Homework 1: Review of calculus and separable equations

due Thursday, January 21

Solutions must be uploaded to gradescope as a .pdf file by Thursday midnight. To learn how to upload your solutions and request regrades, see:

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https://youtu.be/u-pK4GzpId0
https://youtu.be/TOHCkI12mh0

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Each problem is worth 5 points. (5 points: complete solution, 4 points: generally correct solution with minor computational mistakes, 3 points: approximately half of the problem solved, 2 points: some good ideas but less than half of the problem solved, 1 point: incorrect solution but shows some knowledge of basic ideas involved).

You don't need to solve problems marked (Bonus) to receive full points for homework. Each of them will give you 5 extra points.

Problem 1. Compute the derivatives of the functions

$$f(x) = \ln(\sin x) \quad \text{and} \quad f(x) = e^{2x+1}.$$

Problem 2. Given a function f and a number t_0, define a function F by

$$F(t) = \int_{t_0}^{t} f(x)\mathrm{d}x.$$

What is the derivative of F? How about if we define

$$F(t) = \int_{t_0}^{t^2} f(x)\mathrm{d}x?$$

Problem 3. Given a constant $k > 0$, compute the second derivative of

$$f(x) = \sin(kx) \quad \text{and} \quad f(x) = \cos(kx).$$

Can you guess a function which solves the differential equation

$$f'' + 4f = 0$$

and satisfies $f(0) = 1$ and $f'(0) = 1$?

Problem 4. Find the general solution of the differential equations

1. $y'(t) = \sin(4t) + t^2$,
2. $y''(t) = e^{2t}$.

Problem 5. Find the general solution of the differential equations

1. $y'(t) + y(t)^2 \sin t = 0$,
2. (Bonus) $y'(t) = (t + y)/(t - y)$.

Problem 6. Let a and y_0 be real numbers and let $y(t)$ be the solution of

$$y'(t) + ay(t) = 0 \quad \text{and} \quad y(0) = y_0.$$

Compute $\lim_{t\to\infty} y(t)$, depending on a and y_0.

Problem 7. (Bonus) Let a be a positive constant and let f be a continuous function such that $\lim_{t\to\infty} f(t) = 0$. Show that every solution to the differential equation

$$y'(t) + ay(t) = f(t)$$

satisfies

$$\lim_{t\to\infty} y(t) = 0.$$

Homework 2: Population models and linear differential equations

due Thursday, January 28

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Problem 1 (Malthussian model of population growth)**.** In class, we discussed the simplest, Malthussian model of population growth. If $P(t)$ denotes the population at time t, the Malthussian model is

$$P'(t) = \alpha P(t)$$

for a constant $\alpha > 0$.

1. Given a number $P_0 > 0$, find a solution P such that $P(0) = P_0$.
2. Sketch the graph of the solution P.
3. Compute $\lim_{t\to\infty} P(t)$.

Problem 2 (Logistic model of population growth)**.** A different model of population growth, taking into account that the resources available in the environment are limited, was proposed by the mathematician Pierre Verhulst in 1838:

$$P'(t) = \alpha P(t) \left(1 - \frac{P(t)}{K}\right),$$

where $\alpha > 0$ and $K > 0$ are constant. (K is known as the *carrying capacity*.)

1. Given a number $P_0 > 0$, find a solution P such that $P(0) = P_0$.
2. Sketch the graph of the solution P. (You can use a computer program or a website such as Wolfram Alpha for this.)
3. Compute $\lim_{t\to\infty} P(t)$. How do you interpret this result?

Problem 3 (Types of differential equations)**.** For each of the following differential equations for a function $y = y(t)$, determine its order and whether it is linear or not. If it is linear, determine whether it is homogenous or not and if it has constant coefficients or not.

1. $t^2 y'' + ty + 2y = \sin t$,
2. $(1 + y^2)y'' + ty' + y = e^t$,
3. $y^{(3)} + 4y'' = 0$,
4. $y'' + \sin y = 0$.

Problem 4 (Homogenous linear differential equations)**.** Consider a homogenous linear differential equation

$$a_n(t)y^{(n)} + \ldots + a_1(t)y'(t) + a_0(t)y(t) = 0. \tag{0.1}$$

Prove that

1. if $y = y(t)$ is a solution to (0.1), then so is the function λy for any constant λ;
2. if $x = x(t)$ and $y = y(t)$ are both solutions to(0.1), then so is the function $x + y$.

Problem 5 (Non-homogenous linear differential equations)**.** Consider a non-homogenous linear differential equation

$$a_n(t)y^{(n)} + \ldots + a_1(t)y'(t) + a_0(t)y(t) = b(t), \tag{0.2}$$

and the corresponding homogenous equation (0.1). Prove that

1. if functions $x = x(t)$ and $y = y(t)$ are both solutions to the non-homogenous equation (0.2), then their difference $x - y$ is a solution to the homogenous equation (0.1).
2. conversely, if a function $x = x(t)$ is a solution to the non-homogenous equation (0.2) and $y = y(t)$ is a solution to the homogenous equation (0.1), then $x + y$ is a solution to the non-homogenous equation (0.2).

Problem 6 (Newton's law of cooling)**.** From Wikipedia:

> Newton's law of cooling states that the rate of heat loss of a body is directly proportional to the difference in the temperatures between the body and its surroundings.
> https://en.wikipedia.org/wiki/Newton%27s_law_of_cooling

Denote by $u(t)$ the temperature of a body at time t and by $T(t)$ the temperature of its surroundings at time t. Newton's law of cooling gives us

$$u'(t) = -k(u(t) - T(t)),$$

for some constant $k > 0$. Solve this differential equation, with initial value $u(0) = u_0$, in two cases:

1. when the temperature of the surroundings is constant: $T(t) = T_0$;
2. when the temperature of the surroundings varies periodically according to the formula $T(t) = T_0 + T_1 \cos \omega t$ for some constants T_0, T_1, ω.

In both cases, sketch the graph of $u(t)$ and compare it to the graph of $T(t)$.

Homework 3:
Existence and uniqueness of solutions

due Thursday, February 4

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Problem 1 (Continuous functions)**.** Recall that a function f is said to be continuous at x_0 if

$$\lim_{x \to x_0} f(x) = f(x_0).$$

Show that if f and g are two functions which are continuous at x_0, then their sum $f + g$ and product fg are also continuous at x_0.

Hint: Remind yourself of the basic properties of limits: What is the limit of the sum of functions? What is the limit of the product of functions?

Problem 2 (Partial derivatives)**.** Compute the partial derivatives $\partial f / \partial x$ and $\partial f / \partial y$ of the following functions of two variables:

$$f(x, y) = x^2 y + y^3,$$
$$f(x, y) = \sin(xy),$$
$$f(x, y) = e^{(x-y)^2}.$$

Problem 3 (Existence and uniqueness)**.** State the existence and uniqueness theorem for first order differential equations. Argue that the following initial value problem has a unique solution $y = y(t)$ defined for t close to $t = 0$:

$$\begin{cases} y' = e^{(y-t)^2} \\ y(0) = 1. \end{cases}$$

Problem 4 (Failure of uniqueness)**.** Show that the following initial value problem has two solutions $y = y(t)$ defined for $t \geq 0$:

$$\begin{cases} y' = \sqrt{y}, \\ y(0) = 0. \end{cases}$$

Why does the uniqueness theorem not apply in this case?

The next problem is a bonus problem and you don't have to solve it to receive full credit for homework. It is worth 10 extra points.

Problem 5 (Bonus: Banach fixed point theorem)**.** Let f be a continuous function such that

$$|f(x) - f(y)| \leq \alpha |x - y| \tag{0.1}$$

for a constant α satisfying $0 < \alpha < 1$. The purpose of this exercise is to prove that there is an x such that $f(x) = x$. This is known as the Banach fixed point theorem.

[*Comment.* While this theorem does not seem directly related to differential equations, the method of its proof is used in the proof of the existence and uniqueness theorem for differential equations. You can read more about this in Boyce–DiPrima section 2.8 and Braun section 1.10.]

Write a complete proof of the theorem by filling the gaps in these steps:

1. Choose any x_0 and define a sequence x_n inductively by
$$x_n = f(x_{n-1}).$$
That is: $x_1 = f(x_0)$, $x_2 = f(x_1)$, and so on.
2. Using (0.1), show that for every n
$$|x_n - x_{n-1}| \leq \alpha^{n-1} |x_1 - x_0|.$$
3. Write x_n as
$$x_n = (x_n - x_{n-1}) + (x_{n-1} - x_{n-2}) + \ldots + (x_1 - x_0) + x_0.$$
4. By comparing the above sum to the convergent series $\sum_{n=0}^{\infty} \alpha^n$ conclude that the sequence x_n converges as $n \to \infty$.
5. Let $x = \lim_{n \to \infty} x_n$. Justify that we can pass to the limit $n \to \infty$ in
$$x_n = f(x_{n-1})$$
and conclude that x satisfies $f(x) = x$.
6. Using (0.1), show that if y is any number satisfying $f(y) = y$, then $y = x$, that is: there is only one solution to the equation $f(x) = x$.

Homework 4:
Second order equations

due Sunday, February 21

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Problem 1 (Linear algebraic equations)**.** Consider a linear system of algebraic equations for two unknown numbers x and y:

$$\begin{cases} ax + by = e, \\ cx + dy = f. \end{cases}$$

Solve this system for x and y when $ad - bc \neq 0$. Find examples of numbers a, b, c, d, e, f, for which $ad - bc = 0$ and

1. the system has no solutions,
2. the system has infinitely many solutions.

Hint. The easiest way to find such examples is to set some of the coefficients equal to zero.

Problem 2 (Second order equations as systems)**.** Write each of the following second order differential equations as a system of two first order differential equations for functions y and $x = y'$.

1. $y'' + p(t)y' + q(t)y + r(t) = 0$,
2. $y'' + p(t)y'y + q(t)(y')^2 + r(t)y = 0$.

Problem 3 (Existence and uniqueness)**.** Consider the following initial value problems. Which of them satisfy the hypotheses of the existence and uniqueness theorem for second order equations (and therefore admit a unique solution)? In each case, justify your answer.

1. $y'' + t^2y' + y^2 = 0$ with initial conditions $y(0) = 1$, $y'(0) = 0$;
2. $y'' + t^2y' + y^2 = 0$ with initial condition $y(0) = 1$,
3. $y'' + t^2y' + y^{1/2} = 0$ with initial conditions $y(0) = 1$, $y'(0) = 0$.

Problem 4 (Examples of linear equations)**.** Let $k > 0$ be a constant.

1. Verify that the functions $y_1(t) = e^{\sqrt{k}t}$ and $y_2(t) = e^{-\sqrt{k}t}$ solve the second order linear equation
$$y'' - ky = 0.$$
2. Verify that the functions $y_1(t) = \sin(\sqrt{k}t)$ and $y_2(t) = \cos(\sqrt{k}t)$ solve the second order linear equation
$$y'' + ky = 0.$$

In both cases sketch the graph of y_1 and y_2, and solve the initial value problem $y(0) = 0$, $y'(0) = 1$ by considering functions of the form $C_1y_1 + C_2y_2$ for some constants C_1, C_2.

Problem 5 (Wronskian)**.** In both cases in Problem 4, compute the Wronskian

$$W[y_1, y_2](t) = y_1(t)y_2'(t) - y_1'(t)y_2(t)$$

and show that it is non-zero for all t. (Recall from the lecture that this means that every solution to the given second order equation is of the form $C_1y_1 + C_2y_2$ for some constants C_1, C_2.)

Homework 5:
Second order homogenous equations with constant coefficients; complex numbers

due Thursday, February 25

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Problem 1 (Complex numbers)**.** Write the following complex numbers in the form $a + bi$ with a, b real.

- $5 + i - (2 + 3i)$
- $\frac{1+2i}{3-i}$
- i^{-1},
- $(1 + i)^3$
- $\overline{4 + 2i}$

(Recall that the conjugate of a complex number $z = a + bi$ is defined as the complex number $\bar{z} = a - bi$.)

Problem 2 (Quadratic equations). Solve the quadratic equation

$$\lambda^2 + 2\lambda + 5 = 0.$$

Problem 3 (Homogenous equation with distinct real roots). Find the general solution of the differential equation

$$y'' - 3y' - 4y = 0.$$

Problem 4 (Homogenous equation with repeated roots). Find the general solution of the differential equation

$$y'' + 2y' + y = 0.$$

Problem 5 (Homogenous equation with complex roots). Find the general solution of the differential equation

$$y'' + 2y' + 5y = 0.$$

(You can use your solution to Problem 2.)

You don't have to solve the bonus problems to receive full credit. Each bonus problem is worth 5 extra points.

