coefficients.

Trigonometric right-hand side

We can also consider the case

$$y'' + py' + qy = (r_0 + r_1 t + r_n t^n)\cos(\alpha t) \tag{2.2}$$

or

$$y'' + py' + qy = (r_0 + r_1 t + r_n t^n)\sin(\alpha t). \tag{2.3}$$

This case can be reduced to the exponential case by considering the complex equation

$$y'' + py' + qy = (r_0 + r_1 t + r_n t^n)e^{i\alpha t}. \tag{2.4}$$

Observe that by taking the real and imaginary part we get (2.2) and (2.3). Therefore, as in the exponential case, we can look for a complex solution of the form

$$y(t) = (a_0 + a_1 t + a_n t^n)e^{i\alpha t}.$$

First, we find a complex solution of (2.4) of the above form. Then its real and imaginary part are solutions of (2.2) and (2.3) respectively.

ORDINARY DIFFERENTIAL EQUATIONS (MATH 2030)

Lecture 14: Vibrations

ALEKSANDER DOAN

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REFERENCES

Boyce–DiPrima sections 3.6, 3.7, 3.8; Braun sections 2.5, 2.6

1 IMPORTANCE OF SECOND ORDER EQUATIONS

Second order equations model many natural phenomena. Newton's law $F = ma$ is a second order differential equation: a is the acceleration which is the second derivative of the position with respect to time. In many cases, Newton's law leads a linear second order differential equation. Even when the equation is nonlinear it can be often approximated by a linear one.

An important difference between first and second order linear equation is that for first order linear equations all solutions were either growing to infinity or decaying, such as, for example, e^t and e^{-t}. Second order linear equations admit also solutions which are periodic in time, such as $\sin t$ and $\cos t$. As we have seen, this is intimately related to complex numbers. We will now discuss some examples of linear second order equations describing a system that is vibrating (for example, a body on a spring, or an elastic material, or an electric circuit).

2 FREE VIBRATIONS

The simplest model is the harmonic oscillator, or free vibration. If $y(t)$ is the deformation of the system at time t (for example, the displacement of a body attached to a spring from the equilibrium position), and the force acting on the system is proportional to this deformation and acting in the opposite direction, then Newton's law gives us

$$my'' = -ky$$

where m is the mass of the system and $k > 0$ is some constant depending on the system (for example, on the material comprising the spring). We can write this equation in the form

$$my'' + ky = 0.$$

To solve this equation, we look for roots of the characteristic polynomial

$$\chi(\lambda) = m\lambda^2 + k.$$

The roots are complex: $\lambda_1 = i\sqrt{k/m}$ and $\lambda_2 = -i\sqrt{k/m}$. Introduce the constant

$$\omega_0 = \sqrt{k/m},$$

so that the roots are $i\omega_0$ and $-i\omega_0$. The complex solutions are $e^{i\omega_0 t}$ and $e^{-i\omega_0 t}$. We conclude that the general real solution is

$$y(t) = C_1 \mathrm{Im}(e^{i\omega_0 t}) + C_2 \mathrm{Im}(e^{i\omega_0 t}) = C_1 \cos(\omega_0 t) + C_2 \sin(\omega_0 t).$$

In order to see how this solution behaves and to draw its graph, we use the following algebraic trick. Suppose that the solution is non-zero and write it in the form

$$y(t) = C_1 \cos(\omega_0 t) + C_2 \sin(\omega_0 t) = A(\hat{C}_1 \cos(\omega_0 t) + \hat{C}_2(\omega_0 t))$$

where

$$A = \sqrt{C_1^2 + C_2^2}, \quad \hat{C}_1 = C_1/A, \quad C_2 = C_2/A.$$

We simply divided and multiply both terms by A. We did it so that the new constants $\hat{C}_1$ and $\hat{C}_2$ satisfy the

$$\hat{C}_1^2 + \hat{C}_2^2 = 1.$$

This looks like the formula

$$\sin\theta^2 + \cos\theta^2 = 1.$$

Indeed for any numbers $\hat{C}_1$, $\hat{C}_2$ satisfying $\hat{C}_1^2 + \hat{C}_2^2 = 1$ there is an angle θ such that

$$\sin\theta = \hat{C}_1 \quad \text{and} \quad \cos\theta = \hat{C}_2.$$

You can check that the angle given by $\theta = \arctan(\hat{C}_1/\hat{C}_2)$ has this property. Plugging this to the formula for $y(t)$ we get

$$y(t) = A(\sin\theta\cos(\omega_0 t) + \cos\theta\sin(\omega_0 t)) = A\sin(\omega_0 t + \theta),$$

where we used the formula for sine of the sum of two angles. Therefore, any solution to the harmonic oscillator equation has the form

$$y(t) = A\sin(\omega_0 t + \theta)$$

for some constants A and θ. We can easily graph this function. From the graph we see that A is the *amplitude* of the motion, that is the maximal value of $y(t)$; ω_0 is the *frequency*: it tells us how many oscillation happen within a given period of time, and θ is the *phase shift*, meaning that the graph of $y(t)$ is shifted with respect to the graph of $\sin(\omega_0 t)$ by θ.

3 DAMPED VIBRATIONS

We now consider a different model

$$my'' + cy' + ky = 0$$

with m and k as before, and $c > 0$. Here the term cy' corresponds to *damping*, that is resistance to the motion of the system which is proportional to the velocity. For example, this could be the resistance of the surroundings of the system. We will see that damping causes the system to eventually stop.

As before, to find solutions we look for roots of the characteristic polynomial

$$\chi(\lambda) = m\lambda^2 + c\lambda + k.$$

There are three cases, depending on m, c, and k:

1. *Overdamped*. If $c^2 - 4km > 0$ there are two distinct real roots λ_1, λ_2. You can check from the formula that they are negative. The general solution is
$$y(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}.$$
Since the roots are negative, we see that the solution decays exponentially with time. Damping is so strong that the system doesn't vibrate at all.

2. *Critically damped*. If $c^2 - 4km = 0$, there is a single repeated root λ_1, which again is negative. The general solution is
$$y(t) = C_1 e^{\lambda_1 t} + C_2 t e^{\lambda_2 t}.$$
When t is small, the second term can cause $y(t)$ to grow for some time (depending on the values of C_1 and C_2). However, for large values of t the exponential decay dominates and the system behaves similarly to the previous case; the movement eventually decays.

3. *Underdamped*. If $c_2 - 4km < 0$, there are two distinct complex roots λ_1 and $\lambda_2 = \overline{\lambda}_1$. Write them as $\lambda_1 = a + bi$. We can check from the formula that $a < 0$. The general solution is
$$y(t) = C_1 e^{at} \cos(bt) + C_2 e^{at} \sin(bt).$$
Using the same algebraic trick as in the harmonic oscillator case, we can write it simply as
$$y(t) = A e^{at} \sin(bt + \theta)$$
for some A and θ. We see that the system vibrates. However, because $a < 0$, the amplitude of the vibration decays exponentially with time. Eventually, the vibration decays as in the previous two cases.

4 FORCED VIBRATIONS

We have seen that in the presence of damping, all vibrations eventually disappears. We can sustain it by acting on the system with external force. Such an external force adds another term to the equation
$$my'' + cy' + ky = F(t)$$
where $F(t)$ is the external force at time t. This is a second order linear non-homogenous equation. Suppose that the external force is of the form
$$F(t) = F_0 \sin(\omega t),$$
that is it's a simple sine vibration with frequency ω.

Consider first the case when there is no damping, that is $c = 0$ and the equation is
$$my'' + ky = F_0 \sin(\omega_t). \tag{4.1}$$
The general solution is the sum of the general solution of the homogenous equation $my'' + ky = 0$ and any particular solution of the nonhomogenous equation. To find a particular solution, we first find for a particular solution, denote it by y_c, of the complex equation
$$my'' + ky = F_0 e^{i\omega t}.$$

If y_c is a solution of this equation, then its imaginary part $y = \text{Im}(y_c)$ is a solution to (4.1) because

$$\text{Im}(e^{i\omega t}) = \sin(\omega t).$$

We look for y_c of the form

$$y_c(t) = Be^{i\omega t}.$$

We have

$$y_c'(t) = i\omega Be^{i\omega t} \quad \text{and} \quad y_c''(t) = -\omega^2 Be^{i\omega t}.$$

Therefore,

$$my_c'' + ky_c = B(-m\omega^2 + k)e^{i\omega t}.$$

In order for this expression to be equal to $F_0 e^{i\omega t}$, we need

$$B = \frac{F_0}{-m\omega^2 + k} = \frac{F_0/m}{\omega_0^2 - \omega^2}. \tag{4.2}$$

Here $\omega_0 = \sqrt{k/m}$ is the frequency of the free vibrations of the system. This expression makes sense only when $\omega_0 \neq \omega$. The complex solution is then

$$y_c(t) = Be^{i\omega t},$$

with B given by formula (4.2), and the imaginary part

$$y(t) = \text{Im} y_c(t) = B\cos(\omega t)$$

is a particular solution of (4.1). We see that if the frequency of the external force ω is different than the frequency of the system ω_0, then the general solution of (4.1) is

$$y(t) = A\sin(\omega_0 t + \theta) + B\sin(\omega t).$$

Here the first term is the general solution of the harmonic oscillator equation $my'' + ky = 0$: A and θ are any constants, and B is given by formula (4.2). Therefore, if $\omega_0 \neq \omega$, the solution is simply a composition of two simple sine vibrations with different frequencies. (You can plot the solution for different values of ω and ω_0 using a website such as Wolfram Alpha to see the graph.)

The more interesting case is $\omega = \omega_0$, that is when the frequency of the external force is the same as that of the system. In that case, there is no particular solution of the complex equation of the form $B\cos(\omega t)$. As you can see, (4.2) does not produce a solution in this case. In that case, we look for a complex solution of the form

$$y_c(t) = Bte^{i\omega t}.$$

Indeed, a somewhat tedious calculation shows that we can find such a solution, with B given by an explicit formula depending on F_0, m, ω_0, and ω:

$$B = -i\frac{F_0}{2m\omega_0}.$$

In that case, the imaginary part is

$$y(t) = |B|t\cos(\omega_0 t)$$

(we take the absolute value to get a real number since B is imaginary) and the general solution is

$$y(t) = A\sin(\omega_0 t + \theta) + |B|t\sin(\omega_0 t).$$

The second term grows linearly in time! In particular, the amplitude of the vibration increases to infinity. This is the phenomenon of *resonance*: when the frequency of the external force is equal to the frequency of the system, the resulting vibrations become larger and larger with time. This phenomenon can have catastrophic results, as it happened in the case of the Tacoma Narrows Bridge. You can read about this in section 2.6 of Braun's textbook.

This phenomenon can be prevented by introducing some damping in the system. The differential equation for vibration with external sinusoidal force and damping is

$$my'' + cy' + ky = F_0 \sin(\omega t).$$

You can compute that when $c > 0$, that is we have some damping, the equation has a particular solution of the form

$$B \sin(\omega t - \delta)$$

for some constants B and δ. Therefore, the general solution is the sum of the general solution to the homogenous equation $my'' + cy' + ky = 0$, discussed in the previous system, and the above particular solution. We see that the solution is always a bounded vibration, regardless of ω. You can read the derivation in Boyce–DiPrima section 3.8 and Braun section 2.6.

Lecture 15: Higher order linear equations

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REFERENCES

Boyce–DiPrima sections 4.1, 4.2; Braun sections 2.15.

1 HIGHER ORDER EQUATIONS AS SYSTEMS

A general differential equation of order n has the form

$$y^{(n)} = f(t, y, y', y'', \dots, y^{(n-1)})$$

for a function of n variables f. Recall that in Lecture 9 we reinterpreted a second order equation as a system of two first order equations. In the same way, a differential equation of order n is equivalent to a system of n first order equations. If y is a solution, we can define n functions

$$x_1(t) = y(t), \quad x_2(t) = y'(t), \quad x_3(t) = y''(t), \quad x_n(t) = y^{(n-1)}(t).$$

We compute that the derivatives of $x_1, \dots, x_n$ satisfy

$$\begin{cases} x_1'(t) = y'(t) = x_2(t), \\ x_2'(t) = y''(t) = x_3(t), \\ \dots \\ x_n'(t) = y^{(n)} = f(t, y, y', \dots, y^{(n-1)}) = f(t, x_1, \dots, x_{n-1}). \end{cases}$$

We see that the collection of functions $x_1, \dots, x_n$ solves the system of first order equations

$$\begin{cases} x_1' = x_2, \\ x_2' = x_3, \\ \dots \\ x_n' = f(t, x_1, \dots, x_{n-1}). \end{cases}$$

2 EXISTENCE AND UNIQUENESS

This is pretty much the same as for the second order equations. The only difference is that now we have to specify the initial value of the function and its $n-1$ derivatives.

Theorem 1. *Let f be a function of n variables such that f and its partial derivatives are continuous around the point $(t_0, y_0, y_1, \dots, y_{n-1})$. The initial value problem*

$$\begin{cases} y^{(n)} = f(t, y, y', y'', \dots, y^{(n-1)}) \\ y(t_0) = y_0, \\ y'(t_0) = y_1, \\ \dots, \\ y^{(n-1)}(t_0) = y_{n-1} \end{cases}$$

has a unique solution $y = y(t)$ defined for t from the interval $(t_0 - \epsilon, t_0 + \epsilon)$ for some $\epsilon > 0$.

3 LINEAR EQUATIONS

A linear differential equation of order n has the form

$$y^{(n)} + p_{n-1}(t)y^{(n-1)} + \dots + p_1(t)y' + p_0(t)y = r(t),$$

where $p_1, \dots, p_{n-1}$ and r are given functions. We say that such an equation is *homogenous* if $r(t) = 0$. Otherwise we say it is *nonhomogenous*. We say that it has *constant coefficients* if all functions $p_1, \dots, p_{n-1}$ are constant (if the equation is nonhomogenous we don't require r to be constant).

As for second order equations, we have the following.

Theorem 2. *If $y_1, \dots, y_k$ are solutions of a homogenous linear equation*

$$y^{(n)} + p_{n-1}(t)y^{(n-1)} + \dots + p_1(t)y' + p_0(t)y = 0.$$

Then any linear combination

$$y(t) = C_1 y_1(t) + \dots + C_k y_k(t)$$

is also a solution.

Theorem 3. *Let x be a solution of a nonhomogenous linear equation*

$$y^{(n)} + p_{n-1}(t)y^{(n-1)} + \dots + p_1(t)y' + p_0(t)y = r(t).$$

Any other solution of this equation is of the form

$$y(t) = x(t) + z(t)$$

where z is a solution of the corresponding homogenous equation

$$y^{(n)} + p_{n-1}(t)y^{(n-1)} + \dots + p_1(t)y' + p_0(t)y = 0.$$

This can be summarized by saying that the general solution of a nonhomogenous equation is the sum of a particular solution of the nonhomogenous equation and the general solution of the corresponding homogenous equations. This is exactly the same as for second order linear equations.

From now on, we focus on homogenous equations. For second order equations, we wanted to find two linearly independent solutions and then the general solution was their linear combination. Similarly, for linear homogenous equations of order n, we want to find n linearly independent solutions $y_1, \dots, y_n$. By linearly independent I mean that there is no linear combination

$$C_1 y_1(t) + \dots + C_n y_n(t)$$

which is zero for all t, except for the trivial combination $C_1 = \dots = C_n = 0$. Once we find n linearly independent solutions, the general theory tells us that any other solution is their linear combination. (We will prove this when we talk about linear systems of first order equations.)

How do we find n linearly independent solutions? This is, in general, a difficult task. However, we can solve this problem for homogenous equations with constant coefficients, that is equations of the form

$$y^{(n)} + p_{n-1}y^{(n-1)} + \ldots + p_1 y' + p_0 y = 0, \tag{4.1}$$

where $p_0, \ldots, p_{n-1}$ are constant. As for second order equations, we will look for solutions of the form

$$y(t) = e^{\lambda t}$$

for some constant λ. It is convenient to introduce the following notation. Given a function y, define a function $L[y]$ by

$$L[y] = y^{(n)} + p_{n-1}y^{(n-1)} + \ldots + p_1 y' + p_0 y$$

Therefore, solutions of (4.1) are exactly those functions y for which $L[y] = 0$. Now we can compute $L[y]$ for $y(t) = e^{\lambda t}$. We find, after using the chain rule, that

$$L[e^{\lambda t}] = \chi(\lambda)e^{\lambda t},$$

where

$$\chi(\lambda) = \lambda^n + p_{n-1}\lambda^{n-1} + p_{n-2}\lambda^{n-2} \ldots + p_1\lambda + p_0$$

is the *characteristic polynomial* of the equation. Therefore, the function $y(t) = e^{\lambda t}$ is a solution if and only if λ is a root of the characteristic polynomial. Remember that we look for n different solutions, so it would be nice to have n roots. This is, indeed, the case, except we have to allow complex roots and also we need to count roots with multiplicity. This is the content of the Fundamental Theorem of Algebra.