Problem 6 (Bonus: Complex roots of polynomials). Let

$$p(\lambda) = a_n\lambda^n + \cdots + a_1\lambda + a_0$$

be a polynomial whose coefficients $a_0, \ldots, a_n$ are real numbers. Show that if a complex number λ is a root of p, that is: $p(\lambda) = 0$, then its conjugate $\overline{\lambda}$ is also a root of p.

(*Hint:* Recall that the conjugate of a complex number $\lambda = a + bi$ is defined by $\overline{\lambda} = a - bi$. Observe that for two complex numbers λ and μ we have $\overline{\lambda + \mu} = \overline{\lambda} + \overline{\mu}$ and $\overline{\lambda \cdot \mu} = \overline{\lambda} \cdot \overline{\mu}$.)

Problem 7 (Bonus: De Moivre's formula). Show that for any integer n,

$$(\cos(x) + i\sin(x))^n = \cos(nx) + i\sin(nx)$$

Using this formula, express $\cos(3x)$ in terms of $\cos x$ and $\sin x$.

(*Hint:* Use the fact that $e^{ix} = \cos(x) + i\sin(x)$ and the properties of the exponential function.)

Homework 6:
Non-homogenous equations

due Sunday, March 14

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Problem 1 (Non-homogenous equation with polynomial right-hand side)**.** Find a particular solution to the equation

$$y'' + 3y = 3t^2 - 1.$$

As in the lecture, since the right-hand side is a polynomial, look for a particular solution which is a polynomial of the same degree.

Problem 2 (Non-homogenous equation with exponential right-hand side)**.** Find a particular solution to the equation

$$y'' - y' = te^{2t}.$$

Observe that the right-hand side is of the form $p(t)e^{\alpha t}$ where $p(t)$ is a polynomial in t. Therefore, we look for a particular solution of the form $q(t)e^{\alpha t}$ where $q(t)$ is a polynomial of the same degree as $p(t)$.

Problem 3 (Non-homogenous equation with trigonometric right-hand side)**.** Find a particular solution to the equation

$$y'' + 2y = t\cos t. \tag{0.1}$$

This is done using the method from Problem 2. First, consider instead of (0.1) the complex equation

$$y'' + 2y = te^{it}. \tag{0.2}$$

Using the method from Problem 2 find a particular complex solution to equation (0.2) of the form $y_c(t) = (a + bt)e^{it}$, with a and b complex numbers. Now observe that

$$t\cos t = \mathrm{Re}(te^{it})$$

and conclude that $y(t) = \mathrm{Re}(y_c(t))$ is a solution to the original equation (0.1). Write the formula for $y(t)$.

Problem 4 (Free vibrations)**.** Find the general solution to the harmonic oscillator equation

$$my'' + ky = 0$$

where $m > 0$ and $k > 0$ are constant. Write the general solution in the form

$$y(t) = A\cos(\omega_0 t + \theta),$$

where A and θ are some constants and $\omega_0 = \sqrt{k/m}$. Sketch the graph of the solution. How do the constants A, ω_0, and θ affect the shape of the graph?

You do not have to solve the following bonus problem to receive full homework credit. The bonus problem is worth 10 extra points.

Problem 5 (Bonus: Vibrations with external force)**.** The harmonic oscillator equation describes vibrations without any external force applied to the system. We can consider a model in which we apply to the system a periodic external force $F(t) = F_0\cos(\omega t)$. Here ω is a constant, the frequency of the external force (which is not necessarily the same as the frequency as the system $\omega_0 = \sqrt{k/m}$). The differential equation we get is

$$my'' + ky = F_0\cos(\omega t). \tag{0.3}$$

Find the general solution to this equation. There are two cases.

1. When $\omega \neq \omega_0$, look for a particular solution using the method from Problem 3, by first solving the corresponding complex problem

$$my'' + ky = F_0 e^{i\omega t} \tag{0.4}$$

   and then taking the real part of the complex solution. Look for a solution to (0.4) of the form $y(t) = ae^{i\omega t}$ where a is a complex number.

2. When $\omega = \omega_0$ the complex equation (0.4) has no solutions of the form $y(t) = ae^{i\omega t}$. Instead, look for a complex solution of the form $y(t) = ate^{i\omega t}$ for a complex number a. Then take the real part to find a real solution to (0.3).

In both cases, sketch the graph of the solution. What is the difference between the behavior of the system in these two cases?

Homework 7:
Higher order equations; Linear algebra

due Thursday, April 1

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Problem 1 (Higher order linear equations: real roots)**.** Find the general solution to the differential equation

$$y''' - y'' - 2y' = 0.$$

Problem 2 (Higher order linear equations: one real and two complex roots)**.** Find the general solution to the differential equation

$$y''' - 8y = 0.$$

[Look for roots of the characteristic polynomial $\lambda^3 - 8$ of the form $\lambda = 2e^{i\theta}$.]

Problem 3 (Operations on vectors)**.** Let

$$\mathbf{x} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}, \quad \mathbf{y} = \begin{bmatrix} -1 \\ 0 \\ 4 \end{bmatrix}.$$

Compute $3\mathbf{x} - 2\mathbf{y}$.

Problem 4 (Multiplying vectors by matrices)**.** Given

$$\mathbf{A} = \begin{bmatrix} -1 & 5 & 0 \\ 2 & 1 & 3 \\ -1 & -0 & 2 \end{bmatrix}$$

compute $\mathbf{Ax}$ for the following vectors

$$\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}.$$

Problem 5 (Linear equations in a matrix form)**.** Write the following system of linear equations for unknowns x_1, x_2, x_3 in the matrix form $\mathbf{Ax} = \mathbf{a}$, for a matrix $\mathbf{A}$ and vectors $\mathbf{x}$ and $\mathbf{a}$.

$$\begin{aligned} 5x_1 - 7x_2 + 10x_3 &= 0, \\ x_2 + 2x_3 &= 1, \\ 3x_1 - x_2 + 4x_3 &= -2. \end{aligned}$$

Problem 6 (Linear independence)**.** For each of the following collections of vectors, determine if it is linearly independent or not.

1. $$\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

2. $$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} -4 \\ 0 \\ 4 \end{bmatrix}$$

3. $$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

The next problem is worth 15 points.

Problem 7 (Vector spaces)**.** In the lecture we said that the set of all functions on a real line is a vector space. Which of the following sets of functions form a vector space, with the usual operations of adding functions and multiplying them by numbers? In each case, explain why or why not.

1. the set of polynomials of degree smaller than or equal to k;
2. the set of polynomials of degree equal to k;
3. the set of functions f such that $\lim_{x\to\infty} f(x) = 0$;
4. the set of functions f such that $f(0) = 1$;
5. the set of solutions of the differential equation $f'' + f^2 = 0$.

Comments:

1. In order to be a vector space, the set should contain 0, for every f and g from the set $f + g$ should be also in that set, and for every f in the set and real number λ the element λf should also be in the set.
2. Recall that a polynomial is a function of the form

   $$f(x) = a_0 + a_1 x + \dots a_n x^n.$$

   If $a_n \neq 0$, then the degree of f is n. For example $f(x) = 2$ is a polynomial of degree zero, $f(x) = 1 - 2x + x^3$ is a polynomial of degree three.

Homework 8:
Linear algebra

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Problem 1 (Matrix multiplication)**.** Given matrices

$$\mathbf{A} = \begin{bmatrix} 3 & 2 & 0 \\ 1 & 1 & 1 \\ 2 & 6 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} -1 & 0 & 9 \\ 2 & 1 & -1 \\ 0 & 6 & 2 \end{bmatrix}$$

compute $\mathbf{AB}$ and $\mathbf{BA}$.

Problem 2 (Composition of linear transformations)**.** Let $L\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the counterclockwise rotation around the point $(0,0)$ by angle θ and let $K\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the reflection with respect to the x–axis. Compute the matrix of the composition LK of these two linear transformations.

Problem 3 (Linear transformations as matrices)**.** Given n, let V be the vector space consisting of functions of the form

$$\begin{aligned} f(x) = a_0 + a_1 \sin(x) + a_2 \sin(2x) + \ldots + a_n \sin(nx) \\ + b_1 \cos(x) + b_2 \cos(2x) + \ldots + b_n \cos(nx), \end{aligned}$$

that is: linear combinations of the functions

$$1, \sin(x), \cos(x), \sin(2x), \cos(2x), \ldots, \sin(nx), \cos(nx). \tag{0.1}$$

Show that the map L defined by

$$L(f) = \frac{\mathrm{d}f}{\mathrm{d}x}$$

is a linear transformation from V to V. Compute the matrix of L in the basis of V given by functions (0.1).

Problem 4 (Determinants)**.** Compute the determinant of

$$\begin{bmatrix} 1 & 0 & 2 \\ 1 & 2 & 5 \\ 6 & 8 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 1 & 5 \end{bmatrix}$$

(Here is one way to do it: in the first case, reduce the matrix to an upper diagonal matrix by subtracting multiples of rows. In the second case, get rid of the ones in the first row by subtracting other rows and write as a sum of determinants of 4 x 4 matrices by expanding with respect to the last column.)

Problem 5 (Eigenvectors)**.** Find eigenvalues and eigenvectors of the matrix

$$\begin{bmatrix} 1 & -3 & 2 \\ 0 & -1 & 0 \\ 0 & -1 & -2 \end{bmatrix}.$$

(You can immediately see one root $\lambda = 1$ of the characteristic polynomial $\det(\mathbf{A} - \lambda\mathbf{I})$ if you compute the determinant by expanding it into a sum of 2x2 determinants with respect to the first column.)

Problem 6 (Homogenous system of differential equations)**.** Solve the initial value problem

$$\mathbf{y}' = \begin{bmatrix} 1 & -3 \\ -2 & 2 \end{bmatrix} \mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 0 \\ 5 \end{bmatrix}.$$

The next two problems are bonus problems and you don't have to solve them to receive full homework credit. Each is worth 5 additional points.

Problem 7 (Bonus: Commutator)**.** Let $L\colon V \to V$ and $K\colon V \to V$ be two linear transformations. The *commutator* of L and K is the linear transformation $[L, K]$ defined by

$$[L, K] = LK - KL.$$

Let V be the space of functions $f(x)$ on the real line which have derivatives of any order. Define $L\colon V \to V$ and $K\colon V \to V$ by

$$L(f) = xf \quad \text{and} \quad K(f) = \frac{\mathrm{d}f}{\mathrm{d}x},$$

that is: L multiplies a given function by the function x, and K takes the derivative of a function. Show that L and K are linear transformations and compute their commutator $[L, K]$.

Remark: L and K are known as the *position* and *momentum operators* in quantum mechanics. The fact that $[L, K] \neq 0$ is a mathematical expression of Heisenberg's uncertainty principle.

Problem 8 (Bonus: Invertible transformations)**.** Let $L\colon V \to W$ be a linear transformation between two vector spaces. Suppose that L is invertible. Prove that

1. If $v_1, \ldots, v_n$ is a linearly independent collection of vectors in V, then $Lv_1, \ldots, Lv_n$ is a linearly independent collection of vectors in W.
2. If $v_1, \ldots, v_n$ is a basis of V, then $Lv_1, \ldots, Lv_n$ is a basis of W.
3. V and W have the same dimension.

Homework 9: Equilibria and stability

due Thursday, April 15

Solutions must be uploaded to gradescope as a .pdf file by Sunday midnight. To learn how to upload your solutions and request regrades, see:

https://gradescope-static-assets.s3.amazonaws.com/help/submitting_hw_guide.pdf
https://youtu.be/u-pK4GzpId0
https://youtu.be/TOHCkI12mh0

No late homework will be accepted. You are encouraged to discuss the problems with your peers. However, you must write your solutions individually and, if you worked on some of them with other students, please mention this in your solutions and write the names of your collaborators.

Each problem is worth 5 points. (5 points: complete solution, 4 points: generally correct solution with minor computational mistakes, 3 points: approximately half of the problem solved, 2 points: some good ideas but less than half of the problem solved, 1 point: incorrect solution but shows some knowledge of basic ideas involved).

Problem 1 (Homogenous linear systems with constant coefficients)**.** Find the general solution of each of the following linear systems and determine if the solutions are stable or unstable:

$$\mathbf{y}' = \begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix} \mathbf{y},$$

$$\mathbf{y}' = \begin{bmatrix} -2 & 1 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -1 \end{bmatrix} \mathbf{y}.$$

Problem 2 (Integral curves)**.** Sketch the vector field and integral curves of each of the following linear systems. In each case, use the picture to determine whether its solutions are stable or unstable. If they are stable, are they asymptotically stable?

$$\mathbf{y}' = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \mathbf{y}, \qquad \mathbf{y}' = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \mathbf{y}, \qquad \mathbf{y}' = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \mathbf{y}$$

Problem 3 (Equilibria of nonlinear systems)**.** For each of the following nonlinear autonomous system, find its equilibria. For each equilibrium, the corresponding linear system near that equilibrium and determine, if possible, whether the equilibrium is stable or unstable.

$$\begin{cases} x' = 1 - y, \\ y' = x^2 - y^2 \end{cases}$$

$$\begin{cases} x' = 1 - xy, \\ y' = x - y^3. \end{cases}$$

Homework 1

Problem 1

Use $(h(g(x)))' = h'(g(x)) \cdot g'(x)$

- For $f(x) = \ln(\sin x) \Rightarrow f'(x) = \frac{1}{\sin(x)} \cdot \sin'(x)$

$\Leftrightarrow f'(x) = \frac{\cos(x)}{\sin(x)}$

- For $f(x) = e^{2x+1} \Rightarrow f'(x) = e^{2x+1} \cdot (2x+1)'$

$\Leftrightarrow f'(x) = e^{2x+1} \cdot 2$

Problem 2

- $F(t) = \int_{t_0}^{t} f(x)\,dx \Rightarrow F'(t) = \frac{d}{dt}\left(\int_{t_0}^{t} f(x)\,dx\right)$

Let $g(x)$ be an antiderivative of $f(x)$, such that $g'(x) = f(x)$.

$\Rightarrow \int_{t_0}^{t} f(x)\,dx = g(t) - g(t_0)$

$\Rightarrow F'(t) = \frac{d}{dt}\left(g(t) - g(t_0)\right) = g'(t) = f(t)$

So $F'(t) = f(t)$

- $F(t) = \int_{t_0}^{t^2} f(x)\,dx \Rightarrow F'(t) = \frac{d}{dt}\left(\int_{t_0}^{t^2} f(x)\,dx\right)$

Let $g(x)$ be an antiderivative of $f(x) \Leftrightarrow g'(x) = f(x)$

$\Rightarrow \int_{t_0}^{t^2} f(x)\,dx = g(t^2) - g(t_0)$

$\Rightarrow F'(t) = \frac{d}{dt}\left(g(t^2) - g(t_0)\right) = g'(t^2) = 2t\,f(t^2)$

Problem 3

Use $(h(g(x)))' = h'(g(x)) \cdot g'(x)$

- For $f(x) = \sin(Kx) \Rightarrow f'(x) = \cos(Kx) \cdot (Kx)' = K\cos(Kx)$

$\Rightarrow f''(x) = K \cdot (\cos(Kx))' = K \cdot \left(-\sin(Kx) \cdot (Kx)'\right) = -K^2 \sin(Kx)$

So $f''(x) = -K^2 \sin(Kx)$

- For $f(x) = \cos(Kx) \Rightarrow f'(x) = -\sin(Kx) \cdot K \Rightarrow f''(x) = -K^2 \cos(Kx)$

- For $f'' + 4f = 0$ with $f(0) = 1$ and $f'(0) = 1$

$f(x) = \cos(2x) + \frac{1}{2}\sin(2x) = \cos(2x) + \sin x \cos x$ is a solution.

Problem 4

1. $y'(t) = \sin(4t) + t^2 \Leftrightarrow \frac{dy}{dt} = \sin(4t) + t^2$

$\Leftrightarrow \int dy = \int \sin(4t) + t^2 \, dt \Leftrightarrow y(t) = \frac{t^3}{3} - \frac{\cos(4t)}{4} + c$

with c constant.

2. $y''(t) = e^{2t} \Leftrightarrow \frac{d^2y}{dt^2} = e^{2t} \Leftrightarrow \frac{dy}{dt} = \int e^{2t}\,dt = \frac{e^{2t}}{2} + c$

$\Rightarrow y(t) = \int \frac{e^{2t}}{2} + c \, dt \Leftrightarrow y(t) = \frac{e^{2t}}{4} + ct + d$ with c and d constants.

Problems 5

1. $y'(t) + y(t)^2 \sin t = 0$; note separable equation

$\Leftrightarrow \dfrac{dy}{dt} = -y(t)^2 \sin t \Leftrightarrow \dfrac{\frac{dy(t)}{dt}}{y(t)^2} = -\sin(t)$

$\Leftrightarrow \displaystyle\int \frac{dy}{y(t)^2} = \int -\sin(t)\, dt \Rightarrow -\frac{1}{y(t)} = \cos(t) + c$

$\Leftrightarrow y(t) = \dfrac{-1}{\cos(t) + c}$ with c constant

2. $y'(t) = \dfrac{t+y}{t-y} = \dfrac{1 + \frac{y}{t}}{1 - \frac{y}{t}}$ $(*)$

Let $\dfrac{y(t)}{t} = f(t) \Leftrightarrow y(t) = t \cdot f(t) \Rightarrow y'(t) = t f'(t) + f(t)$

Substitute in $(*)$ and get:

$t f'(t) + f(t) = \dfrac{1 + f(t)}{1 - f(t)} \Leftrightarrow t \cdot f'(t) = \dfrac{1 + f(t)}{1 - f(t)} - f(t) \Leftrightarrow$

$\Leftrightarrow f'(t) = \dfrac{1}{t}\left(\dfrac{1 + f(t) - f(t) + f^2(t)}{1 - f(t)}\right) = \dfrac{1}{t}\left(\dfrac{1 + f^2(t)}{1 - f(t)}\right)$

$\Leftrightarrow \dfrac{f'(t)}{\frac{1 + f^2(t)}{1 - f(t)}} = \dfrac{1}{t} \Leftrightarrow \displaystyle\int \frac{df}{\frac{1 + f^2(t)}{1 - f(t)}} = \int \frac{1}{t}\, dt \Leftrightarrow$

$\Leftrightarrow \displaystyle\int \frac{1 - f(t)}{1 + f^2(t)}\, df = \ln(t) + c$ with c constant

$\Leftrightarrow \arctan(f(t)) - \frac{1}{2}\ln(f^2(t) + 1) = \ln(t) + c$

But $f(t) = \frac{y(t)}{t}$

$\Rightarrow \arctan\left(\frac{y(t)}{t}\right) - \frac{1}{2}\ln\left(\frac{y(t)^2}{t^2}+1\right) = \ln(t) + c$

Problem 6

$y'(t) + a\,y(t) = 0$ and $y(0) = y_0$.

$\Rightarrow \frac{dy}{dt} = -a\,y(t) \Leftrightarrow \int \frac{dy}{y(t)} = \int -a\,dt$

$\Leftrightarrow \ln(y(t)) = -at + c \Leftrightarrow y(t) = e^{-at} \cdot d$ with d constant.

$y(0) = e^0 \cdot d = d = y_0$

$\Rightarrow y(t) = y_0 \cdot e^{-at}$

- If $y_0 \in \mathbb{R}$, $a > 0 \Rightarrow \lim_{t\to\infty} e^{-at} = 0$

  $\Rightarrow \lim_{t\to\infty} y(t) = 0$

- If $y_0 > 0$, $a < 0 \Rightarrow \lim_{t\to\infty} e^{-at} = \infty \Rightarrow \lim_{t\to\infty} y(t) = \infty$
- If $y_0 < 0$, $a < 0 \Rightarrow \lim_{t\to\infty} e^{-at} = \infty \Rightarrow \lim_{t\to\infty} y(t) = -\infty$
- If $y_0 = 0$, $a < 0 \Rightarrow y(t) \equiv 0$ and $\lim_{t\to\infty} y(t) = 0$
- If $a = 0 \Rightarrow y(t) = y_0 \Rightarrow \lim_{t\to\infty} y(t) = y_0$

Homework 2

Problem 1

$$P'(t) = \alpha P(t), \quad \alpha > 0$$

1. $P_0 > 0, \quad P(0) = P_0$

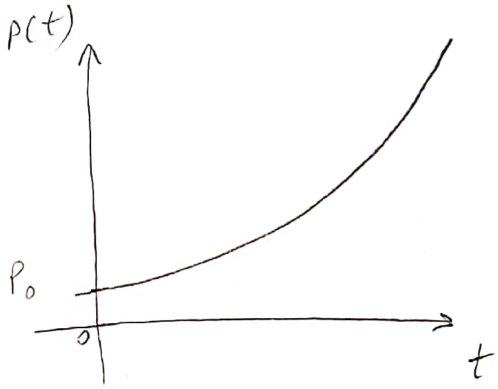
$$\frac{P'(t)}{P(t)} = \alpha \Leftrightarrow \int \frac{dP}{P(t)} = \int \alpha \, dt \Rightarrow \ln|P(t)| = \alpha t + c$$

$$\Rightarrow P(t) = e^{\alpha t} \cdot d \quad \text{with } d \text{ constant}$$

$$P(0) = d = P_0$$

$$\Rightarrow P(t) = P_0 e^{\alpha t}$$

2.



3. $\alpha > 0, \; P_0 > 0$

$$\Rightarrow \lim_{t \to \infty} e^{\alpha t} = \infty \Rightarrow \lim_{t \to \infty} P(t) = \infty$$

Problem 2

$$P'(t) = \alpha P(t) \left(1 - \frac{P(t)}{K}\right)$$

1. $P_0 > 0; \quad P(0) = P_0$

$$\frac{P'(t)}{P(t) - \frac{P(t)^2}{K}} = \alpha \Rightarrow \int \frac{dP}{P(t) - \frac{P(t)^2}{K}} = \int \alpha \, dt \qquad (*)$$

Note: $$\frac{1}{P(t)\left(1 - \frac{P(t)}{K}\right)} = \frac{\frac{1}{K}}{1 - \frac{P(t)}{K}} + \frac{1}{P(t)} = \frac{\cancel{\frac{P(t)}{K}} + 1 - \cancel{\frac{P(t)}{K}}}{P(t)\left(1 - \frac{P(t)}{K}\right)}$$

$$\Rightarrow \int \frac{dP}{P(t)\left(1 - \frac{P(t)}{K}\right)} = \int \frac{\frac{1}{K}}{1 - \frac{P(t)}{K}} dP + \int \frac{1}{P(t)} dP = \int \frac{1}{K - P(t)} dP + \int \frac{1}{P(t)} dP$$

$$\Leftrightarrow \int \frac{dP}{P(t)\left(1-\frac{P(t)}{K}\right)} = \ln\left(K-P(t)\right) + \ln\left(P(t)\right) + c.$$

$\Rightarrow$ equation $(*)$ is equivalent to

$\ln(K-P(t)) + \ln(P(t)) = \alpha t + d.$ with d constant

$$\Leftrightarrow \alpha t + d = \ln(P(t)) - \ln(P(t)-K) = \ln \frac{P(t)}{P(t)-K}$$

$$\Rightarrow \frac{P(t)}{P(t)-K} = e^{\alpha t} \cdot A \quad \text{with } A \text{ constant}$$

$$\Leftrightarrow P(t) = Ae^{\alpha t} \cdot P(t) - Ae^{\alpha t} \cdot K \Leftrightarrow P(t)\left(Ae^{\alpha t}-1\right) = Ae^{\alpha t} \cdot K$$

$$\Leftrightarrow P(t) = \frac{Ae^{\alpha t} \cdot K}{Ae^{\alpha t}-1} = \frac{-Be^{\alpha t} \cdot K}{-Be^{\alpha t}-1} = \frac{Be^{\alpha t} \cdot K}{Be^{\alpha t}+1} \quad \text{for } B = -A$$

Now $P(0) = P_0 = \dfrac{A \cdot K}{A-1} \Leftrightarrow P_0 A - P_0 = A \cdot K \Leftrightarrow A(P_0 - K) = P_0$

$$\Leftrightarrow A = \frac{P_0}{P_0 - K}$$

$$\Rightarrow P(t) = \frac{\frac{P_0}{P_0-K} \cdot e^{\alpha t} \cdot K}{\frac{P_0}{P_0-K} e^{\alpha t} - 1} = \frac{P_0 e^{\alpha t} \cdot K}{P_0 e^{\alpha t} - P_0 + K}$$

$$\Leftrightarrow P(t) = \frac{P_0 K}{P_0 + (K - P_0)e^{-\alpha t}}$$

2.

3. $P(t) = \dfrac{K P_0}{P_0 + (K - P_0) e^{-\alpha t}}$; $\alpha > 0$, $K > 0$, $P_0 > 0$.

$\lim_{t \to \infty} e^{-\alpha t} = 0 \Rightarrow \lim_{t \to \infty} P + (K - P_0) e^{-\alpha t} = P_0$

$\Rightarrow \lim_{t \to \infty} P(t) = K$

As $t \to \infty$, the population increases up to the carrying capacity K ; P(t) does not excede K, and the model shows how population growth is limited by resources available

Problem 3

1. linear, second order ; non homogenous ; constant coefficient is 2

2. non linear, second order

3. linear, third order, homogeneous, constant coefficients: 4, 1 and 0 ; all coefficients are constant.