Theorem 4 (Fundamental Theorem of Algebra)**.** *Any polynomial*

$$\chi(\lambda) = \lambda^n + p_{n-1}\lambda^{n-1} + p_{n-2}\lambda^{n-2} \ldots + p_1\lambda + p_0$$

of degree n has n complex roots, counted with multiplicity. That is, there are complex numbers $\lambda_1, \ldots, \lambda_n$ such that

$$\chi(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2) \cdots (\lambda - \lambda_n).$$

Note that in the above decomposition some of the roots might be equal to each other. So another way would be to say that there are k distinct complex numbers $\lambda_1, \ldots, \lambda_k$ for some $k \leq n$, and positive integers $m_1, \ldots, m_k$ (multiplicities of the roots) such that

$$\chi(\lambda) = (\lambda - \lambda_1)^{m_1}(\lambda - \lambda_2)^{m_2} \cdots (\lambda - \lambda_k)^{m_k}$$

and

$$m_1 + m_2 + \ldots + m_k = n.$$

Now we have an algorithm for finding the general solution of a homogenous linear equation with constant coefficients. First, we look for roots of the characteristic polynomial $\lambda_1, \ldots, \lambda_n$. If they are all different and real, we get n different real solutions $e^{\lambda_1 t}, \ldots, e^{\lambda_n t}$ and the general solution is

$$y(t) = C_1 e^{\lambda_1 t} + \ldots + C_n e^{\lambda_n t}.$$

If some of the roots are complex, they come in conjugate pairs $\lambda_0, \bar{\lambda}_0$. Then the function $e^{\lambda_0 t}$ is a complex solution and to get two real solutions, we take its real and imaginary part:

$$\text{Re}(e^{\lambda_0 t}), \quad \text{Im}(e^{\lambda_0 t}).$$

We repeat this process for every pair of conjugate complex roots to get two real solutions for every such pair. Finally, if a root λ_0 is repeated with multiplicity m, that is the factor $(\lambda - \lambda_0)^m$ appears in the decomposition of χ, then, similarly to what we did for second order equations, we can produce m solutions of the form:

$$e^{\lambda_0 t}, \quad te^{\lambda_0 t}, \quad t^2 e^{\lambda_0 t}, \quad \ldots, t^m e^{\lambda_0 t}.$$

We repeat this process for every eigenvalue so that in total we produce n different solutions. The general solution is their linear combination.

Remark 5. For polynomials of degree two, three, and four there are explicit formulae for the roots, in terms of the coefficients $p_0, \ldots, p_{n-1}$. However, for a general polynomial of degree higher than four there are not such formulae. Interestingly, it's not only that we don't know a formula. In fact, the French mathematician Evariste Galois proved in 1832, that there cannot exist such a formula. This is a fascinating part of mathematics and if you want to learn more about it, you can take a class on abstract algebra and Galois theory.

Example 6. Consider the third order equation

$$y^{(3)} - 3y'' + 2y' = 0.$$

The characteristic polynomial

$$\chi(\lambda) = \lambda^3 - \lambda^2 + 2\lambda$$

has roots $\lambda = 0, 1, 2$. The general solution is

$$y(t) = C_1 + C_2 e^t + C_3 e^{2t}.$$

Example 7. For the following fourth order equation

$$y^{(4)} + y = 0$$

the characteristic polynomial is

$$\chi(\lambda) = \lambda^4 + 1.$$

How to find all λ such that $\lambda^4 = -1$? It follows from this equation that $|\lambda| = 1$. Here $|\lambda|$ is the norm of the complex number, that is, if $\lambda = a + bi$, then $|\lambda| = \sqrt{a^2 + b^2}$. Any complex number with $|\lambda| = 1$ is of the form $\lambda = e^{i\theta}$ for some real number θ. To find θ, we use the equation

$$e^{4i\theta} = \lambda^4 = -1 = e^{\pi i}.$$

Therefore, $\theta = \pi/4$ is a solution. But since $e^{i\theta}$ is 2π-periodic, so is

$$\theta = \pi/4 + 2k\pi$$

for any integer k. Taking $k = 1, 2, 3$, we get $\theta = 3\pi/4, 5\pi/4, 7\pi/4$. Taking larger k won't give us new roots because, for example $e^{i9\pi/4} = e^{i\pi/4}$ by periodicity. So we get roots

$$e^{i\pi/4}, \quad e^{i3\pi/4}, \quad e^{i5\pi/4}, \quad e^{i7\pi/4}.$$

Observe that the first and fourth one are conjugate to each other, and so are the second and the third one. So to get four linearly independent solutions we take the functions

$$\mathrm{Re}(e^{i\pi t/4}), \quad \mathrm{Im}(e^{i\pi t/4}), \quad \mathrm{Re}(e^{i3\pi t/4}), \quad \mathrm{Im}(e^{i3\pi t/4}).$$

You can write them in terms of trigonometric functions using Euler's formula.

Lecture 16: Systems of first order differential equations

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REFERENCES

Boyce–DiPrima sections 7.1, 7.2, 7.4; Braun section 3.1.

1 SYSTEMS OF DIFFERENTIAL EQUATIONS

So far we have studied differential equations for one unknown function $y(t)$. In the remaining part of the course, we will study systems of first order differential equations for a collection of unknown functions $y_1(t), \dots, y_n(t)$. The general form of such a system is

$$\begin{cases} y_1' = f_1(t, y_1, \dots, y_n), \\ y_2' = f_2(t, y_1, \dots, y_n), \\ \dots \\ y_n' = f_n(t, y_1, \dots, y_n) \end{cases}$$

where $f_1, \dots, f_n$ are known functions of n variables. Observe, that like for algebraic equations, if we have n unknowns we also need to have n equations.

Such systems are ubiquitous in science. For example, in physics Newton's law $F = ma$ can be applied to systems of many bodies. For each body we will have one equation, relating its acceleration (i.e. the second derivative of the position function) to the total force acting on the body. This total force, in general, will depend on the positions, velocities, etc. of all bodies since they can all interact (for example, through electromagnetic or gravitational interactions). An interesting example is our Solar system: all planets and the Sun attract each other so the trajectory of each body in the Solar system affected by the positions of the other bodies. This leads to a complicated system of differential equations.

At the beginning of the course, we discussed various models of population growth. However, in reality populations of different species interact with each other. For example, you can consider models for population growth of two different species: predator and prey. The growth of the predator population will depend on the availability of food, so on the prey population. Similarly, the growth of the prey population will depend on the predator population, since predators hunt prey. This leads to a system of two differential equations for two functions, known as the Lotka–Volterra system.

Another example are epidemics models which we will discuss towards the end of the class. In such models you are interested in how the percentage of infected individuals in the population, denoted by $I(t)$, changes in time. But this number depends also on the percentage of susceptible individuals

$S(t)$, since if there are more susceptible individuals, the number of infected individuals will grow faster. You can also introduce the percentage of recovered individuals $R(t)$. Together, these numbers give us the entire population $S + I + R = 1$. The SIR model of epidemics describes how these quantities vary in time by a system of three differential equations for functions I, S, and R.

In essence, any time your model contains a number of quantities which affect each other, you are likely to use systems of differential equations rather than a single differential equation.

Finally, as we have already learned, every order n differential equation is equivalent to a system of n first order differential equations. A general differential equation of order n has the form

$$y^{(n)} = f(t, y, y', y'', \ldots, y^{(n-1)})$$

for a function of n variables f. If y is a solution, we can define n functions

$$x_1(t) = y(t), \quad x_2(t) = y'(t), \quad x_3(t) = y''(t), \quad x_n(t) = y^{(n-1)}(t).$$

We compute that the derivatives of $x_1, \ldots, x_n$ satisfy

$$\begin{cases} x_1'(t) = y'(t) = x_2(t), \\ x_2'(t) = y''(t) = x_3(t), \\ \ldots \\ x_n'(t) = y^{(n)} = f(t, y, y', \ldots, y^{(n-1)}) = f(t, x_1, \ldots, x_{n-1}). \end{cases}$$

We see that the collection of functions $x_1, \ldots, x_n$ solves the system of first order equations

$$\begin{cases} x_1' = x_2, \\ x_2' = x_3, \\ \ldots \\ x_n' = f(t, x_1, \ldots, x_{n-1}). \end{cases}$$

In some situations it is convenient to study the order n equation, and in some the system of n first order equations.

2 EXISTENCE AND UNIQUENESS

There is an existence and uniqueness theorem for systems of differential equations, which is similar to the theorems we already discussed. Note that in order for the initial value problem to have a unique solution, we need to specify n initial conditions, where n is the number of unknown functions (and also the number of equations).

Theorem 1. *Consider the system of first order differential equations*

$$\begin{cases} y_1' = f_1(t, y_1, \ldots, y_n), \\ y_2' = f_2(t, y_1, \ldots, y_n), \\ \ldots \\ y_n' = f_n(t, y_1, \ldots, y_n) \end{cases}$$

with the initial value problem

$$\begin{cases} y_1(t_0) = a_1, \\ y_2(t_0) = a_2, \\ \dots \\ y_n(t_0) = a_n. \end{cases}$$

for given $t_0, a_1, \dots, a_n$. If the functions $f_1, \dots, f_n$ and all their partial derivatives with respect to $y_1, \dots, y_n$ are continuous around the point $(t_0, a_1, \dots, a_n)$, then the initial value problem has a solution $y_1(t), \dots, y_n(t)$ defined in an interval $(t_0 - \epsilon, t_0 + \epsilon)$ for some $\epsilon > 0$. Moreover, this solution is unique in that interval.

3 LINEAR SYSTEMS

General systems of differential equations are hard to solve. However, many models are described (or approximated) by linear systems. A linear system of differential equations has the form

$$\begin{cases} y_1' = a_{11}(t)y_1 + a_{12}(t)y_2 + \ldots + a_{1n}(t)y_n + r_1(t), \\ y_2' = a_{21}(t)y_2 + a_{22}(t)y_2 + \ldots + a_{2n}(t)y_n + r_2(t), \\ \dots \\ y_n' = a_{n1}(t)y_1 + a_{n2}(t)y_2 + \ldots + a_{nn}(t)y_n + r_n(t), \end{cases}$$

for given functions $a_{ij}(t)$, $r_i(t)$. Here the indices i and j can be any numbers from 1 to n. The system is called *homogenous* if

$$r_1(t) = \ldots = r_n(t) = 0.$$

Otherwise we say that the system is *non-homogenous*. The following is a direct analog of what we proved for second order linear equations.

Theorem 2.

1. *If functions $(y_1(t), \dots, y_n(t))$ and $(x_1(t), \dots, x_n(t))$ solve a homogenous linear system of differential equations, then so do $(y_1(t) + x_1(t), \dots, y_n(t) + x_n(t))$ and $(cy_1(t), \dots, cy_n(t))$ for any constant c.*
2. *Every solution of a non-homogenous linear system of differential equation is the sum of any particular solution to that equation and a solution of the corresponding homogenous linear system.*

In the next few lectures we will discuss a general method of finding solutions to homogenous linear systems of differential equations with *constant coefficients*

$$\begin{cases} y_1' = a_{11}y_1 + a_{12}y_2 + \ldots + a_{1n}y_n, \\ y_2' = a_{21}y_2 + a_{22}y_2 + \ldots + a_{2n}y_n, \\ \dots \\ y_n' = a_{n1}y_1 + a_{n2}y_2 + \ldots + a_{nn}y_n, \end{cases}$$

Here a_{ij} are constant. Recall that to solve second order equations we were looking for solutions of the form $y(t) = e^{\lambda t}$. This led us to the characteristic equation for λ. Similarly, for systems we can look for solutions of the form

$$y_1(t) = x_1 e^{\lambda t}, \quad y_2(t) = x_2 e^{\lambda t}, \dots, \quad y_n = x_n e^{\lambda t}$$

for some constants $x_1, \ldots, x_n$ and λ. We are left with the task of finding appropriate constants for which the above functions are indeed solutions. Plugging the formula for y_i to the equations and dividing by $e^{\lambda t}$, we see that to get solutions we need the constants x_i, λ to satisfy the following

$$\begin{cases} \lambda x_1 = a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n, \\ \lambda x_2 = a_{21}x_2 + a_{22}x_2 + \ldots + a_{2n}x_n, \\ \ldots \\ \lambda x_n = a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n. \end{cases}$$

This is a system of *algebraic equations*, namely equations for numbers rather than functions. So we are naturally led to the problem: for what λ does the above system of linear algebraic equations have a solution, which is a collection of numbers $(x_1, \ldots, x_n)$. And how do we find this solution? Such problems are studied by the field of mathematics called *linear algebra*. In the next two lectures we will discuss basic concepts and results of linear algebra.

Lecture 17: Review of linear algebra

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REFERENCES

Boyce–DiPrima sections 7.2, 7.3; Braun sections 3.2, 3.3, 3.5, 3.6, 3.7.

1 VECTORS AND MATRICES

The starting point of linear algebra is this question: how do we solve a system of linear algebraic equations

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1, \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2, \\ \ldots \\ a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n = b_n. \end{cases} \tag{1.1}$$

Here a_{ij} and b_i are given and a solution consists of a collection of n numbers $(x_1, \ldots, x_n)$. Because there are so many variables and equations, this looks like a complicated problem in general. The first step is to introduce notation which will simplify it and helps us deal with such all these expressions.

To do that, we introduce the notion of a vector. An n–dimensional *vector* is simply a collection of n numbers $(x_1, \ldots, x_n)$. In order to distinguish vectors from numbers, we will denote them by bold font: $\mathbf{x} = (x_1, \ldots, x_n)$. It is customary to put all these numbers in a column and to write

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \ldots \\ x_n \end{bmatrix}.$$

Note that a vector can be interpreted as a point in the n-dimensional space. For example, for $n = 2$, a vector has two coordinates, so it defines a point on the plane. For $n = 3$ we get three numbers, so they define a point in the three-dimensional space which has these numbers as coordinates. For general n, you can think of a vector as a collection of n coordinates which gives us a point in the n-dimensional space. The set of all n-dimensional vector is denoted by $\mathbb{R}^n$. To say that $\mathbf{x}$ is an element of this set, we write $\mathbf{x} \in \mathbb{R}^n$ (read: '$\mathbf{x}$ is an element of $\mathbb{R}^n$').

A vector can be multiplied by a number (which in linear algebra is also called a *scalar*). Given $\lambda \in \mathbb{R}$ and $\mathbf{x} \in \mathbb{R}^n$ as above, define

$$\lambda \mathbf{x} = \begin{bmatrix} \lambda x_1 \\ \lambda x_2 \\ \dots \\ \lambda x_n \end{bmatrix}.$$

Similarly, given another vector

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \dots \\ y_n \end{bmatrix}$$

we define the sum of vectors $\mathbf{x}$ and $\mathbf{y}$ by

$$\mathbf{x} + \mathbf{y} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \dots \\ x_n + y_n \end{bmatrix}$$

Example 1. If $\lambda = 2$ and $\mathbf{x}, \mathbf{y} \in \mathbb{R}^3$ are given by

$$\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix} \quad \text{and} \quad \mathbf{y} = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$$

then

$$\lambda \mathbf{x} = \begin{bmatrix} 2 \\ 0 \\ -6 \end{bmatrix} \quad \text{and} \quad \lambda \mathbf{y} = \begin{bmatrix} 0 \\ 4 \\ 2 \end{bmatrix},$$

and

$$\mathbf{x} + \mathbf{y} = \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}.$$

Let us return to the problem of rewriting equations (1.1) in a more compact form. The given collection of numbers b_i also defines a vector in $\mathbb{R}^n$

$$\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \dots \\ b_n \end{bmatrix}.$$

How do we interpret the collection of n^2 numbers a_{ij}? Instead of putting them in one column, we put them in a *matrix*. In general, a matrix with n columns and m rows (or an $n \times m$ matrix) is a table of the form

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

where a_{ij} are all numbers, for $i = 1, 2, \dots, m$ and $j = 1, 2, \dots n$. Matrices can be added to each other (term by term) and multiplied by scalars in the same way as vectors. Moreover, gven an n vector $\mathbf{x}$ and an $n \times m$ matrix $\mathbf{A}$ as above, we define an m vector $\mathbf{Ax}$ by multiplying each of the rows of $\mathbf{A}$ by the column $\mathbf{x}$:

$$\mathbf{Ax} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \\ \ldots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \end{bmatrix}.$$

So an n vector can be multiplied by an $n \times m$ matrix and the result is an m vector. This multiplication has some nice properties. For two matrices $\mathbf{A}$ and $\mathbf{B}$ and for two vectors $\mathbf{x}$ and $\mathbf{y}$ of the right dimension, we have

$$(\mathbf{A} + \mathbf{B})\mathbf{x} = \mathbf{Ax} + \mathbf{Bx} \quad \text{and} \quad \mathbf{A}(\mathbf{x} + \mathbf{y}) = \mathbf{Ax} + \mathbf{Ay}.$$

To summarize: we defined a notion of an n vector and an $n \times m$ matrix. Vectors of the same dimension can be added to each other and multiplied by scalars. Matrices of the same dimension can be added to each other and multiplied by scalar. An n vector can be multiplied by an $n \times m$ matrix and the result is an m vector.