4. nonlinear, second order

Problem 4

1. $y = y(t)$ solves $0 = a_n(t)\, y^{(n)}(t) + \dots + a_1(t)\, y'(t) + a_0(t)\, y(t)$

Let $g(t) = \lambda y(t)$ for $\lambda \in \mathbb{R}$ constant

$\Rightarrow g'(t) = \lambda y'(t)$ and generally $g^{(m)}(t) = \lambda y^{(m)}(t)$ for $m \in \mathbb{Z}_+^*$

$\Rightarrow f(t) \cdot g^{(m)}(t) = f(t) \cdot \lambda \cdot y^{(m)}(t)$ for any function $f(t)$

and in particular for $a_0(t), \dots, a_n(t)$

$$\left.\begin{aligned} \Rightarrow a_n(t)\, g^{(n)}(t) &= a_n(t)\, y^{(n)}(t) \cdot \lambda \\ &\dots \\ a_1(t)\, g'(t) &= a_1(t)\, y'(t) \cdot \lambda \\ a_0(t)\, g(t) &= a_0(t)\, y(t) \cdot \lambda \end{aligned}\right\} \Rightarrow$$

$\Rightarrow a_n(t)\, g^{(n)}(t) + \dots + a_1(t)\, g'(t) + a_0(t)\, g(t) =$

$= \left(a_n(t)\, y^{(n)}(t) + \dots + a_1(t)\, y'(t) + a_0(t)\, y(t) \right) \cdot \lambda$

$= 0 \cdot \lambda$ using the fact that $y(t)$ solution to (0.1)

$\Rightarrow g(t) = \lambda y(t)$ is also a solution to (0.1) for $\lambda \in \mathbb{R}$

2. $x = x(t)$ and $y = y(t)$ solve (0.1)

Let $g(t) = x(t) + y(t) \Rightarrow g^{(m)}(t) = x^{(m)}(t) + y^{(m)}(t)$ for any $m \in \mathbb{Z}_+^*$

$\Rightarrow f(t) \cdot g^{(m)}(t) = f(t)\left(x^{(m)}(t) + y^{(m)}(t) \right)$ for any function $f(t)$

$\Rightarrow$ in particular for $a_0(t), \dots, a_n(t)$

$$\Rightarrow \left.\begin{aligned} a_0(t)\, \rho(t) &= a_0(t)\,(x(t)+y(t)) \\ a_1(t)\, \rho'(t) &= a_1(t)\,(x'(t)+y'(t)) \\ &\;\;\vdots \\ a_n(t)\, \rho^{(n)}(t) &= a_n(t)\,(x^{(n)}(t)+y^{(n)}(t)) \end{aligned}\right\} \Rightarrow$$

$\Rightarrow a_n(t)\rho^{(n)}(t) + \dots + a_1(t)\rho'(t) + a_0(t)\rho(t) =$

$= a_n(t)\left(x^{(n)}(t)+y^{(n)}(t)\right) + \dots + a_1(t)\left(x'(t)+y'(t)\right) + a_0(t)(x(t)+y(t)) =$

$= \left(a_n(t)\, x^{(n)}(t) + \dots + a_1(t)\, x'(t) + a_0(t)\, x(t)\right) + \left(a_n(t)\, y^{(n)}(t) + \dots + a_1(t)\, y'(t) + a_0(t)\, y(t)\right) = 0+0$ using the fact that $x(t)$ and $y(t)$ are solutions to (0.1)

$\Rightarrow a_n(t)\rho^{(n)}(t) + \dots + a_1(t)\rho'(t) + a_0(t)\rho(t) = 0$

$\Rightarrow \rho(t) = x(t)+y(t)$ is also a solution to (0.1)

Problem 5

1. $x(t)$ and $y(t)$ solutions to (0.2) $\Leftrightarrow$ $a_n(t)\, x^{(n)}(t) + \dots + a_1(t)\, x'(t) + a_0(t)\, x(t) = b(t)$ and $a_n(t)\, y^{(n)}(t) + \dots + a_1(t)\, y'(t) + a_0(t)\, y(t) = b(t)$

$\Rightarrow a_n(t)\left(x^{(n)}(t) - y^{(n)}(t)\right) + \dots + a_1(t)(x'(t)-y'(t)) + a_0(t)(x(t)-y(t)) = b(t) - b(t) = 0. \quad (*)$

Now let $g(t) = x(t) - y(t) \Rightarrow g^{(m)}(t) = x^{(m)}(t) - y^{(m)}(t)$ $\forall m \in \mathbb{Z}_+$ $\Rightarrow$ in particular $x^{(n)}(t) - y^{(n)}(t) = g^{(n)}(t), \dots$

$\dots\; x'(t) - y'(t) = g'(t)$

$\Rightarrow$ equation $(*)$ is equivalent to:

$a_n(t)\, g^{(n)}(t) + \dots + a_1(t)\, g'(t) + a_0(t)\, g(t) = 0.$

$\Leftrightarrow g(t) = x(t) - y(t)$ is a solution for (0.1)

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2. $x(t)$ is a solution for (0.2)

$\Leftrightarrow a_n(t)\, x^{(n)}(t) + \dots + a_1(t)\, x'(t) + a_0(t)\, x(t) = b(t)$

$y(t)$ is a solution for (0.1)

$\Leftrightarrow a_n(t)\, y^{(n)}(t) + \dots + a_1(t)\, y'(t) + a_0(t)\, y(t) = 0$

$\Rightarrow a_n(t)\left(x^{n}(t) + y^{n}(t)\right) + \dots + a_1(t)\left(x'(t) + y'(t)\right) + a_0(t)\left(x(t)+y(t)\right)$

$= b(t) + 0 = b(t)$ $\quad (**)$

Similarly to part 1, let $g(t) = x(t) + y(t)$ and get

$g^{(n)}(t) = x^{(n)}(t) + y^{(n)}(t)$; ... ; $g'(t) = x'(t) + y'(t)$;

$g(t) = x(t) + y(t)$

$\Rightarrow$ substitute in $(**)$ and get

$a_n(t)\, g^{(n)}(t) + \dots + a_1(t)\, g'(t) + a_0(t)\, g(t) = b(t)$

$\Rightarrow g(t) = x(t) + y(t)$ is a solution for (0.2)

Problem 6

1. $T(t) = T_0$; $u(0) = u_0$ $\Rightarrow \frac{u'(t)}{u(t) - T_0} = -K \Rightarrow$

$\Rightarrow u'(t) = -K\left(u(t) - T_0\right)$

$\Rightarrow \int \frac{du}{u(t) - T_0} = \int -K\, dt \Leftrightarrow \ln|u(t) - T_0| = -Kt + C$ with C const.

$\Rightarrow |u(t) - T_0| = A e^{-Kt} \Rightarrow u(t) = A e^{-Kt} + T_0$

Note that if $u(t) > T_0$, $A > 0$ and if $u(t) \le T_0$, $A \le 0$

Now $u(0) = A + T_0 = u_0 \Rightarrow A = u_0 - T_0$

$\Rightarrow u(t) = (u_0 - T_0)\, e^{-Kt} + T_0$

$\lim_{t \to \infty} e^{-Kt} = 0$ as $K > 0$

$\Rightarrow \lim_{t \to \infty} u(t) = T_0$

Graph if $u_0 - T_0 = A > 0$

We want to solve

$$u'(t) + ku(t) = k(T_0 + T_1 \cos \omega t).$$

This is a first order linear equation. The integrating factor is a solution to

$$\mu' = k\mu.$$

This is a separable equation and we find that

$$\mu(t) = e^{kt}.$$

(The general solution is Ce^{kt} but for the integrating factor we only need to find any non-zero solution.) We multiply the original equation by μ :

$$\mu(t)u' + k\mu(t)u = k\mu(t)(T_0 + T_1 \cos \omega t).$$

The left-hand side can be rewritten as

$$(\mu u)' = \mu(t)(T_0 + T_1 \cos \omega t),$$

so after integrating and dividing by μ,

$$\begin{aligned} u(t) &= \frac{k}{\mu(t)} \int \mu(t)(T_0 + T_1 \cos \omega t) \mathrm{d}t \\ &= ke^{-kt} \int e^{kt}(T_0 + T_1 \cos \omega t) \mathrm{d}t \\ &= kT_0 e^{-kt} \int e^{kt} \mathrm{d}t + kT_1 e^{-kt} \int e^{kt} \cos \omega t \mathrm{d}t. \end{aligned}$$

Integrating by substitution $s = kt$,

$$\int e^{kt} \mathrm{d}t = \frac{1}{k} e^{kt} + C.$$

Integrating by parts twice

$$\begin{aligned} \int e^{kt} \cos \omega t \mathrm{d}t &= \frac{1}{k} e^{kt} \cos \omega t + \omega \int e^{kt} \sin \omega t \mathrm{d}t \\ &= \frac{1}{k} e^{kt} \cos \omega t + \frac{1}{k} \omega e^{kt} \sin \omega t - \omega^2 \int e^{kt} \cos \omega t \mathrm{d}t. \end{aligned}$$

Solving this equation for the integral $\int e^{kt} \cos \omega \mathrm{d}t$, we get

$$\int e^{kt} \cos \omega t \mathrm{d}t = \frac{e^{kt}}{k(1+\omega^2)} (\cos \omega t + \omega \sin \omega t) + C.$$

Putting all this together, we get

$$u(t) = T_0 + \frac{T_1}{1+\omega^2} (\cos \omega t + \omega \sin \omega t) + Ce^{-kt},$$

for some constant C (this is not the same C as in the previous equations, we just keep renaming whatever constant appear as C). Note that using trigonometric identities this can be written as

$$u(t) = T_0 + T_1 \sqrt{1+\omega^2} \cos(\omega t - \theta) + Ce^{-kt}$$

for an angle θ that can be computed from ω. Therefore, as $t \to \infty$ the function $u(t)$ is asymptotic to the function

$$T_0 + T_1 \sqrt{1+\omega^2} \cos(\omega t - \theta)$$

which oscillates around T_0 with the same frequency as the temperature of the surroundings $T(t) = T_0 + T_1 \cos \omega t$, but phase shifted by θ.

Homework 3

Problem 1

f, g continuous at x_0 $\Rightarrow$ $\begin{cases} \lim_{x \to x_0} f(x) = f(x_0) \\ \lim_{x \to x_0} g(x) = g(x_0) \end{cases}$

$$(f+g)(x_0) = f(x_0) + g(x_0) = \lim_{x \to x_0} f(x) + \lim_{x \to x_0} g(x) =$$

$$= \lim_{x \to x_0} (f+g)(x) \Leftrightarrow (f+g)(x_0) = \lim_{x \to x_0} (f+g)(x)$$

$\Leftrightarrow$ $f+g$ is also continuous at x_0

$$(f \cdot g)(x_0) = f(x_0) \cdot g(x_0) = \left(\lim_{x \to x_0} f(x)\right) \cdot \left(\lim_{x \to x_0} g(x)\right) = \lim_{x \to x_0} (f \cdot g)(x)$$

$\Leftrightarrow$ $f \cdot g$ is also continuous at x_0.

Problem 2

- $f(x,y) = x^2 y + y^3$

  $\frac{df}{dx} = 2xy$; $\frac{df}{dy} = x^2 + 3y^2$

- $f(x,y) = \sin(xy)$

  $\frac{df}{dx} = y \cos(xy)$; $\frac{df}{dy} = x \cos(xy)$

- $f(x,y) = e^{(x-y)^2}$

  $\frac{df}{dx} = e^{(x-y)^2} \cdot (2x - 2y)$; $\frac{df}{dy} = e^{(x-y)^2} (2y - 2x)$

Problem 3

Theorem: Consider initial value problem $\begin{cases} y' = f(t,y) \\ y(t_0) = y_0 \end{cases}$

If f and $\partial_y f$ are both continuous in a neighborhood of (t_0, y_0), then there exists a unique solution $y = y(t)$ defined for t from the interval $(t_0 - \varepsilon, t_0 + \varepsilon)$ for some $\varepsilon > 0$.

In this problem: $\begin{cases} y' = e^{(y-t)^2} \\ y(0) = 1 \end{cases}$

Clearly $f(y,t) = e^{(y-t)^2}$ continuous

$\frac{df}{dy} = 2e^{(y-t)^2}(y-t)$ continuous

$\Rightarrow$ We can use the existence and uniqueness theorem

$\Rightarrow$ the initial value problem $\begin{cases} y' = e^{(y-t)^2} \\ y(0) = 1 \end{cases}$ has a unique solution defined for $(t_0 - \varepsilon ; t_0 + \varepsilon)$ with $t_0 = 0$.

($\Leftrightarrow$ for $(-\varepsilon, \varepsilon)$ or in a neighborhood of $t = 0$)

Problem 4

$\begin{cases} y' = \sqrt{y} \\ y(0) = 0 \end{cases}$

$\Rightarrow f(y,t) = \sqrt{y}$ continuous for $y \geq 0$.

$\frac{df}{dy} = \frac{1}{2\sqrt{y}}$ continuous for $y > 0$.

Now $\frac{df}{dy}$ not continuous at $y(t_0) = y(0) = 0 \Rightarrow$ the uniqueness theorem does not apply

- One solution : $y(t) = 0$
- Second solution : $\frac{\frac{dy}{dt}}{\sqrt{y}} = 1 \Leftrightarrow \int \frac{1}{\sqrt{y}} dy = \int dt$

$$\Leftrightarrow \quad 2\sqrt{y} = t + c \quad \Leftrightarrow \quad y = \frac{(t+c)^2}{4}$$

$$y(0) = 0 = \frac{c^2}{4} \Rightarrow c = 0$$

$$\Rightarrow y(t) = \frac{t^2}{4}$$

Problem 5

$\alpha \in (0,1)$; f continuous ; $|f(x) - f(y)| \le \alpha |x - y|$

Let $x_0 \in \mathbb{R}$ and $x_n = f(x_{n-1})$ for $n > 0$, n integer

For any such n $|x_n - x_{n-1}| = |f(x_{n-1}) - f(x_{n-2})| \le$

$\le \alpha |x_{n-1} - x_{n-2}|$ using $(0,1)$

$$\Rightarrow \begin{cases} |x_n - x_{n-1}| \le \alpha |x_{n-1} - x_{n-2}| \\ |x_{n-1} - x_{n-2}| \le \alpha |x_{n-2} - x_{n-3}| \\ \dots \\ |x_2 - x_1| \le \alpha |x_1 - x_0| \end{cases}$$

$$\Rightarrow |x_n - x_{n-1}| \le \alpha^{n-1} |x_1 - x_0|$$

Now $x_n = (x_n - x_{n-1}) + (x_{n-1} - x_{n-2}) + \dots + (x_1 - x_0) + x_0$

$$|x_n| = |(x_n - x_{n-1}) + (x_{n-1} - x_{n-2}) + \dots + (x_1 - x_0) + x_0|$$

Using triangle inequality:

$|x_n| \leq |x_n - x_{n-1}| + \dots + |x_1 - x_0| + |x_0|$

Since $|x_n - x_{n-1}| \leq \alpha^{n-1} |x_1 - x_0|$ $\forall$ n positive integer

$\Rightarrow |x_n| \leq \alpha^{n-1} |x_1 - x_0| + \alpha^{n-2} |x_1 - x_0| + \dots + |x_1 - x_0| + |x_0|$

$\Leftrightarrow |x_n| \leq |x_1 - x_0| \cdot \sum_{j=0}^{n-1} \alpha^j + |x_0|$

$\sum_{j=0}^{\infty} \alpha^j = \frac{1}{1-\alpha}$ since $0 < \alpha < 1$

$\Rightarrow |x_1 - x_0| \cdot \sum_{j=0}^{\infty} \alpha^j + |x_0| = \frac{|x_1 - x_0|}{1-\alpha} + |x_0|$ convergent

$\Rightarrow$ x_n also converges as $n \to \infty$

Let $\lim_{n\to\infty} x_n = x$.

As f is continuous and $x_n = f(x_{n-1})$ as defined in step 1

$\Rightarrow \lim_{n\to\infty} x_n = \lim_{n\to\infty} f(x_{n-1}) \Leftrightarrow x = f(x)$

Now suppose there is y such that $f(y) = y$ and $x \neq y$.

Since $f(x) = x$

$f(x) - f(y) = x - y$. $\Rightarrow |f(x) - f(y)| = |x - y|$ (*)

But from (0.1): $|f(x) - f(y)| \leq \alpha |x - y| < |x - y|$ since $0 < \alpha < 1$, which contradicts (*)

$\Rightarrow$ there is only one solution to $f(x) = x$.

Homework 4

Problem 1

$$\begin{cases} ax+by=e \\ cx+dy=f \end{cases}$$

For $ad-bc \neq 0$

$$\left.\begin{array}{l} ax+by=e \quad | \cdot c \Rightarrow acx+bcy=ec \\ cx+dy=f \quad | \cdot a \Rightarrow acx+ady=af \end{array}\right\} \Rightarrow y(ad-bc)=af-ec$$

$$\Rightarrow y=\frac{af-ec}{ad-bc}.$$

$$\left.\begin{array}{l} ax+by=e \quad | \cdot d \Rightarrow adx+bdy=ed \\ cx+dy=f \quad | \cdot b \Rightarrow bcx+bdy=bf \end{array}\right\} \Rightarrow x(ad-bc)=ed-bf$$

$$\Rightarrow x=\frac{ed-bf}{ad-bc}$$

For $ad-bc=0$

1) no solution: $a=c$, $b=d$, $e\neq f$

example: $a=c=2$; $b=d=1$; $e=5$, $f=6$.

$$\begin{cases} 2x+y=5 \\ 2x+y=6 \end{cases}$$

2) infinitely many solutions: $c=\alpha a$, $d=\alpha b$, $f=\alpha e$ for $\alpha\in\mathbb{R}^*$

example: $a=1$, $b=2$, $e=1$, $\alpha=3$.

$$\begin{cases} x+2y=1 \\ 3x+6y=3 \end{cases}$$

Problem 2

1) $x = y' \to x' = y''$

$y'' + p(t)y' + q(t)y + r(t) = 0 \iff x' + p(t)x + q(t)y + r(t) = 0$

$\iff x' = -p(t)x - q(t)y - r(t)$

The system becomes $\begin{cases} y' = x \\ x' = -p(t)x - q(t)y - r(t) \end{cases}$

2) $x = y' \Rightarrow x' = y''$

$y'' + p(t)y'y + q(t)(y')^2 + r(t)y = 0 \iff$

$\iff x' + p(t)xy + q(t)x^2 + r(t)y = 0$

$\iff x' = -p(t)xy - q(t)x^2 - r(t)y.$

The system becomes $\begin{cases} y' = x \\ x' = -p(t)xy - q(t)x^2 - r(t)y \end{cases}$

Problem 3

Use the existence and uniqueness theorem (Theorem 3 in lecture 9 pdf file)

1. $y'' + t^2y' + y^2 = 0$; $y(0) = 1$; $y'(0) = 0$.

$y'' = -t^2y' - y^2 = f(t, y, y')$

$\Rightarrow \frac{df}{dy} = -2y$; $\frac{df}{dy'} = -t^2$ both continuous in a neighborhood of $(t_0, y_0, y_0') = (0, 1, 0)$

$\Rightarrow$ the system admits one unique solution.

2. $y'' + t^2y' + y^2 = 0$; $y(0) = 1$,

As $y'(t_0)$ is not defined $\Rightarrow$ not enough initial conditions $\to$

$\Rightarrow$ the system does not admit one unique solution

3. $y'' + t^2 y' + y^{\frac{1}{2}} = 0$; $y(0) = 1$; $y'(0) = 0$

$\Leftrightarrow y'' = -t^2 y' - y^{\frac{1}{2}} = f(t, y, y')$

$\Rightarrow \frac{df}{dy} = -\frac{1}{2} y^{-\frac{1}{2}} = -\frac{1}{2\sqrt{y}}$ $\Rightarrow$ continuous around $(t_0, y_0, y_0') = (0, 1, 0)$ as the only discontinuities are for $y \leq 0$

$\frac{df}{dy'} = -t^2$ $\Rightarrow$ continuous $\Rightarrow$ in particular around $(t_0, y_0, y_0') = (0, 1, 0)$

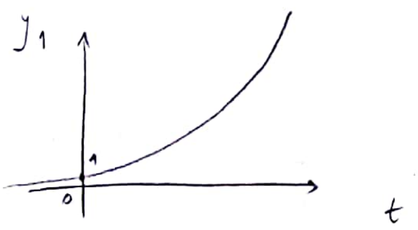
$\Rightarrow$ the system admits one unique solution

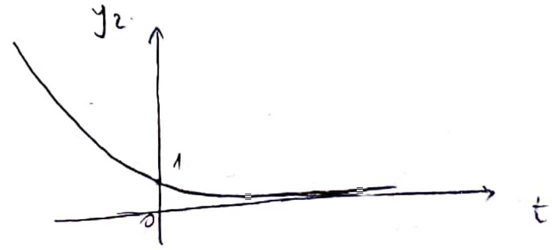
Problem 4

$K > 0$.

1) $y_1(t) = e^{\sqrt{K}t}$ $\Rightarrow y_1'(t) = \sqrt{K} e^{\sqrt{K}t}$ $\Rightarrow y_1''(t) = K e^{\sqrt{K}t} = Ky$.

$y_2(t) = e^{-\sqrt{K}t}$ $\Rightarrow y_2'(t) = -\sqrt{K} e^{-\sqrt{K}t}$ $\Rightarrow y_2''(t) = K e^{-\sqrt{K}t} = Ky$.





$y = c_1 y_1 + c_2 y_2$; $y(0) = 0$; $y'(0) = 1$

$y(0) = c_1 y_1(0) + c_2 y_2(0) = c_1 + c_2 = 0 \Rightarrow c_2 = -c_1$

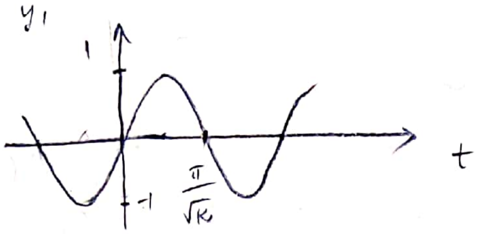
$y'(0) = c_1 y_1'(0) + c_2 y_2'(0) = c_1 \sqrt{K} - c_2 \sqrt{K} = 1$

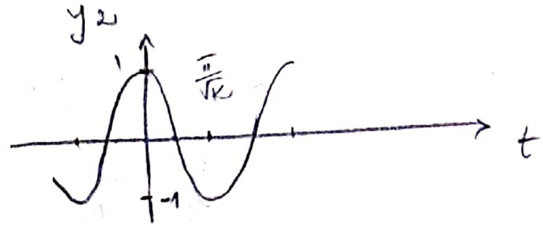
$\Leftrightarrow 2 c_1 \sqrt{K} = 1 \Rightarrow c_1 = \frac{1}{2\sqrt{K}}$; $c_2 = \frac{-1}{2\sqrt{K}}$

$\Rightarrow y(t) = \frac{1}{2\sqrt{K}} \left(e^{\sqrt{K}t} - e^{-\sqrt{K}t} \right)$

2) $y_1(t) = \sin(\sqrt{K}t) \Rightarrow y_1'(t) = \sqrt{K}\cos(\sqrt{K}t) \Rightarrow y_1''(t) = -K\sin(\sqrt{K}t) = -Ky$

$y_2(t) = \cos(\sqrt{K}t) \Rightarrow y_2'(t) = -\sqrt{K}\sin(\sqrt{K}t) \Rightarrow y_2''(t) = -K\cos(\sqrt{K}t) = -Ky$





$y = c_1 y_1 + c_2 y_2$; $y(0) = 0$, $y'(0) = 1$

$y(0) = c_1 y_1(0) + c_2 y_2(0) = c_2 = 0$

$y'(0) = c_1 y_1'(0) + c_2 y_2'(0) = c_1\sqrt{K} = 1 \Rightarrow c_1 = \frac{1}{\sqrt{K}}$

$\Rightarrow y(t) = \frac{1}{\sqrt{K}}\sin(\sqrt{K}t)$

Problem 5

1) $W[y_1, y_2](t) = y_1(t)y_2'(t) - y_1'(t)y_2(t)$

$= e^{\sqrt{K}t}\cdot(-\sqrt{K}e^{-\sqrt{K}t}) - \sqrt{K}e^{\sqrt{K}t}\cdot e^{-\sqrt{K}t}$

$= -\sqrt{K} - \sqrt{K} = -2\sqrt{K} \neq 0$ as $K > 0$, so $K \neq 0$

2) $W[y_1, y_2](t) = y_1(t)y_2'(t) - y_1'(t)y_2(t)$

$= \sin(\sqrt{K}t)\cdot(-\sqrt{K}\sin(\sqrt{K}t)) - \sqrt{K}\cos(\sqrt{K}t)\cdot\cos(\sqrt{K}t)$

$= -\sqrt{K}\cdot\sin^2(\sqrt{K}t) - \sqrt{K}\cdot\cos^2(\sqrt{K}t)$

$= -\sqrt{K} \neq 0$ as $K > 0$, so $K \neq 0$

Homework 5

Problem 1

- $5+i-(2+3i) = 3-2i$
- $\dfrac{1+2i}{3-i} = \dfrac{(1+2i)(3+i)}{10} = \dfrac{3+6i+i-2}{10} = \dfrac{1}{10} + \dfrac{7}{10}i$
- $i^{-1} = \dfrac{1}{i} = \dfrac{i}{-1} = -i$
- $(1+i)^3 = 1^3 + 3\cdot 1\cdot i^2 + 3\cdot 1^2\cdot i + i^3 = 1-3+3i-i = -2+2i$
- $\overline{4+2i} = 4-2i$

Problem 2

$\lambda^2 + 2\lambda + 5 = 0 \Rightarrow \lambda_{1,2} = \dfrac{-2 \pm \sqrt{4-4\cdot 5}}{2} = \dfrac{-2 \pm 4i}{2}$

$\Rightarrow \lambda_1 = -1+2i; \ \lambda_2 = -1-2i$

Problem 3

$y'' - 3y' - 4y = 0 \Rightarrow$ characteristic polynomial $\lambda^2 - 3\lambda - 4 = 0$ and $\Delta = 3^2 - 4\cdot(-4) > 0$.