Example 2. Given a 2×3 matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 3 & 1 \\ 0 & -2 \end{bmatrix}$$

and a vector $\mathbf{x} \in \mathbb{R}^2$

$$\mathbf{x} = \begin{bmatrix} -1 \\ 5 \end{bmatrix}$$

the product $\mathbf{Ax} \in \mathbb{R}^3$ is

$$\mathbf{Ax} = \begin{bmatrix} 1 & 0 \\ 3 & 1 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} -1 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \cdot (-1) + 0 \cdot 5 \\ 3 \cdot (-1) + 1 \cdot 5 \\ 0 \cdot (-1) + (-2) \cdot 5 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ -10 \end{bmatrix}$$

Returning to linear equations (1.1), we see that our collection of coefficients $(a_i j)$ defines an $n \times n$ matrix $\mathbf{A}$ and equation (1.1) can be written in the matrix form as

$$\mathbf{Ax} = \mathbf{b}.$$

Here $\mathbf{A}$ is a given $n \times n$ matrix, $\mathbf{b}$ is a given n vector and $\mathbf{x}$ is an unknown n vector. By introducing vectors and matrices we significantly simplified (1.1). Now we have to learn how to actually solve it. Before we do so, it is convenient to introduce some notions of linear algebra.

2 VECTOR SPACES

The fact that we can always add two vectors and get a vector, or multiply a vector by a number, also called a *scalar*, and get a vector is the starting point of linear algebra. Observe that the same was true for solutions of linear differential equations (for example, second order equations that we studied extensively). A sum of two solutions was also a solution. A solution multiplied by a constant was also a solution.

This leads to the notion of a *vector space*. A vector space is a set V consisting of some elements for which we have an operation of addition $+$ and multiplication by scalars $\cdot$. That is, given elements $\mathbf{x}, \mathbf{y} \in V$ and a number λ we have elements $\mathbf{x} + \mathbf{y} \in V$ and $\lambda \cdot \mathbf{x} \in V$ (we will often drop $\cdot$

from the notation and write simply $\lambda\mathbf{x}$). We require that these operations satisfy the usual properties, for example addition is commutative

$$\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$$

and associative

$$(\mathbf{x} + \mathbf{y}) + \mathbf{z} = \mathbf{x} + (\mathbf{y} + \mathbf{z}),$$

and similarly multiplication by scalars. We also require that V contains a distinguished element 0 such that

$$\mathbf{x} + 0 = \mathbf{x}$$

and

$$\mathbf{x} + (-1) \cdot \mathbf{x} = 0.$$

As with numbers, we will abbreviate this to $\mathbf{x} - \mathbf{x} = 0$. There are also other conditions that the operations $+$ and $\cdot$ have to satisfy – in essence, they should behave exactly like operations $+$ and $\cdot$ defined for $\mathbb{R}^n$. You can find the full list in section 3.2 of Braun's textbook.

An important thing to keep in mind that V does not have to consist of vectors in $\mathbb{R}^n$. It can be a collection of arbitrary elements, and the operations $+$ and $\cdot$ can be defined arbitrarily and don't have to do anything with the standard addition and multiplication. (Although in this course we will mostly deal with operations $+$ and $\cdot$ that are actually given by the standard operations.)

To summarize, for a set V to be a vector space it has to have

1. an operation $+$ which takes two elements $\mathbf{x}$ and $\mathbf{y}$ in V and produces a new element $\mathbf{x} + \mathbf{y}$ in V,
2. an operation $\cdot$ which takes a number λ and an element $\mathbf{x}$ in V and produces a new element $\lambda \cdot \mathbf{x}$ in V,
3. an element 0 in V,

such that all the usual properties of addition and multiplication by scalars are satisfied.

Remark 3. Given a vector space V we will often refer to its elements as vectors. From the context it will be usually clear whether we talk about usual vectors in $\mathbb{R}^n$ or elements of some other vector space.

Example 4. $V = \mathbb{R}^n$ with the standard addition of vectors and multiplication by numbers is a vector space. Similarly, if V is the set of all $n \times m$ matrices, the operation of addition of matrices and multiplication by numbers make V into a vector space.

Given a matrix $\mathbf{A}$, the set of all $x \in \mathbb{R}^n$ satisfying the equation $\mathbf{A}\mathbf{x} = 0$ is also a vector spaces. Indeed, $\mathbf{A}0 = 0$, and if $\mathbf{A}\mathbf{x} = 0$ and $\mathbf{A}\mathbf{y} = 0$, then $\mathbf{A}(\mathbf{x} + \mathbf{y}) = \mathbf{A}\mathbf{x} + \mathbf{A}\mathbf{y} = 0$. Moreover, $\mathbf{A}\lambda\mathbf{x} = \lambda\mathbf{A}\mathbf{x} = \lambda \cdot 0 = 0$.

Example 5. If V is the set of all continuous functions on an interval $[a, b]$, we can define the sum of two functions $f + g$ and multiplication of a function by a number λf in the obvious way. Observe that if f and g are continuous, then so is $f + g$ so by taking sums we don't leave the class of continuous functions. Similarly, λf is continuous. Finally, the constant function 0 is continuous, so 0 is in V. We conclude that the set of all continuous functions on an interval is a vector space. Similarly, the set of differentiable, twice differentiable, etc. functions is a vector space.

Example 6. Let V be the set of solutions to a linear homogenous differential equation

$$y'' + p(t)y' + q(t)y = 0.$$

The constant function 0 is a solution. Moreover, a linear combination of two solutions is also a solution. Therefore, V with the operations of adding two functions and multiplying them by scalars is a vector space.

It is also instructive to see some examples of sets which do not form a vector space in a natural way.

Example 7. Let V be the set of vectors $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$ whose first coordinate satisfies $x_1 \geq 0$. We can ask if this set, with the standard operations of addition and multiplication by scalars, forms a vector space. It does contain the vector $0 = (0, \ldots, 0)$. Moreover, if $\mathbf{x}$ and $\mathbf{y}$ are in V, then so is $\mathbf{x} + \mathbf{y}$ because the first coordinate of $\mathbf{x} + \mathbf{y}$ is $x_1 + y_1 \geq 0$. However, if $\mathbf{x}$ is in V and satisfies $x_1 > 0$, then $\lambda \mathbf{x}$ is not in V for any $\lambda < 0$. So a multiplication of an element of V by a negative scalar does produce an element in V and V is not a vector space.

Example 8. Let V be the set of all continuous functions on the interval $[a, b]$ such that $f(a) = 1$. The standard operations of addition and multiplication by scalars do not make V into a vector space. First of all, the zero function is not in V. Moreover, the sum of two functions in V is not in V because, if f and g are two functions in V, so that $f(a) = 1$ and $g(b) = 1$, then $(f + g)(a) = 2$. Similarly V is not preserved by multiplication by scalars.

Example 9. Let V be the set of solutions of the non-linear differential equation

$$y' = y^2.$$

Is V a vector space, with the standard operations of addition and multiplication by scalars? The function $y(t) = 0$ is a solution. However, since the equation is nonlinear, the sum of two solutions or a solution multiplied by a constant are no longer solutions, in general. For example, by separation of variables we find that the function $y(t) = -t^{-1}$ is a solution, but it is easy to check that $\lambda y(t)$ is not a solution for any $\lambda \neq 0$.

Similarly, the set of solutions of a linear nonhomogenous equation such as

$$y'' + p(t)y' + q(t)y = r(t)$$

with $r(t) \neq 0$, is not a vector space in a natural way.

3 LINEAR INDEPENDENCE

Let V be a vector space. We say that a collection of vectors $v_1, \ldots, v_n$ in V is *linearly dependent* if there are numbers $\lambda_1, \ldots, \lambda_n$, not all equal to zero, such that

$$\lambda_1 v_1 + \lambda_2 v_2 + \ldots + \lambda_n v_n = 0. \tag{3.1}$$

(Of course, for any collection of vectors this sum is zero if we take $\lambda_1 = \ldots = \lambda_n = 0$, that's why in the definition of linear dependence we add the condition that not all λ_i are zero.) A collection of vectors is linearly dependent if and only if one of them can be expressed as a linear combination of the others. For example, if $\lambda_1 \neq 0$, then we can divide the equation by λ_1 and express v_1 as the linear combination of $v_2, \ldots, v_n$:

$$v_1 = -\frac{\lambda_2}{\lambda_1} v_2 - \ldots - \frac{\lambda_n}{\lambda_1} v_n.$$

If there exist no numbers $\lambda_1, \ldots, \lambda_n$ other than $\lambda_1 = \ldots = \lambda_n$ such that the sum (3.1) is zero, we say that the collection of vectors $v_1, \ldots, v_n$ is *linearly independent*.

A *basis* of a vector space is a linearly independent collection of vectors $v_1, \ldots, v_n$ with the property that any other vector in V is a linear combination of $v_1, \ldots, v_n$, that is for every $v \in V$ there are numbers $\lambda_1, \ldots, \lambda_n$ such that

$$v = \lambda_1 v_1 + \ldots + \lambda_n v_n.$$

Observe that such numbers $\lambda_1, \ldots, \lambda_n$ are unique. Indeed, if we can write v as a linear combination of $v_1, \ldots, v_n$ with any other coefficients

$$v = \mu_1 v_1 + \ldots + \mu_n v_n$$

then

$$0 = v - v = (\lambda_1 - \mu_1)v_1 + \ldots + (\lambda_n - \mu_n)v_n$$

which, since $v_1, \ldots, v_n$ are linearly independent, implies that

$$\lambda_1 = \mu_1, \quad \ldots, \quad \lambda_n = \mu_n.$$

It can be shown that any vector space has a basis and, moreover, that any two bases consist of the same numbers of vectors; in the notation above this is the number n. The number of vectors in the basis is called the *dimension* of a vector space. Since any vector of V is a linear combination of $v_1, \ldots, v_n$ in a unique way, it follows that vectors of V correspond to collections of n numbers, i.e. they can be identified with the usual vectors in $\mathbb{R}^n$:

$$v = \lambda_1 v_1 + \ldots + \lambda_n v_n \quad \leftrightarrow \quad \begin{bmatrix} \lambda_1 \\ \ldots \\ \lambda_n \end{bmatrix}.$$

An important point, however, is that this identification between vectors in V and vectors in $\mathbb{R}^n$ is not unique. It depends on the choice of basis, and there are infinitely many such choices.

Example 10. The vectors

$$e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

are linearly independent and form a basis of $\mathbb{R}^2$. Indeed, any other vector in $\mathbb{R}^2$ can be written in a unique way as a linear combination of e_1 and e_2:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 e_1 + x_2 e_2.$$

Therefore, $\mathbb{R}^2$ is a vector space of dimension two because it has a basis consisting of two vectors. However, we could choose another basis of $\mathbb{R}^2$, consisting, for example, of vectors

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{and} \quad e_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Example 11. In general, for $i = 1, 2, \ldots, n$, let e_i be the vector whose i-th coordinate is equal to 1 and all other coordinates are 0. Then the collection of n vectors $e_1, \ldots, e_n$ is a basis of $\mathbb{R}^n$. Therefore, $\mathbb{R}^n$ is a vector space of dimension n, as expected.

Example 12. The space of solutions of the harmonic oscillator equation

$$y'' + y = 0$$

has a basis consisting of functions

$$y_1(t) = \cos t \quad \text{and} \quad y_2(t) = \sin t.$$

Therefore, the space of solution has dimension two. Of course, there are other choices of bases, for example, the functions

$$y_1(t) = \cos t - \sin t \quad \text{and} \quad y_2(t) = 2 \sin t$$

also form a basis of the space of solutions.

ORDINARY DIFFERENTIAL EQUATIONS (MATH 2030)

Lectures 18 and 19: Review of linear algebra, part 2

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REFERENCES

Boyce–DiPrima sections 7.2, 7.3; Braun sections 3.2, 3.3, 3.5, 3.6, 3.7.

1 LINEAR TRANSFORMATIONS

Let V and W be vector spaces. A map $L\colon V \to W$ (read: a map L from V to W) is a way of associating to every vector v of V a vector $L(v)$ in W. We say that such L is a *linear map* if it satisfies the following conditions:

1. $L(v+w) = L(v) + L(w)$ for all $v, w \in V$, and
2. $L(\lambda v) = \lambda L(v)$ for all $v \in V$ and $\lambda \in \mathbb{R}$.

Observe that the third condition, for $\lambda = 0$ implies that $L(0) = 0$, that is: the zero vector in V is mapped to the zero vector in W.

Remark 1. Linear maps are also called *linear operators* or *linear transformations*. We will use all of these names. If L is a linear transformation, it is customary to write Lv instead of $L(v)$.

Example 2. For every vector space V the *identity map* $I\colon V \to V$:

$$I(v) = v \quad \text{for all } v \in V$$

is obviously a linear map. More generally, for any $\lambda \in \mathbb{R}$ we can consider the map λI given by

$$\lambda I(v) = \lambda v.$$

Example 3. Let $V = W = \mathbb{R}$ be the vector space of real numbers. The map

$$L(x) = 5x$$

is a linear map. The map

$$K(x) = x^2$$

is not a linear map. Indeed, we have $K(x+y) = (x+y)^2$ and $K(x) + K(y) = x^2 + y^2$, so in general $K(x+y) \neq K(x) + K(y)$.

Example 4. The map $L\colon \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$L(\mathbf{x}) = \begin{bmatrix} x_1 - x_2 \\ x_1 + x_2 \end{bmatrix} \quad \text{for } \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

is linear. Observe that this map can be written using matrix multiplication as

$$L(\mathbf{x}) = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \mathbf{x}.$$

Example 5 (Linear maps given by matrices)**.** In general, an $m \times n$ (m rows, n columns) matrix $\mathbf{A}$ gives rise to a linear map $L\colon \mathbb{R}^n \to \mathbb{R}^m$ defined by

$$L(\mathbf{x}) = \mathbf{A}\mathbf{x} \quad \text{for } \mathbf{x} \in \mathbb{R}^n.$$

A vector $\mathbf{x} \in \mathbb{R}^n$ is a column of size n, so we can indeed multiply it by an $m \times n$ matrix and a result is a column of size m, i.e. a vector in $\mathbb{R}^m$. As we said in the last lecture, for two vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ we have

$$\mathbf{A}(\mathbf{x} + \mathbf{y}) = \mathbf{A}\mathbf{x} + \mathbf{A}\mathbf{y}$$

and for any $\lambda \in \mathbb{R}$ we have

$$\mathbf{A}(\lambda\mathbf{x}) = \lambda(\mathbf{A}\mathbf{x})$$

so L is indeed a linear map.

Example 6 (Differentiation)**.** Let V be the vector space of polynomials of degree at most n, for some positive integer n. Define a map

$$L\colon V \to V$$

by

$$L(f) = \frac{\mathrm{d}f}{\mathrm{d}x}$$

for a polynomial $f(x)$ of degree at most n. First, observe that this is indeed a map from V to V, that is if f is a polynomial of degree at most n, then $L(f)$ is also a polynomial of degree at most n. Indeed, if

$$f(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n$$

then

$$L(f) = \frac{\mathrm{d}f}{\mathrm{d}x} = a_1 + 2a_2 x + \ldots + n a_n x^{n-1}$$

so $L(f)$ is a polynomial of degree at most $n - 1$; in particular, of degree smaller than n, so $L(f)$ is also a vector in V. Let us verify that the map $L\colon V \to V$ is linear. For two polynomials f and g we have

$$L(f+g) = \frac{\mathrm{d}(f+g)}{\mathrm{d}x} = \frac{\mathrm{d}f}{\mathrm{d}x} + \frac{\mathrm{d}g}{\mathrm{d}x} = L(f) + L(g).$$

For any constant $\lambda \in \mathbb{R}$ we have

$$L(\lambda f) = \frac{\mathrm{d}(\lambda f)}{\mathrm{d}x} = \lambda\frac{\mathrm{d}f}{\mathrm{d}x} = \lambda L(f),$$

so indeed L is a linear map.

Given three vector spaces U, V, and W and linear maps

$$L\colon U \to V, \quad K\colon V \to W$$

we define their composition

$$KL\colon U \to W$$

by

$$KL(v) = K(L(v)) \quad \text{for } v \in U.$$

Observe that this formula makes sense: for $v \in U$, $L(v)$ is a vector in V and therefore we can apply K to it. The result $K(L(v))$ is a vector in W, so in the end KL defines a map from U to W. An important point to keep in mind that in the composition KL you first apply L and then K. It is easy to check that if both L and K are linear maps, then their composition KL is also a linear map.

Example 7. Let $U = V = W = \mathbb{R}$ and let

$$L(x) = 5x \quad K(x) = 2x.$$

Then

$$KL(x) = K(L(x)) = K(5x) = 2 \times 5x = 10x.$$

Example 8 (Position and momentum operators)**.** Let V be the vector space of functions f of variable x which have derivatives of any order. Define $L\colon V \to V$ by

$$L(f) = \frac{\mathrm{d}f}{\mathrm{d}x}.$$

(Since we assume here that f has derivatives of any order, so does $L(f)$, and so $L(f)$ is also an element of V.) As in Example 6, we verify easily that L is a linear map. Define another map $K\colon V \to V$ by

$$K(f) = xf$$

Since the function x has derivatives of any order, so does xf if f does, so indeed the function xf is an element of V. We easily verify that K is a linear map. To see a concrete example, for $f(x) = x^2 - 1$, we have

$$L(f) = 2x, \quad K(f) = x(x^2 - 1) = x^3 - x.$$

Let us compute the composition KL. For a function f,

$$KL(f) = K(L(f)) = K(\frac{\mathrm{d}f}{\mathrm{d}x}) = x\frac{\mathrm{d}f}{\mathrm{d}x}.$$

Since both K and L are maps from V to V, we can define also the composition LK, that is: first apply K and then L. Let us see that in this example KL and LK are actually different linear maps. Using the product rule, we compute

$$LK(f) = L(K(f)) = L(xf) = \frac{\mathrm{d}}{\mathrm{d}x}(xf) = f + x\frac{\mathrm{d}f}{\mathrm{d}x}.$$

So the difference between LK and KL (which is also called the *commutator* of L and K) is nonzero:

$$LK(f) - KL(f) = f.$$

We see that not only $LK \neq KL$ but their difference is, in fact, the identity map. In quantum mechanics, the linear maps K and L are known as the *position* and *momentum operators*. The fact that $LK \neq KL$ is a mathematical description of Heisenberg's uncertainty principle.