$\Rightarrow \lambda_{1,2} = \dfrac{3 \pm 5}{2} \Rightarrow \lambda_1 = 4, \ \lambda_2 = -1$

$\Rightarrow$ solution has the form $y(t) = c_1 y_1(t) + c_2 y_2(t)$ with

$y_1(t) = e^{\lambda_1 t}, \ y_2(t) = e^{\lambda_2 t}$

$\Leftrightarrow y(t) = c_1 e^{4t} + c_2 e^{-t}$

Problem 4

$y'' + 2y' + y = 0 \Rightarrow$ characteristic polynomial $\lambda^2 + 2\lambda + 1 = 0$

$\Rightarrow \lambda_1 = \lambda_2 = -1 = \lambda$, so double roots

$\Rightarrow y(t) = c_1 e^{\lambda t} + c_2 t e^{\lambda t}$

$\Leftrightarrow y(t) = c_1 e^{-t} + c_2 t e^{-t}$

Problem 5

$y'' + 2y' + 5y = 0 \to$ characteristic polynomial $\lambda^2 + 2\lambda + 5 = 0$

$\Rightarrow \lambda_{1,2} = \dfrac{-2 \pm \sqrt{4 - 4\cdot 5}}{2} = \dfrac{-2 \pm 4i}{2} \Rightarrow \lambda_2 = -1 - 2i, \quad \lambda_1 = -1 + 2i$

and note $2^2 - 4\cdot 5 = p^2 - 4\cdot q < 0$.

$\to y_1(t) = e^{\lambda_1 t} = e^{-1\cdot t} \cdot e^{+2it} = e^{-t}\left(\cos(2t) + i\sin(2t)\right)$

$\to y(t) = c_1 e^{-t}\cos(2t) + c_2 e^{-t}\sin(2t)$

Problem 6

$p(\lambda) = a_n\lambda^n + \dots + a_1\lambda + a_0$.

Let z and w be two complex numbers with

$z = a + bi, \quad w = c + di$

$\Rightarrow \bar{z} = a - bi, \quad \bar{w} = c - di$

$z + w = a + c + (b+d)i \Rightarrow \overline{z+w} = a + c - (b+d)i$

$\bar{z} + \bar{w} = a + c - bi - di = \overline{z+w}$

$z \cdot w = ac + adi + cbi - bd = ac - bd + (ad+cb)i$

$\overline{z\cdot w} = ac - bd - (ad+cb)i$

$\bar{z}\cdot\bar{w} = (a-bi)(c-di) = ac - adi - cbi - bd = ac - bd - (ad+cb)i = \overline{z\cdot w}$

$\Rightarrow$ For any $z, w \in \mathbb{C}$: $\overline{z+w} = \bar{z} + \bar{w}$; $\overline{z \cdot w} = \bar{z} \cdot \bar{w}$

Now let λ be such that $p(\lambda) = 0$.

$\Leftrightarrow$ $p(\lambda) = a_n \lambda^n + \dots + a_1 \lambda + a_0 = 0$

$\overline{p(\lambda)} = \overline{a_n \lambda^n + \dots + a_1 \lambda + a_0} = 0$

$\Leftrightarrow$ $\overline{a_n \lambda^n} + \dots + \overline{a_1 \lambda} + \overline{a_0} = 0$

Now $a_i \in \mathbb{R}$; $i = \overline{0, n}$ $\Rightarrow$ $\overline{a_i} = a_i$

$\Rightarrow$ $\overline{p(\lambda)} = a_n \overline{\lambda^n} + \dots + a_1 \bar{\lambda} + a_0 = a_n \overline{\lambda^n} + \dots + a_1 \bar{\lambda} + a_0 = 0$

Since $\overline{z \cdot w} = \bar{z} \cdot \bar{w}$ $\forall$ $z, w \in \mathbb{C}$ $\Rightarrow$ $\overline{\lambda^k} = (\bar{\lambda})^k$ $\forall k \in \mathbb{Z}_+^*$

$\Rightarrow$ $\overline{p(\lambda)} = a_n (\bar{\lambda})^n + \dots + a_1 (\bar{\lambda}) + a_0 = 0$ $\Big\}$ $\Rightarrow$

But $p(\bar{\lambda}) = a_n (\bar{\lambda})^n + \dots + a_1 (\bar{\lambda}) + a_0$

$\Rightarrow$ $p(\bar{\lambda}) = 0$, so $\bar{\lambda}$ is also a root of p when λ is a root of p.

Problem 7

Use induction for this solution

$n=1$:

$(\cos(x) + i \sin(x))^1 = \cos(1 \cdot x) + i \sin(1 \cdot x)$

Now suppose $(\cos(x) + i \sin(x))^m = \cos(mx) + i \sin(mx)$ for some m integer

$\Rightarrow$ $(\cos x + i \sin x)(\cos x + i \sin x)^m = (\cos x + i \sin x)^{m+1} =$

$= (\cos x + i \sin x)(\cos(mx) + i \sin(mx))$ $(\Rightarrow)$

③

$\Leftrightarrow (\cos x + i \sin x)^{m+1} = \cos x \cdot \cos(mx) - \sin x \sin(mx) + i(\sin x \cos(mx) +$
$+ \sin(mx)\cos x)$ $\quad (*)$

Recall $\cos(a+b) = \cos a \cos b - \sin a \sin b$
$\sin(a+b) = \sin a \cos b + \sin b \cos a$

$\Rightarrow \cos((m+1)x) = \cos(mx + x) = \cos(mx)\cos x - \sin x \sin(mx)$
$\sin((m+1)x) = \sin(mx)\cos x + \sin x \cos(mx)$

Substitute in $(*)$ and get:

$(\cos x + i \sin x)^{m+1} = \cos((m+1)x) + i \sin((m+1)x)$

$\Rightarrow$ the equation $(\cos(x) + i\sin(x))^n = \cos(nx) + i \sin(nx)$ holds for $n=1$ and for $n = m+1$ if it holds for $n = m$

$\Rightarrow$ using induction $(\cos(x) + i\sin(x))^n = \cos(nx) + i\sin(nx)$ $\forall$ n integer

Now for $n = 3$: $(\cos x + i \sin x)^3 = \cos(3x) + i \sin(3x)$

$(\cos x + i \sin x)^3 = (\cos x)^3 + 3(\cos x)^2 \cdot i \sin x + 3\cos x \cdot (i \sin x)^2 + (i \sin x)^3$

(from $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$)

$\Leftrightarrow (\cos x + i \sin x)^3 = (\cos x)^3 + i \cdot 3(\cos x)^2 \sin x - 3\cos x(\sin x)^2 - i(\sin x)^3$

$= (\cos x)^3 - 3\cos x(\sin x)^2 + i(3(\cos x)^2 \sin x - (\sin x)^3)$

$(\cos x + i \sin x)^3 = \cos(3x) + i \sin(3x)$

$\Rightarrow Re(\cos(3x) + i \sin(3x)) = Re[(\cos x)^3 - 3\cos x(\sin x)^2 + i(3(\cos x)^2 \sin x - (\sin x)^3)]$

$\Leftrightarrow \cos(3x) = (\cos x)^3 - 3\cos x(\sin x)^2$

Homework 6

Problem 1

$y'' + 3y = 3t^2 - 1$

Let $y(t) = at^2 + bt + c \Rightarrow y'(t) = 2at + b \Rightarrow y''(t) = 2a$

$y'' + 3y = 2a + 3at^2 + 3bt + 3c = 3t^2 - 1$

$\Rightarrow 3a = 3 \,;\, 3b = 0,\ 2a + 3c = -1$

$\Leftrightarrow a = 1,\ b = 0,\ c = -1$

$\Rightarrow y = t^2 - 1$ particular solution

Problem 2

$y'' - y' = te^{2t}$

Look for $y = g(t)\, e^{\alpha t}$ with $g(t) = at + b$.

$\Leftrightarrow y = (at+b)e^{\alpha t} \Rightarrow y' = ae^{\alpha t} + (at+b)\cdot \alpha e^{\alpha t}$

$\Rightarrow y'' = a\alpha e^{\alpha t} + a\alpha e^{\alpha t} + (at+b)\alpha^2 e^{\alpha t}$

$y'' - y' = 2a\alpha e^{\alpha t} + (at+b)\alpha^2 e^{\alpha t} - ae^{\alpha t} - (at+b)\alpha e^{\alpha t}$

$= 2a\alpha e^{\alpha t} - ae^{\alpha t} + b\alpha^2 e^{\alpha t} + a\alpha^2 te^{\alpha t} - a\alpha te^{\alpha t} - b\alpha e^{\alpha t}$

$= e^{\alpha t}\left(2a\alpha - a + b\alpha^2 - b\alpha\right) + te^{\alpha t}\left(a\alpha(\alpha - 1)\right) = te^{2t}$

$\Rightarrow \alpha = 2,\quad a\alpha(\alpha-1) = 1 \,;\quad 2a\alpha - a + b\alpha^2 - b\alpha = 0$

$\Rightarrow a = \frac{1}{2}$ and $b(\alpha^2 - \alpha) = a - 2a\alpha \Leftrightarrow b = \frac{a - 2a\alpha}{\alpha^2 - \alpha} = \frac{\frac{1}{2} - \cancel{2}\cdot\frac{1}{\cancel{2}}\cdot 2}{4-2}$

$\Leftrightarrow b = \frac{-\frac{3}{2}}{2} = \frac{-3}{4}.$

$\Rightarrow y = \left(\frac{1}{2}t - \frac{3}{4}\right)\cdot e^{2t}$ particular solution

Problem 3

First consider $y'' + 2y = te^{it}$ and $y_c = (a+bt)e^{it}$

$\Rightarrow y_c' = e^{it}(ia + ibt + b) \Rightarrow y_c'' = -e^{it}(a + bt - 2ib)$

$$y_c'' + 2y_c = e^{it}(-a - bt + 2ib) + 2e^{it}(a+bt) =$$
$$= e^{it}(-a + 2ib + 2a) + e^{it}(-bt + 2bt) = te^{it}.$$

$$\Rightarrow \begin{cases} bt = t \Rightarrow b = 1 \\ a + 2ib = 0 \Rightarrow a = -2i \end{cases}$$

$\Rightarrow y_c = (t - 2i)e^{it}$

Now $t\cos t = Re(te^{it}) \Rightarrow y(t) = Re(y_c)$ is a solution for (0.1)

$y_c(t) = (t-2i)e^{it} = (t-2i)(\cos t + i\sin t) = t\cos t - 2i\cos t + it\sin t +$
$+ 2\sin t = t\cos t + 2\sin t + i(-2\cos t + t\sin t)$

$\Rightarrow Re(y_c(t)) = t\cos t + 2\sin t = y(t)$ particular solution for (0.1)

Problem 4

$my'' + Ky = 0 \Leftrightarrow y'' + \frac{K}{m}y = 0 \Rightarrow$ characteristic polynomial is:

$\lambda^2 + \frac{K}{m} = 0 \Rightarrow \lambda_{1,2} = \pm i\sqrt{\frac{K}{m}}$

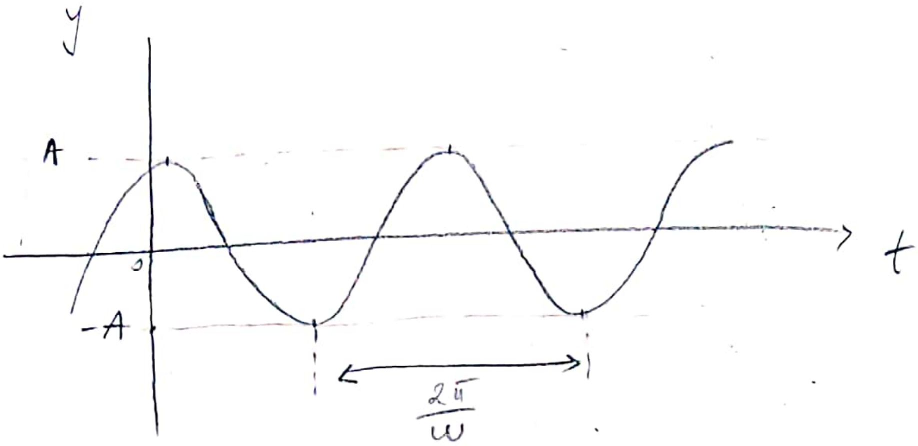
Let $w_0 = \sqrt{\frac{K}{m}} \Rightarrow \begin{cases} y_1 = e^{iw_0t} = \cos(w_0t) + i\sin(w_0t) \\ y_2 = e^{-iw_0t} \end{cases}$

$\Rightarrow y = c_1\cos(w_0t) + c_2\sin(w_0t)$ (*)

Now let $A = \sqrt{c_1^2 + c_2^2} \Rightarrow c_1 = A\cos\theta_1$ and $c_2 = A\sin\theta_1$ for some θ_1

$\Rightarrow$ in (*): $y = A\cos\theta_1\cos(w_0t) + A\sin\theta_1\sin(w_0t)$

$\Leftrightarrow y = A\cos(w_0 t - \theta_1) \Rightarrow y = A\cos(w_0 t + \theta)$ for $\theta = -\theta_1$



A : amplitude ; lower A ⟺ "smaller peaks"

w_0 : frequency ⟺ how often peaks repeat
lower w_0 ⟺ peaks are more distant in time

θ : phase ; describes how the sine function is shifted in time

Homework 7

Problem 1

$y''' - y'' - 2y' = 0$

Observe homogeneous equation, 3^{rd} order, with characteristic equation.

$r^3 - r^2 - 2r = 0 \iff r(r^2 - r - 2) = 0 \iff r(r+1)(r-2) = 0$

$\Rightarrow r_1 = 0;\ r_2 = -1;\ r_3 = 2$; all real

$\Rightarrow y(t) = c_1 e^{0t} + c_2 e^{-1t} + c_3 e^{2t}$

$\iff y(t) = c_1 + c_2 e^{-t} + c_3 e^{2t}$

Problem 2

$y''' - 8y = 0 \Rightarrow$ characteristic equation: $\lambda^3 - 8 = 0 \Rightarrow$ for $\lambda = 2e^{i\theta}$

$8e^{i \cdot 3\theta} - 8 = 0 \iff e^{3i\theta} = 1 \Rightarrow \theta_1 = 2\pi,\ \theta_2 = 2\pi - \frac{2\pi}{3};\ \theta_3 = 2\pi + \frac{2\pi}{3}$

$\iff \theta_1 = 2\pi,\ \theta_2 = \frac{4\pi}{3},\ \theta_3 = \frac{8\pi}{3}$

$\Rightarrow \lambda_1 = 2 \cdot 1 = 2;\ \lambda_2 = 2 \cdot \left(-\frac{1}{2} - \frac{i\sqrt{3}}{2}\right) = -1 - i\sqrt{3};$

$\lambda_3 = 2 \cdot \left(-\frac{1}{2} + \frac{i\sqrt{3}}{2}\right) = -1 + i\sqrt{3}$

$\Rightarrow y(t) = c_1 e^{2t} + c_2 e^{-1} \cos(\sqrt{3}t) + c_3 e^{-1} \sin(\sqrt{3}t)$

Problem 3

$$\vec{x} = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}, \quad \vec{y} = \begin{pmatrix} -1 \\ 0 \\ 4 \end{pmatrix}$$

$$3\vec{x} - 2\vec{y} = 3 \cdot \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} - 2 \begin{pmatrix} -1 \\ 0 \\ 4 \end{pmatrix} = \begin{pmatrix} 3 \\ 9 \\ 6 \end{pmatrix} - \begin{pmatrix} -2 \\ 0 \\ 8 \end{pmatrix}.$$

$$\Leftrightarrow 3\vec{x} - 2\vec{y} = \begin{pmatrix} 5 \\ 9 \\ -2 \end{pmatrix}$$

Problem 4

$$A = \begin{pmatrix} -1 & 5 & 0 \\ 2 & 1 & 3 \\ -1 & 0 & 2 \end{pmatrix}$$

- For $\vec{x} = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} \Rightarrow A\vec{x} = \begin{pmatrix} -1 & 5 & 0 \\ 2 & 1 & 3 \\ -1 & 0 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} = \begin{pmatrix} -1+10 \\ 2+2+12 \\ -1+8 \end{pmatrix} = \begin{pmatrix} 9 \\ 16 \\ 7 \end{pmatrix}$

- For $\vec{x} = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} \Rightarrow A\vec{x} = \begin{pmatrix} -1 & 5 & 0 \\ 2 & 1 & 3 \\ -1 & 0 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} = \begin{pmatrix} -1-5 \\ 2-1-3 \\ -1-2 \end{pmatrix} = \begin{pmatrix} -6 \\ -2 \\ -3 \end{pmatrix}$

Problem 5

$$5x_1 - 7x_2 + 10x_3 = 0 \qquad (1)$$

$$x_2 + 2x_3 = 1 \qquad (2)$$

$$3x_1 - x_2 + 4x_3 = -2. \qquad (3)$$

Let $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

Equation (1) is equivalent to $\begin{pmatrix} 5 & -7 & 10 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0.$

Equation (2) is equivalent to : $\begin{pmatrix} 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 1$

Equation (3) is equivalent to : $\begin{pmatrix} 3 & -1 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = -2$

$\Rightarrow A = \begin{pmatrix} 5 & -7 & 10 \\ 0 & 1 & 2 \\ 3 & -1 & 4 \end{pmatrix}$ and $\vec{a} = \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}$ such that :

$$A\vec{x} = \vec{a} \Leftrightarrow \begin{pmatrix} 5 & -7 & 10 \\ 0 & 1 & 2 \\ 3 & -1 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}$$

Problem 6

1. $A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \Rightarrow \det A = 1 \cdot (-1) - 1 \cdot 1 = -2 \neq 0$

$\Rightarrow \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ linearly independent.

2. $A = \begin{pmatrix} 1 & 1 & -4 \\ 1 & -1 & 0 \\ 1 & 1 & 4 \end{pmatrix} \Rightarrow$

$$\det A = 1(-1)4 + 1 \cdot 0 \cdot 1 + 1 \cdot 1 \cdot (-4) - \left((-4)(-1) \cdot 1 + 0 \cdot 1 \cdot 1 + 4 \cdot 1 \cdot 1 \right)$$

$$= -4 - 4 - (4 + 4) = -16 \neq 0$$

$\Rightarrow \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} -4 \\ 0 \\ 4 \end{pmatrix}$ linearly independent

3. $A = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \Rightarrow \det A = 1 + 0 + 1 - 0 - 0 - 0 = 2 \neq 0$

$\Rightarrow \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ linearly independent

Homework 7 solutions

Problem 7 (Vector spaces)**.** In the lecture we said that the set of all functions on a real line is a vector space. Which of the following sets of functions form a vector space, with the usual operations of adding functions and multiplying them by numbers? In each case, explain why or why not.

1. the set of polynomials of degree smaller than or equal to k;
2. the set of polynomials of degree equal to k;
3. the set of functions f such that $\lim_{x\to\infty} f(x) = 0$;
4. the set of functions f such that $f(0) = 1$;
5. the set of solutions of the differential equation $f'' + f^2 = 0$.

Solution. (1) Given two polynomials of degree $\leq k$, their linear combination is a polynomial of degree $\leq k$, so this set is a vector space.

(2) The sum of two polynomials of degree $= k$ is not necessarily a polynomial of degree $= k$. For example, if $f(x) = x^k$ and $g(x) = -x^k + x^{k-1}$, then $f(x) + g(x) = x^{k-1}$ is a polynomial of degree $k - 1$. So this set is not a vector space.

(3) Let f and g be two functions such that

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = 0.$$

Given any real numbers a and b, we have

$$\lim_{x\to\infty} (af + bg)(x) = a \lim_{x\to\infty} f(x) + b \lim_{x\to\infty} g(x) = 0,$$

so the linear combination $af + bg$ is also in the set. We conclude that the set is a vector space.

(4) The constant function zero is not in this set, so it is not a vector space.

(5) This is a nonlinear equation so we don't expect that the set of solutions is a vector space. We can show that, for example, by showing that there is a function f which satisfies the equation but λf does not satisfy it for some constant λ. Suppose that f is a solution, i.e.

$$f'' + f^2 = 0.$$

Then, if $g = \lambda f$, then

$$g'' = \lambda f'' = -\lambda f^2.$$

On the other hand,

$$g^2 = \lambda^2 f^2.$$

So if g is also a solution, we must have $g'' + g^2 = 0$, that is:

$$-\lambda f^2 + \lambda^2 f^2 = \lambda(1-\lambda) f^2 = 0,$$

which is possible only if $\lambda = 0, \lambda = 1$, or $f = 0$. We know from the existence and uniqueness theorem that there exists a nonzero solution f to the equation. For such a solution and any number λ such that $\lambda \neq 0, 1$, the function $g = \lambda f$ is not a solution. This shows that the set of solutions is not a vector space.

Homework 8

Problem 1

$$A = \begin{pmatrix} 3 & 2 & 0 \\ 1 & 1 & 1 \\ 2 & 6 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} -1 & 0 & 9 \\ 2 & 1 & -1 \\ 0 & 6 & 2 \end{pmatrix}$$

$$AB = \begin{pmatrix} 3 & 2 & 0 \\ 1 & 1 & 1 \\ 2 & 6 & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 & 9 \\ 2 & 1 & -1 \\ 0 & 6 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 25 \\ 1 & 7 & 10 \\ 10 & 6 & 12 \end{pmatrix}$$

$$BA = \begin{pmatrix} -1 & 0 & 9 \\ 2 & 1 & -1 \\ 0 & 6 & 2 \end{pmatrix} \begin{pmatrix} 3 & 2 & 0 \\ 1 & 1 & 1 \\ 2 & 6 & 0 \end{pmatrix} = \begin{pmatrix} 15 & 52 & 0 \\ 5 & -1 & 1 \\ 10 & 18 & 6 \end{pmatrix}$$

Problem 2

$$L = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}; \quad K = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$LK = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$$

Problem 3

$$f(x) = a_0 + a_1 \sin(x) + a_2 \sin(2x) + \dots + a_n \sin(nx) + b_1 \cos(x) + \dots + b_n \cos(nx)$$

$$L(f) = \frac{df}{dx}$$

- $L(0) = f'(0) = 0$
- $L(f+g) = \frac{d(f+g)}{dx} = \frac{df}{dx} + \frac{dg}{dx} = L(f) + L(g)$
- $L(\lambda f) = \frac{d(\lambda f)}{dx} = \lambda \frac{df}{dx} = \lambda L(f)$

$\Rightarrow$ $L(f) = \frac{df}{dx}$ is a linear transformation.