2 MATRIX REPRESENTATION

We have seen in Example 5 that matrices give rise to linear maps. In fact, every linear map can be represented by a matrix. Let V and W be vector spaces of dimension n and m and let $L\colon V \to W$ be a linear map.

Suppose we have chosen bases $v_1, \ldots v_n$ of V and $w_1, \ldots, w_m$ of W. Every vector in V is of the form

$$v = \lambda_1 v_1 + \ldots + \lambda_n v_n$$

for $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$. Let us compute $L(v)$. Since L is a linear map:

$$L(v) = L(\lambda_1 v_1 + \ldots + \lambda_n v_n) = \lambda_1 L(v_1) + \ldots + \lambda_n L(v_n),$$

so in order to compute $L(v)$ for any vector v is suffices to know $L(v_i)$ for every i. Since $L(v_i)$ is a vector in W and every vector in W is a linear combination of the elements of the basis, there are numbers a_{ij} such that

$$\begin{aligned} L(v_1) &= a_{11}w_1 + a_{21}w_2 + \ldots + a_{m1}w_m, \\ L(v_2) &= a_{12}w_1 + a_{22}w_2 + \ldots + a_{m2}w_m, \\ \ldots \\ L(v_n) &= a_{1n}w_1 + a_{2n}w_2 + \ldots + a_{mn}. \end{aligned}$$

Now we can write $L(v)$ in the basis $w_1, \ldots, w_m$ as

$$L(v) = \sum_{i=1}^{n} \lambda_i L(v_i) = \sum_{i=1}^{n} \sum_{j=1}^{n} \lambda_i a_{ji} w_j.$$

This looks similar for the formula for multiplying vectors by matrices. Indeed, introduce the $m \times n$ matrix

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{m1} & a_{m2} & \ldots & a_{mn}. \end{bmatrix}$$

that is: the i–th column contains the numbers appearing when we write $L(v_i)$ in the basis $w_1, \ldots, w_m$. Now a vector in V given by

$$v = \lambda_1 v_1 + \ldots + \lambda_n v_n$$

can be identified with a column of n numbers

$$v = \begin{bmatrix} \lambda_1 \\ \ldots \\ \lambda_n \end{bmatrix}.$$

Similarly, a vector in W given by

$$w = \mu_1 w_1 + \ldots + \mu_m w_m$$

can be identified with a column of m numbers. Under these identifications, the linear map L is simply given by

$$L(v) = \mathbf{A}v$$

where the right-hand side is the multiplication of a column of n numbers by an $m \times n$ matrix. The result is a column of m numbers, which defines a vector in W. This is exactly the situation from Example 5.

The summary of this discussion is: if we choose a basis of V, every vector in V can be identified with a column of n numbers, i.e. a vector in $\mathbb{R}^n$. Similarly, if we choose a basis of W, every vector in W can be identified with a vector in $\mathbb{R}^m$. With respect to these identifications, every linear map $L\colon V \to W$ can be represented by an $m \times n$ matrix and to compute $L(v)$ for any $v \in V$ we simply perform multiplication of the column by the matrix. It is important to keep in mind that the matrix representing the map L depends on the choice of bases of V and W. Different choices of bases will yield different matrices representing the same linear map.

Example 9 (Matrix multiplication)**.** Let $\mathbf{A} = (a_{ij})$ be an $m \times n$ matrix and let $\mathbf{B} = (b_{ij})$ be an $n \times k$ matrix, so that $\mathbf{A}$ defines a linear map $\mathbf{A}\colon \mathbb{R}^n \to \mathbb{R}^m$ and $\mathbf{B}$ defines a linear map $\mathbf{B}\colon \mathbb{R}^k \to \mathbb{R}^n$ as in Example 5. The matrix of the composition $\mathbf{AB}\colon \mathbb{R}^k \to \mathbb{R}^n$ is given by the *product matrix*, that is an $n \times k$ matrix $\mathbf{AB} = (c_{ij})$ whose entry in the i-th row and j-th column is

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \ldots + a_{in}b_{nj}.$$

For example, for

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix}$$

we have

$$\mathbf{AB} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 \times 2 + 2 \times 1 & 1 \times 0 + 2 \times 2 \\ 3 \times 2 + 4 \times 1 & 3 \times 0 + 4 \times 2 \end{bmatrix} = \begin{bmatrix} 4 & 4 \\ 10 & 8 \end{bmatrix}.$$

Example 10 (Rotations)**.** For an angle θ, let $L_\theta\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the counter-clockwise rotation of the plane around the point $(0, 0)$ by θ. This is a linear map. Let us compute its matrix representation in the standard basis

$$e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Using elementary trigonometry, we compute that

$$\begin{aligned} L_\theta e_1 &= \cos\theta e_1 + \sin\theta e_2, \\ L_\theta e_2 &= -\sin\theta e_1 + \cos\theta e_2. \end{aligned}$$

Therefore, the matrix of L_θ is

$$L_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

It is geometrically clear that for two angles θ and φ, the composition of the rotation by φ with the rotation by θ is the rotation by $\theta + \varphi$, that is: $L_\theta L_\varphi = L_{\theta+\varphi}$. As an exercise, let us verify that this is what we get by multiplying the matrices of these linear maps:

$$\begin{aligned} L_\theta L_\varphi &= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix} \\ &= \begin{bmatrix} \cos\theta\sin\varphi - \sin\theta\sin\varphi & -\cos\theta\sin\varphi - \sin\theta\sin\varphi \\ \sin\theta\cos\varphi + \cos\theta\sin\varphi & -\sin\theta\sin\varphi + \cos\theta\cos\varphi \end{bmatrix} \\ &= \begin{bmatrix} \cos(\theta+\varphi) & -\sin(\theta+\varphi) \\ \sin(\theta+\varphi) & \cos(\theta+\varphi) \end{bmatrix} = L_{\theta+\varphi}. \end{aligned}$$

Example 11 (Differentiation)**.** Let V be the vector space of polynomials of degree at most n and let $L = \frac{\mathrm{d}}{\mathrm{d}x}$ be the differentiation map defined in Example 6. Let us find the matrix of L in the basis of V given by the polynomials

$$e_0 = 1,\ e_1 = x,\ e_2 = x^2,\ \ldots,\ e_n = x^n.$$

We have

$$\begin{aligned}
Le_0 &= \frac{\mathrm{d}}{\mathrm{d}x}(1) = 0, \\
Le_1 &= \frac{\mathrm{d}}{\mathrm{d}x}(x) = 1 = e_0, \\
Le_2 &= \frac{\mathrm{d}}{\mathrm{d}x}(x^2) = 2x = 2e_1, \\
&\dots, \\
Le_n &= \frac{\mathrm{d}}{\mathrm{d}x}(x^n) = nx^{n-1} = ne_{n-1}.
\end{aligned}$$

The matrix of L is

$$L = \begin{bmatrix} 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 2 & 0 & \dots & 0 \\ 0 & 0 & 0 & 3 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & 0 & n \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

3 INVERTIBLE TRANSFORMATIONS; DETERMINANT

Let V be a vector space and let $L\colon V \to V$ be a linear map. We say that L is *invertible* if there is a linear map $K\colon V \to V$ such that

$$LK = KL = I$$

where I denotes the identity map: $Iv = v$ for every $v \in V$. The map K is called the *inverse* of L and denoted

$$K = L^{-1}.$$

Remark 12. More generally, we could say that a linear map between two vector spaces $L\colon V \to W$ is invertible if there is a linear map $K\colon W \to V$ such that the above relation holds. In this lecture, we will mostly consider linear maps between the same vector space.

Example 13. The rotation $L_\theta\colon \mathbb{R}^2 \to \mathbb{R}^2$ defined in Example 10 is invertible. Indeed, we have

$$L_\theta L_{-\theta} = L_{-\theta} L_\theta = L_{\theta-\theta} = L_0 = I,$$

since the rotation by angle 0 is the identity map. Therefore, $L_{-\theta}$ is the inverse of L_θ. Geometrically, this is clear: $L_{-\theta}$ is the clockwise rotation by angle θ, which, of course, is the inverse of the counterclockwise rotation L_θ by the same angle.

The notion of invertibility is related to the problem of solving linear equations. Suppose $L\colon V \to V$ is a linear map and $w \in V$. Suppose we want to find $v \in V$ satisfying the equation

$$Lv = w. \tag{3.1}$$

For example, if $V = \mathbb{R}^n$, then L is represented by an $n \times n$ matrix $L = (a_{ij})$ and the equation $Lv = w$ is simply a system of n equations for n unknown numbers $v_1, \ldots, v_n$

$$\begin{aligned} a_{11}v_1 + \ldots + a_{1n}v_n &= w_1, \\ a_{21}v_1 + \ldots + a_{2n}v_n &= w_2, \\ \ldots & \\ a_{n1}v_1 + \ldots + a_{nn}v_n &= w_n. \end{aligned}$$

If L is invertible, we can solve (3.1) by applying L^{-1} to both sides:

$$v = L^{-1}Lv = L^{-1}w.$$

So $v = L^{-1}w$ is the unique solution. In particular, for $w = 0$ we see that if L is invertible, then the equation

$$Lv = 0$$

has a unique solution $v = L^{-1}0 = 0$.

Example 14. As we have seen, the rotation operator $L_\theta \colon \mathbb{R}^2 \to \mathbb{R}^2$ is invertible. Therefore, the equation for $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$

$$L_\theta \mathbf{x} = 0$$

or equivalently,

$$\begin{cases} \cos\theta x_1 - \sin\theta x_2 = 0, \\ \sin\theta x_1 + \cos\theta x_2 = 0. \end{cases}$$

has only one solution $\mathbf{x} = (0, 0)$.

Example 15. Let V be the vector space of polynomials of degree at most n and let $L = \frac{\mathrm{d}}{\mathrm{d}x}$ be the differentiation map defined in Example 6. The constant polynomial 1 is a nonzero element of V. It satisfies the equation

$$L(1) = \frac{\mathrm{d}}{\mathrm{d}x}(1) = 0.$$

Therefore, we have found a nonzero solution of the equation $Lv = 0$. It follows that L is not invertible.

Given a linear map $L \colon V \to V$, how do we determine if it is invertible? After choosing a basis of V, we can write L as an $n \times n$ matrix where n is the dimension of V, so it is enough to consider this problem for matrices, i.e. linear maps from $\mathbb{R}^n$ to $\mathbb{R}^n$. Let us begin with the case $n = 2$.

Proposition 16. *A 2×2 matrix*

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is invertible only if its determinant

$$\det \mathbf{A} = ab - cd$$

is non-zero. In that case, the inverse of $\mathbf{A}$ *is given by*

$$\mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Proof. One way to find the inverse, is to solve the equation

$$\mathbf{A}\mathbf{x} = \mathbf{y}.$$

Here $\mathbf{y} \in \mathbb{R}^2$ is given, and we want to solve for $\mathbf{x} \in \mathbb{R}^2$. Once we find the solution $\mathbf{x}$, it will be given by

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{y},$$

so the formula for the solution will give us the coefficients of $\mathbf{A}^{-1}$. Explicitly, $\mathbf{A}\mathbf{x} = \mathbf{y}$ is the system

$$\begin{cases} ax_1 + bx_2 = y_1 \\ cx_1 + dx_2 = y_2. \end{cases}$$

In Homework 4 we proved that this system has a unique solution if and only if $ab - cd \neq 0$ and in that case, the solution is given by

$$x_1 = \frac{1}{ab - cd}(dy_1 - by_2)$$
$$x_2 = \frac{1}{ab - cd}(-cy_1 + ay_2),$$

which in the matrix form can be written as

$$\mathbf{x} = \frac{1}{\det \mathbf{A}} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \mathbf{y}.$$

This shows that $\mathbf{A}^{-1}$ is the matrix on the right-hand side. □

The determinant can be defined for an $n \times n$ matrix for any n using the following inductive procedure. We have already defined the determinant of a 2×2 matrix, so let us define it for 3×3 matrices. Let $\mathbf{A} = (a_i j)$ be a 3×3 matrix. Here a_{ij} is the entry in the i-th row and j-th column. Define M_{ij} to be the 2×2 matrix obtained by removing from $\mathbf{A}$ the i-th row and the j-th column. Now we can define $\det \mathbf{A}$ by "expanding the determinant with respect to the first row" by

$$\det \mathbf{A} = a_{11}M_{11} - a_{12}M_{12} + a_{13}M_{13}$$

or, equivalently, by expanding it with respect to the first column

$$\det \mathbf{A} = a_{12}M_{11} - a_{21}M_{21} + a_{31}M_{31}.$$

In fact, you can define the determinant by expanding it with respect to any column or row, as long as you keep track of the signs. The result will not depend on which column or row you chose. (For details see section 3.5 of Braun's textbook; the Wikipedia article on determinants is also well-written and has many examples.)

Example 17. Let us compute

$$\det \begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & 0 \\ 2 & 0 & 1 \end{bmatrix} = 1 \det \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} - 2 \det \begin{bmatrix} 0 & 0 \\ 2 & 1 \end{bmatrix} + 0 \det \begin{bmatrix} 0 & 3 \\ 2 & 0 \end{bmatrix}$$
$$= 1 \times 3 - 2 \times 0 + 0 \times (-6) = 3.$$

Alternatively, we could expand with respect to the first column:

$$\det \begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & 0 \\ 2 & 0 & 1 \end{bmatrix} = 1 \det \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} - 0 \det \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} + 2 \det \begin{bmatrix} 2 & 0 \\ 3 & 0 \end{bmatrix}$$
$$= 1 \times 3 - 0 \times 2 + 2 \times 0 = 3.$$

In the same way we can define the determinant of a 4×4 matrix as a sum of determinant of 3×3 matrices, and so on. This gives us a definition of the determinant of an $n \times n$ matrix for any n.

Theorem 18. *A matrix* $\mathbf{A}$ *is invertible if and only if* $\det \mathbf{A} \neq 0$. *In particular, if* $\det \mathbf{A} \neq 0$, *then for every* $\mathbf{y} \in \mathbb{R}^n$ *the equation*

$$\mathbf{A}\mathbf{x} = \mathbf{y}$$

has a unique solution $\mathbf{x} = \mathbf{A}^{-1}\mathbf{y}$.

Computing determinants directly from the definition can be tedious; it is helpful to use various properties of determinants. These properties, together with examples of how to use them, are listed in section 3.5 of Braun's textbook (available on Canvas).

4 Eigenvalues and eigenvectors

Let $L\colon V \to V$ be a linear map. A nonzero vector $v \in V$ is an *eigenvector* of L with *eigenvalue* $\lambda \in \mathbb{R}$ if

$$Lv = \lambda v.$$

Observe that if v is an eigenvector with eigenvalue λ, then so is the vector rv for any nonzero real number r.

Example 19. Let $\mathbf{A}\colon \mathbb{R}^n \to \mathbb{R}^n$ be the linear map given by the matrix

$$\mathbf{A} = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \cdots & \\ & & & \lambda_n \end{bmatrix}$$

(All other entries are 0. Such a matrix is called *diagonal*.) Let $e_1, \ldots, e_n$ be the standard basis of $\mathbb{R}^n$, i.e. e_i is the column whose i-th entry is 1 and all other entries are 0. We see that

$$\mathbf{A}e_i = \lambda_i e_i,$$

that is: e_i is an eigenvector of $\mathbf{A}$ with eigenvalue λ_i.

Example 20. The rotation $L_\theta\colon \mathbb{R}^2 \to \mathbb{R}^2$ has no eigenvectors unless θ is a multiple of 2π (in which case $L_\theta = I$ is the identity map and every vector is an eigenvector with eigenvalue 1).

We will see that finding eigenvalues and eigenvectors of a linear map is crucial for solving linear systems of differential equations. How do we do this? Observe that if v is an eigenvector with eigenvalue λ, then

$$(L - \lambda I)v = Lv - \lambda v = 0.$$

Since v is assumed to be nonzero, that means that the linear map

$$L - \lambda I$$

is not invertible. (Recall: if a linear map K is invertible, then the equation $Kv = 0$ has only one solution $v = 0$.) Therefore,

$$\det(L - \lambda I) = 0.$$

The expression on the left-hand side is called the *characteristic polynomial* of the linear map L and denoted

$$\chi(\lambda) = \det(L - \lambda I).$$

This is a function of λ. It can be shown that, in fact, χ is a polynomial of degree n, where n is the dimension of the vector space V, that is:

$$\chi(\lambda) = a_0 + a_1\lambda + a_2\lambda^2 + \ldots + a_n\lambda^n$$

for some $a_0, \ldots, a_n \in \mathbb{R}$.

Theorem 21. *A real number λ is an eigenvalue of a linear map L if and only if it is a root of the characteristic polynomial:*

$$\chi(\lambda) = 0.$$

This gives us a way of finding eigenvalues of a linear map. Suppose we have found such an eigenvalue λ. To find an eigenvector v with this eigenvalue, we need to solve the linear equation

$$(L - \lambda I)v = 0.$$

Example 22. Let $\mathbf{A}\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a linear map given by the matrix

$$\mathbf{A} = \begin{bmatrix} -2 & 1 \\ -4 & 3 \end{bmatrix}.$$

Let us find its eigenvalues and eigenvectors. The characteristic polynomial is

$$\chi(\lambda) = \det(\mathbf{A} - \lambda I) = \det \begin{bmatrix} -2-\lambda & 1 \\ -4 & 3-\lambda \end{bmatrix} = (-2-\lambda)(3-\lambda) + 4 = \lambda^2 - \lambda - 2.$$

We find that the roots are

$$\lambda_1 = 2, \quad \lambda_2 = -1.$$

To find an eigenvector v_1 with eigenvalue λ_1, we need to solve

$$(A - 2I)v_1 = 0.$$

Write v_1 as a column with entries (a, b), so that:

$$(A - 2I)v_1 = \begin{bmatrix} -4 & 1 \\ -4 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}.$$

Therefore, a and b satisfy

$$-4a + b = 0,$$

so $b = 4a$ and any eigenvector with eigenvalue 2 is of the form

$$v_1 = a \begin{bmatrix} 1 \\ 4 \end{bmatrix} \quad \text{for } a \in \mathbb{R}.$$

Similarly we find that any eigenvector v_2 with eigenvalue -1 is of the form

$$v_2 = a \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{for } a \in \mathbb{R}.$$

Theorem 23. *Let $L\colon V \to V$ be a linear map. If $v_1, \ldots, v_n$ are eigenvectors of L with distinct eigenvalues $\lambda_1, \ldots, \lambda_n$, then the collection $v_1, \ldots, v_n$ is linearly independent. If, moreover, this collection is a basis of V, then with respect to that basis L is given by the diagonal matrix*

$$L = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ldots & \\ & & & \lambda_n \end{bmatrix}.$$

ORDINARY DIFFERENTIAL EQUATIONS (MATH 2030)

Lecture 20: Systems of first order linear equations

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REFERENCES

Boyce–DiPrima sections 7.4–7.8; Braun sections 3.1, 3.4, 3.8, 3.9, 3.10.

1 LINEAR SYSTEMS

In the next two lectures we will consider systems of first order linear equations

$$\begin{cases} y_1' = a_{11}(t)y_1 + \ldots + a_{1n}(t)y_n + b_1(t), \\ y_2' = a_{21}(t)y_2 + \ldots + a_{2n}(t)y_n + b_2(t), \\ \ldots \\ y_n' = a_{n1}(t)y_1 + \ldots + a_{nn}(t)y_n + b_n(t). \end{cases}$$

Here $y_1(t), \ldots, y_n(t)$ are unknown functions and $a_{ij}(t)$ and $b_i(t)$ are given. We can write this system in the matrix form. Set

$$\mathbf{y}(t) = \begin{bmatrix} y_1(t) \\ \ldots \\ y_n(t) \end{bmatrix}, \quad \mathbf{b}(t) = \begin{bmatrix} b_1(t) \\ \ldots \\ b_n(t) \end{bmatrix}.$$

For each t, $\mathbf{y}(t)$ and $\mathbf{b}(t)$ are vectors in $\mathbb{R}^n$. We say that $\mathbf{y}$ and $\mathbf{b}$ are functions with values in $\mathbb{R}^n$, or vector-valued functions. Similarly, we can define

$$\mathbf{A}(t) = \begin{bmatrix} a_{11}(t) & a_{12}(t) & \ldots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \ldots & a_{2n}(t) \\ \ldots & \ldots & \ldots & \ldots \\ a_{n1}(t) & a_{n2}(t) & \ldots & a_{nn}(t) \end{bmatrix}.$$

We say that $\mathbf{A}$ is a matrix-valued function, since for every t it gives us an $n \times n$ matrix $\mathbf{A}(t)$. Now our linear system can be written as

$$\mathbf{y}' = \mathbf{A}(t)\mathbf{y} + \mathbf{b}(t), \tag{1.1}$$

where $\mathbf{y}$ is an unknown vector-valued function and $\mathbf{A}$ and $\mathbf{b}$ are given functions (matrix-valued and vector-valued, respectively). The derivative $\mathbf{y}'$ is defined by taking derivatives of each entry, that is $\mathbf{y}'$ is a vector-valued function given by

$$\mathbf{y}'(t) = \begin{bmatrix} y_1'(t) \\ \ldots \\ y_n'(t) \end{bmatrix}.$$

We say that the linear system (1.1) is *homogenous* if $\mathbf{b}(t) = 0$. We say that it has *constant coefficients* if the matrix-valued function $\mathbf{A}$ is constant i.e. each function a_{ij} does not depend on t.

As for second order linear equations, solving a general linear system (1.1) can be reduced to solving the corresponding homogenous linear system

$$\mathbf{y}' = \mathbf{A}(t)\mathbf{y}, \tag{2.1}$$

provided we happen to know any solution of the nonhomogenous system.

Theorem 1. *If* $\mathbf{x}$ *is a particular solution of the nonhomogenous linear system* (1.1)*, then any other solution of* (1.1) *has the form*

$$\mathbf{y} = \mathbf{x} + \mathbf{z}$$

where $\mathbf{z}$ *is a solution of the corresponding homogenous system* (2.1)*.*

Proof. Let $\mathbf{y}$ be any solution of (1.1). Define $\mathbf{z} = \mathbf{y} - \mathbf{x}$. Since both $\mathbf{x}$ and $\mathbf{y}$ satisfy (1.1), we have

$$\mathbf{z}' = (\mathbf{y} - \mathbf{x})' = \mathbf{y}' - \mathbf{x}' = \mathbf{A}(t)\mathbf{y} + \mathbf{b}(t) - \mathbf{A}(t)\mathbf{x} - \mathbf{b}(t) = \mathbf{A}(t)(\mathbf{y} - \mathbf{x}) = \mathbf{A}(t)\mathbf{z},$$

so $\mathbf{z}$ is a solution of the homogenous system (2.1) and

$$\mathbf{y} = \mathbf{x} + \mathbf{z}. \qquad \square$$

The conclusion is: to find the general solution of a nonhomogenous linear system, we need to know a particular solution and the general solution of the corresponding homogenous system. Therefore, from now on, we will focus on homogenous linear systems. Recall that for second order linear equations, the general solution was a linear combination of two linearly independent solutions. The same thing happens for linear systems. (As we discussed in Lecture 16, a second order equation for one function can be written as a system of first order linear equations for two functions, so the theory we developed for second order linear equations is a special case of the general theory for systems.)

Theorem 2. *Let* $\mathbf{A}$ *be an* $n \times n$ *matrix-valued function.*

1. *If* $\mathbf{y}^1, \dots, \mathbf{y}^k$ *are solutions of the homogenous linear system* (2.1)*, then* $\mathbf{y}^1, \dots, \mathbf{y}^k$ *are linearly independent as functions if and only if for every* t *the vectors* $\mathbf{y}^1(t), \dots, \mathbf{y}^k(t)$ *are linearly independent in* $\mathbb{R}^n$*.*

2. *The set of all functions satisfying* (2.1) *is a vector space of dimension* n*.*

Remark 3. Before we prove the theorem, observe that it is not always true that if functions $\mathbf{y}^1, \dots, \mathbf{y}^k$ are linearly independent, then their values $\mathbf{y}^1(t), \dots, \mathbf{y}^k(t)$ are linearly independent for all t. For example, the vector-valued functions

$$\mathbf{y}^1(t) = \begin{bmatrix} 1 \\ t \end{bmatrix}, \quad \mathbf{y}^2(t) = \begin{bmatrix} 1 \\ t^2 \end{bmatrix}$$

are linearly independent, that is: there is no nontrivial linear combination $C_1\mathbf{y}^1 + C_2\mathbf{y}^2$ which is zero for all values of t. However, for $t = 0$, the vectors

$$\mathbf{y}^1(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \mathbf{y}^2(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

are linearly dependent in $\mathbb{R}^2$. The theorem tells us that this cannot happen if $\mathbf{y}^1, \dots, \mathbf{y}^k$ are all solutions of the same homogenous linear system.

Proof. Let us prove the first part of the theorem. Suppose that $\mathbf{y}^1, \ldots, \mathbf{y}^k$ are linearly dependent, that is there are numbers $C_1, \ldots, C_k$, not all equal to zero, such that

$$C_1\mathbf{y}^1(t) + \ldots + C_k\mathbf{y}^k(t) = 0$$

for all t. Since this equality holds for all t, it follows that for every t the vectors $\mathbf{y}^1(t), \ldots, \mathbf{y}^k(t)$ are linearly dependent in $\mathbb{R}^n$.

Conversely, suppose that $\mathbf{y}^1, \ldots, \mathbf{y}^k$ are linearly independent as functions. Our goal it to show that for every t the vectors $\mathbf{y}^1(t), \ldots, \mathbf{y}^k(t)$ are linearly independent in $\mathbb{R}^n$. Fix any t_* and suppose that there are numbers $C_1, \ldots, C_k$ such that

$$C_1\mathbf{y}^1(t_*) + \ldots + C_k\mathbf{y}^k(t_*) = 0$$

Consider the function

$$\mathbf{y} = C_1\mathbf{y}^1 + \ldots + C_k\mathbf{y}^k.$$

Since the equation $\mathbf{y}' = \mathbf{A}(t)\mathbf{y}$ is linear, and $\mathbf{y}$ is a linear combination of solutions, it is itself a solution of this differential equation. Its value at $t = t_*$ is

$$\mathbf{y}(t_*) = C_1\mathbf{y}^1(t_*) + \ldots + C_k\mathbf{y}^k(t_*) = 0.$$

However, there is another solution of the equation $\mathbf{y}' = \mathbf{A}(t)\mathbf{y}$ whose value at $t = t_*$ is zero: the constant function equal to zero! By the existence and uniqueness theorem, any two solutions which have the same value at any time, must be equal for all times. We conclude that $\mathbf{y}(t) = 0$ for all t. But that means that

$$C_1\mathbf{y}^1(t) + \ldots + C_k\mathbf{y}^k(t) = 0$$

for all t. Since $\mathbf{y}^1, \ldots, \mathbf{y}^k$ are linearly independent as functions, it follows that

$$C_1 = C_2 = \ldots = C_k = 0.$$

Therefore, the only linear combination of the vectors $\mathbf{y}^1(t_*), \ldots, \mathbf{y}^k(t_*)$ which is zero is the trivial combination. That shows that these vectors are linearly independent in $\mathbb{R}^n$.

Let us prove the second part of the theorem. Since the differential equation $\mathbf{y}' = \mathbf{A}(t)\mathbf{y}$ is linear, a linear combination of solutions is again a solution. This shows that the set of solutions is a vector space. To compute its dimension, we need to find a basis of this vector space. Let

$$\mathbf{e}_1, \ldots, \mathbf{e}_n$$

be any basis of $\mathbb{R}^n$ (for example, the standard basis). Let $\mathbf{y}^1, \ldots, \mathbf{y}^n$ be the solutions of $\mathbf{y}' = \mathbf{A}(t)\mathbf{y}$ satisfying the initial condition

$$\mathbf{y}^k(0) = \mathbf{e}_k.$$

Such solutions exist by the existence and uniqueness theorem. By the first part of the theorem, the functions $\mathbf{y}^1, \ldots, \mathbf{y}^n$ are linearly independent since the vectors $\mathbf{y}^1(0), \ldots, \mathbf{y}^n(0)$ are. To show that $\mathbf{y}^1, \ldots, \mathbf{y}^n$ form a basis of the space of solution, it remains to show that any other solution is their linear combination. Let $\mathbf{y}$ be any solution. Since $\mathbf{e}_1, \ldots, \mathbf{e}_n$ is a basis of $\mathbb{R}^n$, the vector $\mathbf{y}(0)$ is their linear combination, that is: there are $C_1, \ldots, C_n$ such that

$$\mathbf{y}(0) = C_1\mathbf{e}_1 + \ldots + C_n\mathbf{e}_n.$$

Now consider the function

$$\mathbf{x} = C_1\mathbf{y}^1(t) + \ldots + C_n\mathbf{y}^n.$$

Since $\mathbf{x}$ is a linear combination of solutions, it is itself a solution of our differential equation. At $t = 0$ it satisfies

$$\mathbf{x}(0) = C_1\mathbf{y}^1(0) + \ldots + C_n\mathbf{y}^n(0) = C_1\mathbf{e}_1 + \ldots + C_n\mathbf{e}_n = \mathbf{y}(0).$$

Therefore, the solutions $\mathbf{x}$ and $\mathbf{y}$ have the same value at $t = 0$. By the uniqueness part of the existence and uniqueness theorem, they must be equal, that is $\mathbf{x}(t) = \mathbf{y}(t)$ for all t. This shows that $\mathbf{y}$ is a linear combination of the solutions $\mathbf{y}^1, \ldots, \mathbf{y}^n$ (since, by construction, $\mathbf{x}$ is). Therefore, $\mathbf{y}^1, \ldots, \mathbf{y}^n$ is a linearly independent collection of solutions such that every other solution is their linear combination, i.e. they form a basis of our space of solutions. □

3 HOMOGENOUS LINEAR SYSTEMS WITH CONSTANT COEFFICIENTS

Theorem 2 tells us that in order to find the general solution of a homogenous linear system given by an $n \times n$ matrix function, we need to find n linearly independent solutions. We will now discuss a method of finding such solutions for homogenous linear systems with constant coefficients:

$$\mathbf{y}' = \mathbf{A}\mathbf{y}. \tag{3.1}$$

Here $\mathbf{A}$ is a constant matrix. If you remember from the lectures about second order linear equation, the exponential function plays a very important role in the theory of differential equations. The simplest example of a linear differential equation is

$$y' = \lambda y,$$

where y is a single function and $\lambda \in \mathbb{R}$. This equation has a solution $y(t) = e^{\lambda t}$. Similarly, for n-dimensional linear systems (3.1) we look for solutions of the form

$$\mathbf{y}(t) = e^{\lambda t}\mathbf{v}$$

for some constant λ and a vector $\mathbf{v} \in \mathbb{R}^n$.

Theorem 4. *A function $\mathbf{y}(t) = e^{\lambda t}\mathbf{v}$ is a solution of the linear system (3.1) if and only if $\mathbf{v}$ is an eigenvector of the matrix $\mathbf{A}$ with eigenvalue λ, that is $\mathbf{A}\mathbf{v} = \lambda\mathbf{v}$.*

Proof. This is a straightforward computation. Let $\mathbf{y}(t) = e^{\lambda t}\mathbf{v}$. We have

$$\mathbf{y}'(t) = \lambda e^{\lambda t}\mathbf{v},$$

so $\mathbf{y}$ satisfies (3.1) if and only if

$$\lambda e^{\lambda t}\mathbf{v} = e^{\lambda t}\mathbf{A}\mathbf{v},$$

that is: $\lambda\mathbf{v} = \mathbf{A}\mathbf{v}$. □

Theorem 5. *If $\mathbf{v}_1, \ldots, \mathbf{v}_k$ are eigenvectors of $\mathbf{A}$ with distinct eigenvalues $\lambda_1, \ldots, \lambda_k$, then the solutions*

$$\mathbf{y}^1(t) = e^{\lambda_1 t}\mathbf{v}_1, \quad \ldots, \quad \mathbf{y}^k(t) = e^{\lambda_k t}\mathbf{v}_k$$

are linearly independent. In particular, if $\mathbf{A}$ is an $n \times n$ matrix with n distinct eigenvalues $\lambda_1, \ldots, \lambda_n$, and eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$, then the general solution of (3.1) has the form

$$\mathbf{y}(t) = C_1 e^{\lambda_1 t}\mathbf{v}_1 + \ldots + C_n e^{\lambda_n t}\mathbf{v}_n.$$

Proof. From Lecture 19, we know that eigenvectors corresponding to different eigenvalues are linearly independent, so $\mathbf{v}_1, \ldots, \mathbf{v}_k$ are linearly independent. Observe that

$$\mathbf{y}^i(0) = e^0 \mathbf{v}_i = \mathbf{v}_i.$$

From Theorem 2 we know that solutions whose value at $t = 0$ are linearly independent, are linearly independent. Therefore, $\mathbf{y}^1, \ldots, \mathbf{y}^k$ are linearly independent.. Since the set of solutions has dimension n, again by Theorem 2, if there are n distinct eigenvectors, we have n linearly independent solutions $\mathbf{y}^1, \ldots, \mathbf{y}^n$, and they must form the basis of the space of solutions, i.e. any other solution is their linear combination. □

Example 6. Consider the linear system

$$\mathbf{y}' = \begin{bmatrix} -2 & 1 \\ -4 & 3 \end{bmatrix} \mathbf{y}.$$

In Lecture 19 we computed its eigenvalues

$$\lambda_1 = 2, \quad \lambda_2 = -1.$$

The corresponding eigenvectors are

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 4 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

(Of course, we can always multiply an eigenvector by a constant, but we can choose any eigenvector.) Since this is a system of two equations and we have two distinct eigenvalues, the general solution, according to our theorem, is

$$\mathbf{y}(t) = C_1 e^{2t} \begin{bmatrix} 1 \\ 4 \end{bmatrix} + C_2 e^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Suppose now that we are interested in finding the particular solution satisfying the initial condition

$$\mathbf{y}(0) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Therefore, we look for constants C_1 and C_2 such that

$$C_1 \begin{bmatrix} 1 \\ 4 \end{bmatrix} + C_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix},$$

or, equivalently,

$$\begin{cases} C_1 + C_2 = 1, \\ 4C_1 + C_2 = -1. \end{cases}$$

We find that $C_1 = -2/3$ and $C_2 = 5/3$ so the solution to our initial value problem is the function

$$\mathbf{y}(t) = -\frac{2}{3} e^{2t} \begin{bmatrix} 1 \\ 4 \end{bmatrix} + \frac{5}{4} e^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Lecture 21: Complex and repeated eigenvalues

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REFERENCES

Boyce–DiPrima sections 7.4–7.8; Braun sections 3.1, 3.4, 3.8, 3.9, 3.10.

1 ROOTS OF THE CHARACTERISTIC POLYNOMIAL

We continue studying homogenous systems with constant coefficients:

$$\mathbf{y}' = \mathbf{A}\mathbf{y}, \tag{1.1}$$

where $\mathbf{A}$ is a constant $n \times n$ matrix. Last time we proved that if $\mathbf{A}$ has n distinct eigenvalues $\lambda_1, \ldots, \lambda_n$, then the corresponding eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are linearly independent. Consequently, the solutions

$$\mathbf{y}_1(t) = e^{\lambda_1 t}\mathbf{v}_1, \quad \ldots, \quad \mathbf{y}_n(t) = e^{\lambda_n t}\mathbf{v}_n$$

are linearly independent and the general solution of (1.1) is a linear combination of $\mathbf{y}_1, \ldots, \mathbf{y}_n$, that is:

$$\mathbf{y}(t) = C_1\mathbf{y}_1(t) + \ldots + C_n\mathbf{y}_n(t).$$

Recall that the eigenvalues of $\mathbf{A}$ are the roots of the characteristic polynomial:

$$\chi(\lambda) = \det(\mathbf{A} - \lambda\mathbf{I})$$

where $\mathbf{I}$ is the identity matrix. This is a degree n polynomial. As you remember from the discussion of second order linear equation, not every polynomial of degree n has n distinct real roots. Two things can happen:

- some roots are complex numbers, and/or
- some roots are repeated.

In this lecture we will discuss how to find n linearly independent solutions if one of these situations happen.

2 COMPLEX ROOTS

Since the characteristic polynomial has real coefficients, its roots come in conjugate pairs, that is: if $\lambda = a + bi$ is a complex root of χ, then so is its conjugate $\bar{\lambda} = a - bi$. Given such complex roots, we can look for complex eigenvectors with eigenvalue λ. A complex eigenvector is a vector

$$\mathbf{v} = \begin{bmatrix} z_1 \\ z_2 \\ \dots \\ z_n \end{bmatrix}$$

where now $z_1, \dots, z_n$ are complex numbers, such that

$$\mathbf{A}\mathbf{v} = \lambda \mathbf{v}. \tag{2.1}$$

Here we define the multiplication of vectors by matrices and by complex numbers in the same way as for real numbers. We can similarly look for an eigenvector with eigenvalue $\overline{\lambda}$ but this is not necessary. If $\mathbf{v}$ is an eigenvector with eigenvalue λ, then, by conjugating equation (2.1) and using the fact that $\mathbf{A}$ is a real matrix, so that $\bar{\mathbf{A}} = \mathbf{A}$, we get

$$\mathbf{A}\bar{\mathbf{v}} = \overline{\mathbf{A}\mathbf{v}} = \overline{\lambda \mathbf{v}} = \bar{\lambda}\bar{\mathbf{v}},$$

so the vector

$$\bar{\mathbf{v}} = \begin{bmatrix} \bar{z}_1 \\ \bar{z}_2 \\ \dots \\ \bar{z}_n \end{bmatrix}$$

is an eigenvector with eigenvalue $\bar{\lambda}$. This is convenient: once we find one complex eigenvector $\mathbf{v}$ with complex eigenvalue λ, then automatically $\bar{\mathbf{v}}$ is an eigenvector with eigenvalue $\bar{\lambda}$. Moreover, since λ is not real, then $\lambda \neq \bar{\lambda}$ so, by the general principle, $\mathbf{v}$ and $\bar{\mathbf{v}}$ are linearly independent.

Example 1. Let us find eigenvectors of the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & 2 & -1 \end{bmatrix}.$$

The characteristic polynomial is

$$\begin{aligned} \chi(\lambda) = \det(\mathbf{A} - \lambda \mathbf{I}) &= \det \begin{bmatrix} 1-\lambda & 0 & 0 \\ 0 & -1-\lambda & -1 \\ 0 & 2 & -1-\lambda \end{bmatrix} \\ &= (1-\lambda) \det \begin{bmatrix} -1-\lambda & -1 \\ 2 & -1-\lambda \end{bmatrix} \\ &= (1-\lambda)((1+\lambda)_2 + 2) \\ &= (1-\lambda)(\lambda + 1 - i\sqrt{2})(\lambda + 1 + i\sqrt{2}), \end{aligned}$$

so there is one real root and two conjugate complex roots

$$\lambda_1 = 1, \quad \lambda_2 = -1 + i\sqrt{2}, \quad \lambda_3 = \bar{\lambda}_2 = -1 - i\sqrt{2}.$$

From the first column of the matrix we see that

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

is an eigenvector with eigenvalue $\lambda_1 = 1$. Let us find a complex eigenvector $\mathbf{v}_2$ with eigenvalue λ_2. We need to solve the equation

$$(\mathbf{A} - \lambda_2 \mathbf{I})\mathbf{v}_2 = 0,$$

that is, if

$$\mathbf{v}_2 = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

for complex numbers a, b, c, then

$$\begin{bmatrix} 2 - i\sqrt{2} & 0 & 0 \\ 0 & -i\sqrt{2} & -1 \\ 0 & 2 & -i\sqrt{2} \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0.$$

We find that

$$a = 0 \quad \text{and} \quad c = -i\sqrt{2}b$$

and b can be any complex number (nonzero, since we want $\mathbf{v}_2$ to be a nonzero vector). In particular, for $b = 1$, we get

$$\mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ -i\sqrt{2} \end{bmatrix}.$$

Since $\lambda_3 = \bar{\lambda}_2$ we automatically get an eigenvector for λ_3:

$$\mathbf{v}_3 = \bar{\mathbf{v}}_2 = \begin{bmatrix} 0 \\ 1 \\ i\sqrt{2} \end{bmatrix}.$$

Once we find a pair of complex eigenvalues $\lambda, \bar{\lambda}$ of the characteristic polynomial, and the corresponding pair of complex eigenvectors $\mathbf{v}, \bar{\mathbf{v}}$, this gives us two complex solutions of the differential equation (1.1):

$$\mathbf{y}_c(t) = e^{\lambda t}\mathbf{v}, \quad \bar{\mathbf{y}}_c(t) = e^{\bar{\lambda} t}\bar{\mathbf{v}}.$$

To get two real solutions, we take the real and imaginary part of one of the complex ones, for example $\mathbf{y}_c$:

$$\text{Re}\mathbf{y}_c(t), \quad \text{Im}\mathbf{y}_c(t).$$

The real and imaginary part are simply defined by taking the real and imaginary part of each entry in the column $\mathbf{y}_c(t)$. Of course, we can also have other real solutions, corresponding to other real or complex eigenvalues. The point is that we always want to find n linearly independent solutions.

Example 2. Consider the homogenous system corresponding to the matrix from Example 1:

$$\mathbf{y}' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & 2 & -1 \end{bmatrix} \mathbf{y}.$$

The real eigenvector $\mathbf{v}_1$ with eigenvalue $\lambda_1 = 1$ gives us a real solution:

$$\mathbf{y}_1(t) = e^{\lambda_1 t}\mathbf{v}_1 = e^t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

The complex eigenvector $\mathbf{v}_2$ with eigenvalue $\lambda_2 = -1 + i\sqrt{2}$ gives us two complex solutions, one is

$$\mathbf{y}_c(t) = e^{\lambda_2 t}\mathbf{v}_2 = e^{-t+i\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ -i\sqrt{2} \end{bmatrix}$$

and the other is its conjugate $\bar{\mathbf{y}}_c(t)$. To find two real solutions, we take the real and imaginary part of $\mathbf{y}_c(t)$. We use Euler's formula to expand and group together all terms with and without i:

$$\mathbf{y}_c(t) = e^{-t}(\cos(\sqrt{2}t) + i\sin(\sqrt{2}t)) \begin{bmatrix} 0 \\ 1 \\ -i\sqrt{2} \end{bmatrix} = e^{-t} \begin{bmatrix} 0 \\ \cos\sqrt{2} \\ \sqrt{2}\sin(\sqrt{2}t) \end{bmatrix} + ie^{-t} \begin{bmatrix} 0 \\ \sin(\sqrt{2}t) \\ -\sqrt{2}\cos(\sqrt{2}t) \end{bmatrix}.$$

Therefore

$$\mathrm{Re}\mathbf{y}_c(t) = e^{-t} \begin{bmatrix} 0 \\ \cos\sqrt{2} \\ \sqrt{2}\sin(\sqrt{2}t) \end{bmatrix},$$
$$\mathrm{Im}\mathbf{y}_c(t) = e^{-t} \begin{bmatrix} 0 \\ \sin(\sqrt{2}t) \\ -\sqrt{2}\cos(\sqrt{2}t) \end{bmatrix}.$$

In total, we have found three linearly independent real solutions $\mathbf{y}_1$, $\mathrm{Re}\mathbf{y}_c$, and $\mathrm{Im}\mathbf{y}_c$. The general solution is their linear combination:

$$\mathbf{y}(t) = C_1 e^t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + C_2 e^{-t} \begin{bmatrix} 0 \\ \cos\sqrt{2} \\ \sqrt{2}\sin(\sqrt{2}t) \end{bmatrix} + C_3 e^{-t} \begin{bmatrix} 0 \\ \sin(\sqrt{2}t) \\ -\sqrt{2}\cos(\sqrt{2}t) \end{bmatrix}.$$

3 REPEATED EIGENVALUES

It can also happen that some of the eigenvalues are repeated, that is the characteristic polynomial has the form

$$\chi(\lambda) = (\lambda - \lambda_1)^k p(\lambda)$$

for some $k > 1$ and p a polynomial such that λ_1 is not a root of p. In this case, we may still be able to find k linearly independent eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ with the same eigenvalue λ_1. This would produce k linearly independent solutions

$$\mathbf{y}_1(t) = e^{\lambda_1 t}\mathbf{v}_1, \quad \ldots, \quad \mathbf{y}^k(t) = e^{\lambda_1 t}\mathbf{v}_k.$$

Example 3. Let

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

The characteristic polynomial is

$$\chi(\lambda) = \det \begin{bmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{bmatrix}.$$

We can simplify this determinant to immediately see one of the roots. After subtracting the third row from the first row (which does not change the determinant), we get

$$\chi(\lambda) = \det \begin{bmatrix} -\lambda - 1 & 0 & 1+\lambda \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{bmatrix} = (\lambda+1)\det \begin{bmatrix} -1 & 0 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{bmatrix}.$$

We see that $\lambda = -1$ is a root. We can simplify things further by adding the first row to the second and third rows and expanding with respect to the first column:

$$\begin{aligned} \chi(\lambda) &= (\lambda+1) \begin{bmatrix} -1 & 0 & 1 \\ 0 & -\lambda & 2 \\ 0 & 1 & -\lambda+1 \end{bmatrix} \\ &= (\lambda+1)\det \begin{bmatrix} -\lambda & 2 \\ 1 & -\lambda+1 \end{bmatrix} \\ &= (\lambda+1)(\lambda_2 - \lambda - 2) \\ &= (\lambda+1)_2(\lambda-2). \end{aligned}$$

So $\lambda_1 = -1$ is a repeated eigenvalue, and another eigenvalue is $\lambda_2 = 2$. We find, as in the previous examples, that the equation $(\mathbf{A} - \lambda_1 \mathbf{I})\mathbf{v} = 0$ has two linearly independent solutions

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \quad \text{and} \quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

so there are two linearly independent eigenvectors with eigenvalue λ_1. We also find an eigenvector with eigenvalue $\lambda_2 = 2$,

$$\mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

This gives us three linearly independent solutions of the differential equation

$$\mathbf{y}' = \mathbf{A}\mathbf{y},$$

namely

$$\mathbf{y}_1(t) = e^{-t} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \quad \mathbf{y}_2(t) = e^{-t} \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \quad \mathbf{y}^3(t) = e^{2t} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

The general solution is their linear combination.

However, in some cases it happens that λ_1 is a repeated root of the characteristic polynomial with multiplicity k:

$$\chi(\lambda) = (\lambda - \lambda_1)^k p(\lambda)$$

but there are no k linearly independent eigenvectors with eigenvalue λ_1. Therefore, we cannot produce k linearly independent solutions of $\mathbf{y}' = \mathbf{A}\mathbf{y}$ of the form $\mathbf{y}(t) = e^{\lambda t}\mathbf{v}$.

Example 4. A simple computation shows that the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix}$$

has characteristic polynomial

$$\chi(\lambda) = (\lambda - 2)_2$$

and every eigenvector with eigenvalue 2 is of the form

$$c \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

for some constant c. Therefore, we cannot find two linearly independent eigenvectors.

Suppose that λ_1 is a repeated eigenvalue and we don't have enough eigenvectors to construct n linearly independent solutions of the differential equation $\mathbf{y}' = \mathbf{A}\mathbf{y}$. Since λ_1 is an eigenvalue, there is still at least one eigenvector $\mathbf{v}_1$, so we can find at least one solution:

$$\mathbf{y}_1(t) = e^{\lambda_1 t}\mathbf{v}_1.$$

How do we find more solutions? If you remember from the lecture about second order linear equations, sometimes we also look for solutions that have the function $te^{\lambda_1 t}$ or, more generally, a polynomial in t times $e^{\lambda_1 t}$. Here we do something similar. A natural guess would be that maybe the function

$$\mathbf{y}(t) = te^{\lambda_1 t}\mathbf{v}_1$$

is also a solution. This, however, does not work. Let us try to correct this initial guess by adding another function:

$$\mathbf{y}(t) = te^{\lambda_1 t}\mathbf{v}_1 + e^{\lambda_1 t}\mathbf{v}$$

for some vector $\mathbf{v}$. What vector $\mathbf{v}$ do we need to choose so that this function is a solution? We compute

$$\mathbf{y}'(t) = e^{\lambda_1 t}\mathbf{v}_1 + \lambda_1 te^{\lambda_1 t}\mathbf{v}_1 + \lambda_1 e^{\lambda_1 t}\mathbf{v}.$$

We want this to be equal to $\mathbf{A}\mathbf{y}(t)$, that is:

$$e^{\lambda_1 t}\mathbf{v}_1 + \lambda_1 te^{\lambda_1 t}\mathbf{v}_1 + \lambda_1 e^{\lambda_1 t}\mathbf{v} = te^{\lambda_1 t}\mathbf{A}\mathbf{v}_1 + e^{\lambda_1 t}\mathbf{A}\mathbf{v}.$$

Using the fact that $\mathbf{A}\mathbf{v}_1 = \lambda_1\mathbf{v}_1$ and dividing by $e^{\lambda_1 t}$, we get that this equation is equivalent to

$$(\mathbf{A} - \lambda_1\mathbf{I})\mathbf{v} = \mathbf{v}_1. \tag{3.1}$$

This equation is similar to the equation for an eigenvector with eigenvalue λ_1. Now, instead of looking for a solution of $(\mathbf{A} - \lambda_1)\mathbf{v} = 0$, we look for a solution of the above equation, where $\mathbf{v}_1$ is a given eigenvector with eigenvalue λ_1. Note that this equation implies that

$$(\mathbf{A} - \lambda_1\mathbf{I})^2\mathbf{v} = (\mathbf{A} - \lambda_1\mathbf{I})(\mathbf{A} - \lambda_1\mathbf{I})\mathbf{v} = (\mathbf{A} - \lambda_1\mathbf{I})\mathbf{v}_1 = 0.$$

In general, a vector $\mathbf{v}$ satisfying equation

$$(\mathbf{A} - \lambda_1\mathbf{I})^k\mathbf{v} = 0$$

for some k, is called a *generalized eigenvector* with eigenvalue λ_1.

Suppose that we have found a vector $\mathbf{v}$ satisfying (3.1). It follows from our calculation that then the function

$$\mathbf{y}_2(t) = te^{\lambda_1 t}\mathbf{v}_1 + e^{\lambda_1 t}\mathbf{v}$$

is a solution of the differential equation $\mathbf{y}' = \mathbf{A}\mathbf{y}$. It is easy to show that this solution is linearly independent from $\mathbf{y}_1$. Thus, we have achieved our goal: we have found a new solution.

Example 5. Consider a differential equation given by the matrix from Example 4:

$$\mathbf{y}' = \mathbf{A} = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix}.$$

The eigenvalue $\lambda_1 = 2$ and eigenvector

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

give us one solution

$$\mathbf{y}_1(t) = e^{2t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

To find another solution, we look for a generalized eigenvector, that is $\mathbf{v}_2$ satisfying

$$(\mathbf{A} - \lambda_1)\mathbf{v}_2 = \mathbf{v}_1,$$

so if we write $\mathbf{v}_2$ as a column with entries a and b, we want to solve

$$\begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

We find that $a = -1 - b$ and b can be anything, so for example we can take $b = 0$ and

$$\mathbf{v}_2 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}.$$

This gives us a new solution

$$\mathbf{y}_2(t) = te^{2t}\begin{bmatrix} 1 \\ -1 \end{bmatrix} + e^{2t}\begin{bmatrix} -1 \\ 0 \end{bmatrix}.$$

We have found two linearly independent solutions so the general solution is their linear combination:

$$\mathbf{y}(t) = C_1 e^{2t}\begin{bmatrix} 1 \\ -1 \end{bmatrix} + C_2\left(te^{2t}\begin{bmatrix} 1 \\ -1 \end{bmatrix} + e^{2t}\begin{bmatrix} -1 \\ 0 \end{bmatrix}\right).$$

For $n \times n$ matrices, where $n \geq 2$ we might have to continue the process of finding generalized eigenvector to produce n linearly independent solutions. That is, given a repeated eigenvalue λ_1, we look for a chain of generalized eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_l$ satisfying

$$\begin{aligned}
(\mathbf{A} - \lambda_1\mathbf{I})\mathbf{v}_1 &= 0, \\
(\mathbf{A} - \lambda_1\mathbf{I})\mathbf{v}_2 &= \mathbf{v}_1, \\
(\mathbf{A} - \lambda_1\mathbf{I})\mathbf{v}_3 &= \mathbf{v}_2, \\
&\cdots \\
(\mathbf{A} - \lambda_1\mathbf{I})\mathbf{v}_l &= \mathbf{v}_{l-1}.
\end{aligned}$$

This gives us l linearly independent solutions:

$$\begin{aligned}
\mathbf{y}_1(t) &= e^{\lambda_1 t}\mathbf{v}_1, \\
\mathbf{y}_2(t) &= e^{\lambda_1 t}\left(\frac{t}{1!}\mathbf{v}_1 + \mathbf{v}_2\right), \\
\mathbf{y}_3(t) &= e^{\lambda_1 t}\left(\frac{t^2}{2!}\mathbf{v}_1 + t\mathbf{v}_2 + \mathbf{v}_3\right), \\
&\dots \\
\mathbf{y}_l(t) &= e^{\lambda_1 t}\left(\frac{t^{l-1}}{(l-1)!}\mathbf{v}_1 + \frac{t^{l-2}}{(l-2)!}\mathbf{v}_2 + \ldots + \frac{t}{1!}\mathbf{v}_{l-1} + \mathbf{v}_l\right).
\end{aligned}$$

Here $l! = 1 \times 2 \times \cdot \times (l-1) \times l$ is the factorial of l. For each repeated eigenvalue with multiplicity k, we find as many of such chains as we can in order to find k linearly independent solution of the differential equation. A general theorem of linear algebra tells us that this is always possible. This process is related to finding the so-called *Jordan normal form* of the matrix $\mathbf{A}$.

The appearance of factorials in the formula is related to the exponential function. Recall from calculus, that the function e^x has a power series expansion

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots.$$

In the process of finding generalized eigenvectors, we essentially try to find the exponential of the matrix $\mathbf{A}$, that is a matrix defined as the infinite sum

$$e^{\mathbf{A}} = \mathbf{I} + 1 + \frac{\mathbf{A}}{1!} + \frac{\mathbf{A}^2}{2!} + \frac{\mathbf{A}^3}{3!} + \ldots.$$

It turns that computing such a matrix for a given $\mathbf{A}$ can be reduced to finding n linearly independent vectors which are either eigenvectors or sequences of generalized eigenvectors, as described above. Once we find $e^{\mathbf{A}}$, the general solution of the differential equation

$$\mathbf{y}' = \mathbf{A}\mathbf{y}$$

has the form

$$\mathbf{y}(t) = e^{\mathbf{A}t}\mathbf{v},$$

for any vector $\mathbf{v} \in \mathbb{R}^n$, which has the interpretation of the initial condition of the solution $\mathbf{v} = \mathbf{y}(0)$. Therefore, once we can define and compute the matrix $e^{\mathbf{A}t}$, solving linear system is very similar to solving first order linear equation for one function: recall that the solution to the equation $y' = ay$ for a constant a, is $y(t) = e^{at}y_0$.

You can read more about the Jordan normal form and the exponential of a matrix in section 3.10 and 3.11 of Braun's textbook. This is a fascinating and important topic in linear algebra, theory of computations, differential equations, and other fields of mathematics. However, we will not develop this general theory any further in this course.

ORDINARY DIFFERENTIAL EQUATIONS (MATH 2030)

Lectures 22, 23: Nonlinear systems; Stability of linear systems

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REFERENCES

Boyce–DiPrima sections 9.1–9.5, Braun sections 4.1, 4.2, 4.3, 4.4, 4.6, 4.7, 4.10.

1 NONLINEAR SYSTEMS

A general system of n first order differential equations has the form

$$\begin{cases} y_1' = f_1(t, y_1, \dots, y_n), \\ y_2' = f_2(t, y_1, \dots, y_n), \\ \dots \\ y_n' = f_n(t, y_1, \dots, y_n), \end{cases}$$

where $y_1, y_2, \dots, y_n$ are unknown functions of one variable t and $f_1, \dots, f_n$ are given functions of $n+1$ variables. We can succinctly write such a system in a vector form

$$\mathbf{y}' = \mathbf{f}(t, \mathbf{y}), \tag{1.1}$$

where $\mathbf{y} = (y_1, \dots, y_n)$ is a vector-valued function of one variable t, and $\mathbf{f} = (f_1, \dots, f_n)$ is a vector-valued functions of $n+1$ variables. (Following the usual convention, we will typically write them as columns with n elements.)

We say that such a system is *linear* if there is function $\mathbf{A}$ of one variable t with values in $n \times n$ matrices (that is: for every t, $\mathbf{A}(t)$ is an $n \times n$ matrix), such that

$$\mathbf{f}(t, \mathbf{y}) = \mathbf{A}(t)\mathbf{y}$$

where the right-hand side is given by multiplying the n vector $\mathbf{y}$ by the $n \times n$ matrix. A system is called *nonlinear* if it is not linear.

For linear systems with constant coefficients, which we studied in the last few lectures, the matrix-valued function $\mathbf{A}$ is constant. More generally, we say that the system is *autonomous* if the function $\mathbf{f}(t, \mathbf{y}) = \mathbf{f}(\mathbf{y})$ does not depend on t.

Unlike for linear systems with constant coefficients, there is no general method for finding solutions of a nonlinear system (1.1). Moreover, for nonlinear systems, a linear combination of solutions is typically not a solution, so we can't use methods of linear algebra to study the space of solutions, as we did for linear systems. Nevertheless, we still can ask various *qualitative* questions about the behavior of solutions, even though we cannot typically write down an explicit formula for solutions. Here are some typical questions we ask about nonlinear systems.

Equilibria

Are there any solutions which are constant in t? Such a solution is called an *equilibrium* of the system. These solutions are important because many natural system "want" to achieve an equilibrium. Think of some of the examples we studied in the first part of the course: using Newton's law of cooling we computed that if the temperature of the surroundings of a body is constant, then the temperature of the body converges to the temperature of the surroundings as $t \to \infty$. Another example is the logistic model of population growth: if we start with a non-zero population, the model predicts that the population grows and as $t \to \infty$ converges to the constant solution given by the carrying capacity of the environment.

Suppose that a constant function $\mathbf{y}(t) = \mathbf{y}_*$ is a solution of (1.1); then

$$0 = \mathbf{y}' = \mathbf{f}(t, \mathbf{y}_*)$$

so the condition for such a function to be a solution is that the vector $\mathbf{y}_*$ satisfies $\mathbf{f}(t, \mathbf{y}_*) = 0$ for all t. This is an algebraic rather than a differential equation, so we can often solve it and find all equilibria of the system.

Example 1. Consider a homogenous linear system with constant coefficients:

$$\mathbf{y}' = \mathbf{A}\mathbf{y}.$$

For what vectors $\mathbf{y}$ is the right-hand side zero? Certainly for $\mathbf{y} = 0$, so zero is always an equilibrium of such a system. We know from linear algebra that zero is the only solution of the equation $\mathbf{A}\mathbf{y} = 0$ if $\mathbf{A}$ is invertible. In that case, $\mathbf{y} = 0$ is the only equilibrium. If $\mathbf{A}$ is not invertible, then there is an entire vector space of eigenvector with eigenvalue zero (the so-called *kernel* of $\mathbf{A}$). All vectors $\mathbf{y}$ from this vector space are equilibria of the system. This is, however, a very special case, because if you pick a random matrix $\mathbf{A}$, it will be invertible.

Stability

Are the solutions stable, that is: can the behavior of a solution change drastically when we change the initial conditions slightly? Let $\mathbf{y}$ be a solution of (1.1). We say that it is *stable* if every other solution $\mathbf{x}$ of (1.1) which starts at $t = 0$ at point $\mathbf{x}(0)$ close to $\mathbf{y}(0)$ remains close to $\mathbf{y}$ for all $t \geq 0$. A precise definition is this: $\mathbf{y}$ is stable if for every $\epsilon > 0$ there is $\delta > 0$ such that if $\mathbf{x}$ is another solution and

$$|\mathbf{x}(0) - \mathbf{y}(0)| \leq \delta$$

then

$$|\mathbf{x}(t) - \mathbf{y}(t)| \leq \epsilon \quad \text{for all } t \geq 0.$$

If a solution is stable, we say that it is *asymptotically stable* if every solution $\mathbf{x}$ which starts at $t = 0$ at a point $\mathbf{x}(0)$ close to $\mathbf{y}(0)$, converges to $\mathbf{y}$ as $t \to \infty$. A precise definition is: $\mathbf{y}$ is asymptotically stable if for every $\epsilon > 0$ there is $\delta > 0$ such that if $\mathbf{x}$ is another solution and

$$|\mathbf{x}(0) - \mathbf{y}(0)| \leq \delta$$

then

$$\lim_{t \to \infty} |\mathbf{x}(t) - \mathbf{y}(t)| = 0.$$

We say that a solution is *unstable* if it is not stable.

Long-time behavior

Given a solution $\mathbf{y}$, what happens to it as $t \to \infty$ or $t \to -\infty$, i.e. in infinite future or infinite past? How does the long-time behavior of the solution depend on the initial condition $\mathbf{y}(0)$? This problem is closely related to the previous two questions.

Example 2. It follows from Newton's law $F = ma$ that the motion of a pendulum is described by the second order equation

$$y'' + k \sin y = 0.$$

Here y is the angle between the pendulum and vertical axis, and $k > 0$ is a constant depending on the mass of the pendulum and the gravitational constant. Introduce the velocity function $x(t) = y'(t)$. Then this second order equation is equivalent to the nonlinear system of first order equations:

$$\begin{cases} x' = -k \sin y, \\ y' = x. \end{cases}$$

Equilibria are the pairs (x, y) such that the right-hand side is zero, that is: $x = 0, y = 0$, and $x = 0, y = \pi$. (We have $\sin y = 0$ for $y = n\pi$ for any integer n, but since y denotes an angle, the equilibrium $y = 2\pi$ is the same as $y = 0$, and so on.) Equation $x = 0$ says that in the equilibrium the velocity of the pendulum is zero. The first equilibrium $y = 0$ corresponds to the pendulum being in the lowest position. The second equilibrium $y = \pi$ corresponds to the pendulum being in the highest position. One can show that the first equilibrium is stable, while the second isn't. This is physically clear: if you move the pendulum slightly from the position $y = 0$, it starts swinging and remains close to the angle $y = 0$. However, if you move it only slightly from the highest position $y = \pi$, then gravity immediately causes it to move downward so $y(t)$ goes away from $y = \pi$.

By the way, observe that for y small, $\sin y$ is very close to y. So as long as we are interested in the movement of the pendulum close to the stable equilibrium $y = 0$, we can approximate the nonlinear equation

$$y'' + k \sin y = 0$$

by the linear equation

$$y'' + ky = 0,$$

or equivalently, by the system:

$$\begin{cases} x' = -ky, \\ y' = x, \end{cases}$$

that is: the harmonic oscillator equation. Of course, we can solve this equation: we know that solutions are given by sine and cosine, so describe a periodic oscillation around $y = 0$. The movement of the actual pendulum will be different, but as long as we stay close to $y = 0$ the sine/cosine solution will be a good approximation. This is a general principle: we often try to understand nonlinear systems by approximating them by linear systems.

2 STABILITY OF LINEAR SYSTEMS

As we just saw in Example 2, we can try to understand nonlinear systems by approximating them by linear ones. I will talk more about this idea in the next section. For now, let us accept this as a motivation for studying stability of linear systems with constant coefficients. Since for such systems we can write explicit formulae for solutions, it is easy to study whether solutions are stable or not.

Theorem 3. *Let* $\mathbf{A}$ *be an* $n \times n$ *matrix. Consider the homogenous linear system with constant coefficients*

$$\mathbf{y}' = \mathbf{A}\mathbf{y}.$$

1. *If all eigenvalues of* $\mathbf{A}$ *have negative real part, then every solution of the system is asymptotically stable.*
2. *If at least one eigenvalue of* $\mathbf{A}$ *have positive real part, then every solution of the system is unstable.*

Note that the theorem does not tell us what happens when all eigenvalues of $\mathbf{A}$ have real part zero (that is: they are imaginary numbers). There is, in fact, a refined version of the theorem which covers also that case. Since the statement is a bit more complicated, and this is a rather special situation, we will not discuss it. You can read about it in Braun's textbook: see Theorem 1 in section 4.2. Let us outline the proof of the theorem to see how eigenvalues of $\mathbf{A}$ are related to stability.

Outline of the proof. First, even though the theorem is about stability of all solutions, it is, in fact, sufficient to consider only the constant solution $\mathbf{y}_0(t) = 0$. Let us show that the following statement is true: *the zero solution is stable if and only all solutions are stable* (and similarly for asymptotic stability).

Of course, if all solutions are stable, then, in particular, the zero solution is stable, so we need to show that if the zero solution is stable, then all solutions are. Let $\mathbf{y}$ be any solution. In order to show that $\mathbf{y}$ is stable, we need to show that given any other solution $\mathbf{x}$ such that $\mathbf{x}(0)$ is close to $\mathbf{y}(0)$, we have that $\mathbf{x}(t)$ is close to $\mathbf{y}(t)$ for all $t \geq 0$. So, for any $\epsilon > 0$ we can find $\delta > 0$ such that if

$$|\mathbf{x}(0) - \mathbf{y}(0)| \leq \delta \quad \text{then} \quad |\mathbf{x}(t) - \mathbf{y}(t)| \leq \epsilon \text{ for all } t \geq 0. \tag{2.1}$$

Since we have assumed that the zero solution is stable, there is a $\delta > 0$ such that if

$$|\mathbf{z}(0)| \leq \delta \quad \text{then} \quad |\mathbf{z}(t)| \leq \epsilon \text{ for all } t \geq 0. \tag{2.2}$$

Now since the equation is homogenous and linear, $\mathbf{z}(t) = \mathbf{x}(t) - \mathbf{y}(t)$ is a solution. Plugging in this choice of $\mathbf{z}$ to equation (2.2) above gives us (2.1), which is the condition for $\mathbf{y}$ to be stable. So we have shown that stability of the zero solution implies stability of any solution $\mathbf{y}$. In the same way we can show that asymptotic stability of the zero solution implies asymptotic stability of any solution.

In order to prove the theorem, it remains to investigate stability of the zero solution. Let us focus on the case when $\mathbf{A}$ has n linearly independent eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ with n eigenvalues $\lambda_1, \ldots, \lambda_n$, possibly complex and possibly repeated. (As you remember, it is not always true that we have n linearly independent eigenvectors when the roots are repeated, but let us ignore that and focus on the proof in this special case, as other cases are similar.) The general complex solution is

$$\mathbf{y}(t) = C_1 e^{\lambda_1 t}\mathbf{v}_1 + C_2 e^{\lambda_2 t}\mathbf{v}_2 + \ldots + C_n e^{\lambda_n t}\mathbf{v}_n. \tag{2.3}$$

Let $\lambda_k = a_k + ib_k$ where a_k and b_k are real and imaginary parts. Suppose first that all a_k are negative. In that case, the expression

$$e^{\lambda_k t} = e^{a_k t}(\cos(b_k t) + i \sin(b_k t))$$

converges to zero as $t \to \infty$. So no matter what constants $C_1, \dots, C_n$ we choose in (2.3), we always have that

$$\lim_{t\to\infty} \mathbf{y}(t) = 0.$$

This shows that if all a_k are negative, then the zero solution is asymptotically stable. Suppose now that at least one a_k is positive. Then $e^{\lambda_k t}$ goes to infinity as $t \to \infty$. Pick a solution (2.3) with all C_i being zero apart from C_k:

$$\mathbf{y}(t) = C_k e^{\lambda_k t}\mathbf{v}_k.$$

This solution goes to infinity as $t \to \infty$. By making C_k as small as we want, we can make $\mathbf{y}(0)$ as close to zero as we want. So we have found solutions which start arbitrarily close to zero, but as $t \to \infty$ go away from it. That means that the zero solution is unstable. To conclude: if all a_k are negative, then the zero solution is asymptotically stable. If at least one a_k is positive, then the zero solution is unstable. This concludes the proof of the theorem. □

We will now look at some examples. In order to understand them, you should look at the pictures of the integral curves of each of the systems. You can find them, with a detailed discussion, in section 9.1 of Boyce and DiPrima's textbook. Alternatively, you can plot them using any of these or similar free online tools:

- https://www.wolframalpha.com/input/?i=plot+a+vector+field
- https://www.geogebra.org/m/QPE4PaDZ
- https://www.desmos.com/calculator/eijhparfmd

Example 4. The matrix of the linear system

$$\mathbf{y}' = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \mathbf{y}$$

has eigenvalues λ_1, λ_2.

- If both of them are positive, all solutions go away from zero, so the system is unstable. (Remember that for homogenous linear systems it is enough to study stability of the zero solution.)
- If, say, $\lambda_1 >$ but $\lambda_2 < 0$, then almost all solutions go away from zero, so the system is unstable as well. The only solutions that go to zero are the ones that start along the horizontal axis (i.e. the line of the eigenvector with eigenvalue λ_2).
- If both eigenvalues are negative, all solutions converge to zero, so the system is asymptotically stable.

Example 5. The matrix of the linear system

$$\mathbf{y}' = \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \mathbf{y}$$

has complex eigenvalues $a + bi$ and $a - bi$.

- If $a = 0$, this is the harmonic oscillator equation. The integral curves are circles centered at zero. If we start close to zero, the solutions stays close to zero but does not converge to it. So the solutions are stable but not asymptotically stable. Observe that this is the case not covered by Theorem 3 when the eigenvalues have real part zero. It turns out that for two-dimensional systems when both the eigenvalues are complex and have zero real part, we always have a picture like this, except when the eigenvectors are not perpendicular, as it is the case here, the integral curves are ellipses rather than circles.

- If $a > 0$, the integral curves are spirals starting at zero and going to infinity. All solutions are unstable.

- If $a < 0$, the integral curves are spirals converging to zero. All solutions are asymptotically stable.

Lectures 24: Stability of nonlinear systems; Predator-prey equations

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REFERENCES

Boyce–DiPrima sections 9.1–9.5, Braun sections 4.1, 4.2, 4.3, 4.4, 4.6, 4.7, 4.10.

1 STABILITY OF NONLINEAR SYSTEMS

Last time we studied stability of linear system. We will now consider general nonlinear autonomous systems

$$\mathbf{y}' = \mathbf{f}(\mathbf{y})$$

where $\mathbf{y}$ is a vector-valued function of t, and $\mathbf{f}\colon \mathbb{R}^n \to \mathbb{R}^n$ is a vector-valued function of n variables, that is, for every $\mathbf{y} \in \mathbb{R}^n$, $\mathbf{f}$ returns a vector $\mathbf{f}(\mathbf{y})$. Recall that we call such a system autonomous because the right-hand side does not depend on t.

Recall also that an equilibrium of such a system is a vector $\mathbf{y}_*$ such that $\mathbf{f}(\mathbf{y}_*) = 0$. In that case, the constant function $\mathbf{y}(t) = \mathbf{y}_*$ is a solution. We are interested in stability of such solutions. This is important because in many cases we can show that solutions of autonomous systems either escape to infinity or converge to an equilibrium as $t \to \infty$. Keep in mind that other things can happen too: we can have periodic solutions whose integral curves are closed. This happens for the harmonic oscillator equation. For nonlinear systems, we can also have integral curves which are asymptotic to a closed solution. What you should take from this discussion is that equilibria and periodic solutions play an important role in understanding nonlinear systems. In this last lecture of the course, we will focus on the equilibria. You can read more about periodic solutions in sections 9.7 and 9.8 of Boyce–DiPrima and section 4.8 of Braun.

We can study stability of an equilibrium of a nonlinear system by approximating the nonlinear system by a linear one, in a neighborhood of the equilibrium. In general, in mathematics we often try to understand nonlinear phenomena by approximating them by linear phenomena. You have probably encountered this idea in your calculus class. Given a general function f and a given point x_0 we can approximate f by its Taylor expansion

$$f(x) \simeq f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \frac{f^{(3)}}{3!}(x - x_0)^3 + \dots$$

The beginning of the expansion

$$f(x_0) + f'(x_0)(x - x_0) \tag{1.1}$$

is a linear function. In fact, the graph of this function is precisely the tangent line to the graph of f at the point $(x_0, f(x_0))$. When x is very close to x_0, the remaining terms are proportionate to $(x - x_0)^k$ where $k \geq 2$, so they are much smaller than the first two terms. (For example, if $|x - x_0| = 0.1$, then the $(x - x_0)^2 = 0.01$, so an order of magnitude smaller.) Therefore, as long as x is close to x_0, we can approximate f by the linear function (1.1). However, as x goes far away from x_0, our approximation will no longer be good. Instead, we will have to linearize the function at another point, close to x.

Going back to nonlinear systems, suppose that $\mathbf{y}_* = 0$ is an equilibrium of the system

$$\mathbf{y}' = \mathbf{f}(\mathbf{y}), \tag{1.2}$$

that is, we have $\mathbf{f}(0) = 0$. In that case, using the Taylor expansion for functions of many variables, we can write $\mathbf{f}$ in the form

$$\mathbf{f}(\mathbf{y}) = \mathbf{f}(0) + \mathbf{A}\mathbf{y} + \mathbf{g}(\mathbf{y}) = \mathbf{A}\mathbf{y} + \mathbf{g}(\mathbf{y}).$$

Here $\mathbf{A}$ is a constant $n \times n$ matrix (consisting of partial derivatives of $\mathbf{f}$ at 0), and $\mathbf{g}$ is another function with the property that for $\mathbf{y}$ close to 0 we have

$$|\mathbf{g}(\mathbf{y})| \leq c|\mathbf{y}|^2 \quad \text{for some constant } c > 0.$$

You should think of it like this: the term $\mathbf{A}\mathbf{y}$ is the linear approximation of $\mathbf{f}$ around 0 and $\mathbf{g}$ is the sum of all nonlinear parts in the Taylor expansion, consisting of terms which depend on second and higher powers of the coordinates of $\mathbf{y}$. The upshot of this discussion is this: if 0 is an equilibrium, we can write our nonlinear system in the form

$$\mathbf{y}' = \mathbf{A}\mathbf{y} + \mathbf{g}(\mathbf{y}),$$

where the second term $\mathbf{g}(\mathbf{y})$ is small as long as $\mathbf{y}$ is close to zero. The *linearization* of the system at zero is then the linear system

$$\mathbf{y}' = \mathbf{A}\mathbf{y}.$$

It turns out that stability of the linear system can tell us something about the stability of the nonlinear system.

Theorem 1. *Consider a nonlinear autonomous system of the form*

$$\mathbf{y}' = \mathbf{A}\mathbf{y} + \mathbf{g}(\mathbf{y})$$

where $\mathbf{A}$ *is a constant matrix and* $\mathbf{g}$ *is a function satisfying*

$$|\mathbf{g}(\mathbf{y})| \leq c|\mathbf{y}|^2$$

for some constant $c > 0$ *and* $\mathbf{y}$ *close to zero.*

- *If all eigenvalues of* $\mathbf{A}$ *have negative real part, then the equilibrium* $\mathbf{y}_* = 0$ *is asymptotically stable.*
- *If at least one eigenvalue of* $\mathbf{A}$ *has positive real part, then the equilibrium* $\mathbf{y}_* = 0$ *is unstable.*

What do we do in the general case, when we an equilibrium $\mathbf{y}_*$ which is nonzero? We simply shift everything by a change of variables so that the equilibrium is at zero. Suppose that $\mathbf{y}_*$ is an equilibrium of (1.2). Introduce $\mathbf{z}(t) = \mathbf{y}(t) - \mathbf{y}_*$. If $\mathbf{y}$ is a solution of (1.2), then $\mathbf{z}$ satisfies

$$\mathbf{z}' = \mathbf{y}' = \mathbf{f}(\mathbf{y}) = \mathbf{f}(\mathbf{y}_* + \mathbf{z}).$$

Now the new system has an equilibrium at $\mathbf{z} = 0$, so we can linearize in the same way as before:

$$\mathbf{z}' = \mathbf{A}\mathbf{z} + \mathbf{g}(\mathbf{z})$$

and apply our theorem to study stability of the equilibrium $\mathbf{y}_*$. The linear system

$$\mathbf{z}' = \mathbf{A}\mathbf{z}$$

obtained in this way is called the *linearization* at $\mathbf{y}_*$. Observe that for every equilibrium we get the corresponding linearization, i.e. the corresponding matrix $\mathbf{A}$ which will be different for different equilibria, so we have to study stability of each equilibrium separately.

Example 2. Consider the nonlinear autonomous system

$$\begin{cases} x' = 1 - xy, \\ y' = x - y^3. \end{cases} \tag{1.3}$$

The equilibria are points (x, y) such that the right-hand side is zero. We find that there are two equilibria: $x = 1, y = 1$ and $x = -1, y = -1$. Let us compute the linearization of the system at the first equilibrium. We introduce new functions $u = x - 1$ and $v = y - 1$, so that the equilibrium $x = 1, y = 1$ corresponds to $u = 0, v = 0$. Now we can write the system in terms of u and v:

$$\begin{cases} u' = x' = 1 - xy = 1 - (u+1)(v+1) = -u - v - uv, \\ v' = y' = x - y^3 = u - (v+1)^3 = u - 3v - 3v^2 - v^3. \end{cases}$$

To find the linearization, we write this system as the sum of a linear system and a term consisting of higher powers of u and v. (Keep in mind that an expression like uv has order two, because if u and v are smaller than some small ϵ, then uv is smaller than ϵ^2.)

$$\begin{bmatrix} u \\ v \end{bmatrix}' = \begin{bmatrix} -1 & -1 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} + \begin{bmatrix} -uv \\ -3v^2 - v^3 \end{bmatrix}.$$

We have written our system as

$$\begin{bmatrix} u \\ v \end{bmatrix}' = \mathbf{A} \begin{bmatrix} u \\ v \end{bmatrix} + \mathbf{g}(u, v),$$

where

$$\mathbf{A} = \begin{bmatrix} -1 & -1 \\ 1 & -3 \end{bmatrix} \quad \text{and} \quad \mathbf{g}(u, v) = \begin{bmatrix} -uv \\ -3v^2 - v^3 \end{bmatrix}.$$

Moreover, $\mathbf{g}$ satisfies $\mathbf{g}(u, v) \le c(u^2 + v^2)$. Therefore, we can apply Theorem 1 to study stability of the equilibrium $u = 0, v = 0$. The matrix $\mathbf{A}$ has a eigenvalue -2, so the theorem tells us that this equilibrium is asymptotically stable. We conclude that the equilibrium $x = 1, y = 1$ of the original system (1.3) is asymptotically stable.

Similarly, to study stability of the second equilibrium $x = -1, y = -1$ we introduce functions $u = x + 1$ and $v = y + 1$. In terms of these functions, (1.3) can be written as

$$\begin{bmatrix} u \\ v \end{bmatrix}' = \begin{bmatrix} 1 & 1 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} + \begin{bmatrix} -uv \\ 3v^2 - v^3 \end{bmatrix}.$$

So the linearization at $x = -1, y = -1$ is given by the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 1 & -3 \end{bmatrix}$$

which has eigenvalues $-1 - \sqrt{5}$ and $-1 + \sqrt{5}$. Since the second eigenvalue is positive, Theorem 1 tells us that the equilibrium $x = -1, y = -1$ is unstable.

2 PREDATOR-PREY EQUATIONS

We will apply these ideas to study the *predator-prey equations*. These are nonlinear differential equations which model the dynamics of populations of two species, one of which (predator) hunts the other (prey). Let $x(t)$ and $y(t)$ be respectively the population of the prey and predator species at time t. Suppose first that there are no predators whatsoever, i.e. $y(t) = 0$. In that case, if there is abundance of food in the environment, the prey species can grow without any constraints according to the exponential growth equation

$$x' = ax$$

for a constant $a > 0$. On the other hand, if there is no prey, i.e. $x(t) = 0$, then the predator species will decay exponentially since there is no food available:

$$y' = -cy$$

for a constant $c > 0$. The general case is a mixture of these two. If both populations are nonzero, then there is some interaction between them. The number of interactions at any given time is proportional to both populations, so it will be proportional to xy. Such interactions benefit predators, i.e. increase the rate of growth of y, and harm prey, i.e. decrease the rate of growth of x. So a simple model is given by the following system of nonlinear differential equations:

$$\begin{cases} x' = ax - bxy, \\ y' = cy - dxy, \end{cases}$$

where a, b, c, d are given positive constants. These are the *predator-prey equations*, also called the Lotka–Volterra equations after the mathematicians who introduced them.

We easily compute that hese equations have two equilibria: $x = 0, y = 0$ and $x = c/d, y = a/b$. The first equilibrium is uninteresting: it describes the situation in which both populations are zero and so happens. The linearization at $x = 0, y = 0$ is given by the linear system

$$\begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} a & 0 \\ 0 & c \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

which has positive eigenvalues $a > 0$, $c > 0$. We conclude that this equilibrium is unstable.

The second equilibrium is more interesting. Let us compute the linearization. Introduce $u = x - c/d$ and $v = y - a/b$ so that the equilibrium corresponds to $u = 0, v = 0$. With some amount of calculation, we get that the differential equation for u and v is

$$\begin{bmatrix} u \\ v \end{bmatrix}' = \begin{bmatrix} 0 & -\mu_1 \\ \mu_2 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} + \mathbf{g}(u, v)$$

where

$$\mu_1 = bc/d \quad \text{and} \quad \mu_2 = ad/b$$

where $\mathbf{g}(u, v)$ consists of nonlinear terms. Since μ_1 and μ_2 are both positive, the linearizatrion matrix has purely imaginary eigenvalues $i\sqrt{\mu_1\mu_2}$ and $-i\sqrt{\mu_1\mu_2}$. Since the real part of all eigenvalues is zero, we are in the situation which is not covered by Theorem 1. However, it is instructive to see what happens for the linear system

$$\begin{bmatrix} u \\ v \end{bmatrix}' = \begin{bmatrix} 0 & -\mu_1 \\ \mu_2 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$$

This is essentially the harmonic oscillator equation, so we know from the last lecture that all solutions are periodic and the integral curves are closed curves around the equilibrium 0, which is stable but not asymptotically stable. It turns out that this linear system is a good approximation of our nonlinear system and the same thing happens for the nonlinear system. The equilibrium $x = c/d, y = a/b$ is stable but not asymptotically stable, and all integral curves with $x > 0$ and $y > 0$ are closed curves going around the equilibrium. They are no longer ellipses, as in the linear case, but more complicated curves. They have an interpretation as cycles of growth and decay: at first, there are few predators so prey can grow freely. Once there is more prey, predators have more food so their population grows. Once the predator population is large, the prey population starts decreasing. Thus, there is less food for predators so their population decreases as well, and this brings us back to the beginning of the cycle. You can show that no matter what the initial population was, as long as we had some predators and some prey, it will always follow such a cycle.

You can show that for any solution the average value of x and y over the period of one cycle is, in fact, given by the value at equilibrium

$$x_{\text{average}} = c/d \quad \text{and} \quad y_{\text{average}} = a/b.$$

Suppose now that there are external factors which cause both populations to decrease at constant rate. For example, we could study two fish populations and try to understand the effect of fishing. If we decrease both populations at constant rate $\epsilon > 0$, then the new equations are

$$\begin{cases} x' = ax - bxy - \epsilon x = (a - \epsilon)x - bxy, \\ y' = cy - dxy - \epsilon y = -(c + \epsilon)y - dxy. \end{cases}$$

We obtain the same equations, except with a replaced by $a - \epsilon$ and c replaced by $c + \epsilon$. So the average population for any solution is now

$$x_{\text{average}} = (c + \epsilon)/d \quad \text{and} \quad y_{\text{average}} = (a - \epsilon)/b.$$

We see that the effect is that on average the prey population will increase while the predactor population will decrease. This can have important consequences. For example, imagine that you have two insect populations, x and y, such that x is harmful to crops you grow, and y is a predator which eats x, so is good for the crops. You could try to get rid of x using insecticide, but if the insecticide kills both x and y at the same rate, this will be counterproductive as the effect will be that on average the population of x increases while the population of y decreases.

We can also make our model more realistic by introducing competition. In practice, resources are not unlimited, so even in the absence of predators,

$y = 0$, the prey population will not grow exponentially. A better model would be the logistic equation

$$x' = ax - ex^2.$$

Now, when we introduce the predator population, we can incorporate competition between prey to get the system

$$\left\{ x' = ax - bxy - ex^2, y' = cy - dxy. \right.$$

It turns out that this changes the behavior of the system. Now it will depend on how strong the competition between the prey is. We now have a new equilibrium $x = a/e$, $y = 0$. You can show that if e is small, then this equilibrium is unstable, and the situation is very similar to the original predator-prey equations, that is each solution is a closed cycle going around a stable equilibrium. However, if e is large, then there are no cycles. In that case, the equilibrium $x = a/e, y = 0$ is asymptotically stable and every solution converges to it as $t \to \infty$. This means that eventually the predator population dies out and the growth of the prey population is governed by the logistic equation $x' = ax - ex^2$ which, as we know from the beginning of the course, grows as $t \to \infty$ to an equilibrium $x = a/e$.