Note $\quad L(\sin(mx)) = m \cos(mx)$ for $m \in \mathbb{Z}_+^*$

$\qquad\quad L(\cos(mx)) = -m \sin(mx)$ for $m \in \mathbb{Z}_+^*$

$\Rightarrow$ For $m \in \{1, \dots, n\}$, $a_m = m$, $b_m = -m$

$$\Rightarrow [L]_B = \begin{pmatrix} 0 & 0 & 0 & \dots & & & 0 \\ 0 & 0 & -1 & 0 & \dots & & 0 \; 0 \\ 0 & 0 & 0 & 0 & -2 & \dots & 0 \; 0 \\ \vdots & & & & & & \ddots \\ \dots & & & & & & 0 \; -m \\ 0 & 0 & \dots & & & & 0 \\ 0 & 1 & 0 & \dots & & & 0 \\ 0 & 0 & 0 & 2 & \dots & & \\ \vdots & & & & & & \\ 0 & 0 & \dots & & & & m \; 0 \end{pmatrix} \quad (2n+1) \times (2n+1)$$

Problem 4

- $$\begin{pmatrix} 1 & 0 & 2 \\ 1 & 2 & 5 \\ 6 & 8 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 2 \\ 0 & 2 & 3 \\ 0 & 8 & -12 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 2 \\ 0 & 2 & 3 \\ 0 & 0 & -24 \end{pmatrix}$$

$$\Rightarrow \det \begin{pmatrix} 1 & 0 & 2 \\ 1 & 2 & 5 \\ 6 & 8 & 0 \end{pmatrix} = 1 \cdot 2 \cdot (-24) = -48$$

- $$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 1 & 5 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & -1 & -1 & -1 & 1 \\ 0 & 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 1 & 5 \end{pmatrix} \rightarrow$$

$$\rightarrow \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & -1 & -1 & -1 & 1 \\ 0 & 0 & -1 & -1 & 4 \\ 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 1 & 5 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & -1 & -1 & -1 & 1 \\ 0 & 0 & -1 & -1 & 4 \\ 0 & 0 & 0 & -1 & 8 \\ 0 & 0 & 0 & 1 & 5 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & -1 & -1 & -1 & 1 \\ 0 & 0 & -1 & -1 & 4 \\ 0 & 0 & 0 & -1 & 8 \\ 0 & 0 & 0 & 0 & 13 \end{pmatrix}$$

$$\Rightarrow \det \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 1 & 5 \end{pmatrix} = 1 \cdot (-1) \cdot (-1) \cdot (-1) \cdot 13 = -13.$$

Problem 5

$$A = \begin{pmatrix} 1 & -3 & 2 \\ 0 & -1 & 0 \\ 0 & -1 & -2 \end{pmatrix} \Rightarrow A - \lambda I = \begin{pmatrix} 1-\lambda & -3 & 2 \\ 0 & -1-\lambda & 0 \\ 0 & -1 & -2-\lambda \end{pmatrix}$$

$$\Rightarrow \det(A - \lambda I) = (1-\lambda)(-1-\lambda)(-2-\lambda) + (-3)\cdot 0 + (-1)\cdot 2 \cdot 0 - \Big[2(-1-\lambda) +$$
$$+ 0 \cdot (-3)(-2-\lambda) + 0 \cdot (-1) \cdot (1-\lambda) \Big]$$

$$\Leftrightarrow \det(A - \lambda I) = (1-\lambda)(-1-\lambda)(-2-\lambda) \Rightarrow \text{Eigenvalues } \lambda_1 = 1, \lambda_2 = -1, \lambda_3 = -2$$

$$\text{For } \lambda_1 = 1 \Rightarrow (A - \lambda_1 I)\vec{x}_1 = \begin{pmatrix} 0 & -3 & 2 \\ 0 & -2 & 0 \\ 0 & -1 & -3 \end{pmatrix} \vec{x}_1 = \vec{0} \Rightarrow \vec{x}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \text{ eigenvector}$$

$$\text{For } \lambda_2 = -1 \Rightarrow (A - \lambda_2 I)\vec{x}_2 = \begin{pmatrix} 2 & -3 & 2 \\ 0 & 0 & 0 \\ 0 & -1 & -1 \end{pmatrix} \vec{x}_2 = \vec{0} \Rightarrow \vec{x}_2 = \begin{pmatrix} -5 \\ -2 \\ 2 \end{pmatrix} \text{ eigenvector}$$

$$\text{For } \lambda_3 = -2 \Rightarrow (A - \lambda_3 I)\vec{x}_3 = \begin{pmatrix} 3 & -3 & 2 \\ 0 & 1 & 0 \\ 0 & -1 & 0 \end{pmatrix} \vec{x}_3 = \vec{0} \Rightarrow \vec{x}_3 = \begin{pmatrix} 2 \\ 0 \\ -3 \end{pmatrix} \text{ eigenvector}$$

$\Rightarrow$ Eigenvalues are $\lambda_1 = 1, \lambda_2 = -1, \lambda_3 = -2$ with corresponding eigenvectors $\vec{x}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\vec{x}_2 = \begin{pmatrix} -5 \\ -2 \\ 2 \end{pmatrix}$, $\vec{x}_3 = \begin{pmatrix} 2 \\ 0 \\ -3 \end{pmatrix}$

Problem 6

$$A = \begin{pmatrix} 1 & -3 \\ -2 & 2 \end{pmatrix} \Rightarrow A - I\lambda = \begin{pmatrix} 1-\lambda & -3 \\ -2 & 2-\lambda \end{pmatrix} \Rightarrow \det(A - \lambda I) = (1-\lambda)(2-\lambda) - 6$$

$$\Leftrightarrow \det(A - \lambda I) = 2 - \lambda - 2\lambda + \lambda^2 - 6 = \lambda^2 - 3\lambda - 4 = (\lambda - 4)(\lambda + 1)$$

$$\Rightarrow \lambda_1 = -1, \ \lambda_2 = 4$$

For $\lambda_1 = -1 \Rightarrow (A - \lambda_1 I)\vec{x}_1 = \begin{pmatrix} 2 & -3 \\ -2 & 3 \end{pmatrix} \vec{x}_1 = \vec{0} \Rightarrow \vec{x}_1 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$ eigenvector

For $\lambda_2 = 4 \Rightarrow (A - \lambda_2 I)\vec{x}_2 = \begin{pmatrix} -3 & -3 \\ -2 & -2 \end{pmatrix} \vec{x}_2 = \vec{0} \Rightarrow \vec{x}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ eigenvector.

$$\Rightarrow y(t) = c_1 e^{4t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

$$y(0) = \begin{pmatrix} 0 \\ 5 \end{pmatrix} = \begin{pmatrix} c_1 \\ -c_1 \end{pmatrix} + \begin{pmatrix} 3c_2 \\ 2c_2 \end{pmatrix}$$

$$\Rightarrow \left.\begin{array}{l} c_1 + 3c_2 = 0 \\ -c_1 + 2c_2 = 5 \end{array}\right\} \Rightarrow c_2 = 1, \ c_1 = -3$$

$$\Rightarrow y(t) = -3 e^{4t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + e^{-t} \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

Problem 7

$L: V \to V; \quad K: V \to V$

$$L(f) = xf \quad ; \quad K(f) = \frac{df}{dx}$$

$L(0) = 0 = K(0)$

- $L(f+g) = x(f+g) = xf + xg = L(f) + L(g)$

  $L(\lambda f) = x\lambda f = \lambda x f = \lambda \cdot L(f)$

- $K(f+g) = \frac{d(f+g)}{dx} = \frac{df + dg}{dx} = \frac{df}{dx} + \frac{dg}{dx} = K(f) + K(g)$

  $K(\lambda f) = \frac{d(\lambda f)}{dx} = \lambda \frac{df}{dx} = \lambda K(f)$

$\Rightarrow$ L and K are linear transformations

For any function $f(x)$:

$[L, K] = LK - KL = x \cdot \frac{d f(x)}{dx} - \frac{d(x f(x))}{dx} = x \cdot \frac{d f(x)}{dx} - \frac{dx}{dx} \cdot f(x) - \frac{d f(x)}{dx} \cdot x$

$\Leftrightarrow [L, K] = -f(x)$

Problem 8

$L: V \to W$ invertible

1. Suppose $\exists\ \lambda_i$ such that $\lambda_1 L(v_1) + \dots + \lambda_m L(v_m) = 0$.

L linear transformation, L invertible.

$\Rightarrow L(\lambda_1 v_1) + L(\lambda_2 v_2) + \dots + L(\lambda_n v_m) = 0 \Leftrightarrow$

$\Leftrightarrow L(\lambda_1 v_1 + \dots + \lambda_n v_n) = 0 \Rightarrow L^{-1} L(\lambda_1 v_1 + \dots + \lambda_n v_n) = 0$

$\Leftrightarrow \lambda_1 v_1 + \dots + \lambda_n v_m = 0$ which contradicts the fact that $v_1, \dots, v_m$ linearly independent

$\Rightarrow L(v_1), \dots, L(v_m)$ also linearly independent

2. $v_1, \dots, v_n$ basis of V.

$\Rightarrow \forall v \in V$, $v = \lambda_1 v_1 + \dots + \lambda_n v_n$ and v_i linearly independent

We know from 1. that $L(v_i)$ linearly independent

Now $L: V \to W$, $L^{-1}: W \to V$

$\Rightarrow L^{-1}(w) \in V$ and v_i basis of V

$\Rightarrow L^{-1}(w) = \lambda_1 v_1 + \dots + \lambda_n v_n$.

$\Rightarrow L \cdot L^{-1}(w) = w = L(\lambda_1 v_1) + \dots + L(\lambda_n v_n) \Leftrightarrow$

$\Leftrightarrow w = \lambda_1 L(v_1) + \dots + \lambda_n L(v_n)$

$\Rightarrow$ any vector in W is linear combination of $L(v_i)$

and $L(v_i)$ linearly independent

$\Rightarrow L(v_1) \dots L(v_n)$ is a basis of W.

3. $v_1, \dots, v_n$ basis of V

$L(v_1), \dots, L(v_n)$ basis of W

$\Rightarrow \dim(V) = \dim(W) = n$.

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Homework 9 solutions: Equilibria and stability

April 17, 2021

Problem 1 (Homogenous linear systems with constant coefficients)**.** Find the general solution of each of the following linear systems and determine if the solutions are stable or unstable:

$$\mathbf{y}' = \begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix} \mathbf{y},$$

$$\mathbf{y}' = \begin{bmatrix} -2 & 1 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -1 \end{bmatrix} \mathbf{y}.$$

Solution. In the first case, the eigenvalues are $\lambda_1 = 1 + 2i$ and $\lambda_2 = \overline{\lambda_1} = 1 - 2i$. The real part is positive, so solutions are unstable. We easily find that

$$\begin{bmatrix} 1+i \\ 2 \end{bmatrix}$$

is an eigenvector with eigenvalue $1 + 2i$. This gives us a complex solution

$$y_c(t) = e^{(1+2i)t} \begin{bmatrix} 1+i \\ 2 \end{bmatrix}.$$

We have

$$y_c(t) = (e^t \cos(2t) + ie^t \sin(2t)) \left(\begin{bmatrix} 1 \\ 2 \end{bmatrix} + i \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right),$$

so we get real solutions

$$\begin{aligned} \mathrm{Re} y_c(t) &= e^t \cos(2t) \begin{bmatrix} 1 \\ 2 \end{bmatrix} - e^t \sin(2t) \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ \mathrm{Im} y_c(t) &= e^t \sin(2t) \begin{bmatrix} 1 \\ 2 \end{bmatrix} + e^t \cos(2t) \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \end{aligned}$$

The general solution is a linear combination of those.

In the second case, we get a repeated eigenvalue $\lambda_1 = -2$ and another eigenvalue $\lambda_2 = -1$. Since both eigenvalues have negative real part, all solutions are asymptotically stable. We see from the form of the matrix that

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

is an eigenvector with eigenvalue -2. This is the only eigenvector. We now look for a generalized eigenvector $\mathbf{v}_2$ such that $(\mathbf{A}+2\mathbf{I})\mathbf{v}_2 = \mathbf{v}_1$. We get a solution

$$\mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

Finally,

$$\mathbf{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

is an eigenvector with eigenvalue -1. So the general solution is

$$y(t) = C_1 e^{-2t} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + C_2 \left(te^{-2t} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + e^{-2t} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right) + C_3 e^{-t} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Problem 2 (Integral curves)**.** Sketch the vector field and integral curves of each of the following linear systems. In each case, use the picture to determine whether its solutions are stable or unstable. If they are stable, are they asymptotically stable?

$$\mathbf{y}' = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \mathbf{y}, \qquad \mathbf{y}' = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \mathbf{y}, \qquad \mathbf{y}' = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \mathbf{y}$$

Solution. The method of drawing vector fields and integral curves is described in detail in section 9.1 of Boyce–DiPrima. You might find it helpful to plot the vector fields using an computer program such as, for example, `https://www.desmos.com/calculator/eijhparfmd`. Either using the picture, or computing eigenvalues, we determine that

1. In the first case, solutions are unstable.
2. In the second case, solutions are unstable.
3. In the third case, solutions are stable.

Problem 3 (Equilibria of nonlinear systems)**.** For each of the following nonlinear autonomous system, find its equilibria. For each equilibrium, the corresponding linear system near that equilibrium and determine, if possible, whether the equilibrium is stable or unstable.

$$\begin{cases} x' = 1 - y, \\ y' = x^2 - y^2 \end{cases}$$

$$\begin{cases} x' = 1 - xy, \\ y' = x - y^3. \end{cases}$$

Solution.

In the first problem, equilibria are $x = 1$, $y = 1$ and $x = -1$, $y = 1$. To find the linearization at $x = 1$, $y = 1$, introduce new functions $u = x - 1$ and $v = y - 1$. We write differential equations for u and v:

$$\begin{cases} u' = x' = 1 - y = -v, \\ v' = x^2 - y^2 = (u+1)^2 - (v+1)^2 = 2u - 2v + u^2 - v^2. \end{cases}$$

We can write it as a linear equation + a non-linear term:

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} + \begin{bmatrix} 0 \\ u^2 - v^2 \end{bmatrix}.$$

So the linearization at $x = 1$ and $y = 1$ is the linear system

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix}.$$

The eigenvalues are $-1 + i$ and $-1 - i$. Since they have negative real part, the equilibrium $u = 0$, $v = 0$ of the linear system is asymptotically stable. We conclude that the equilibrium $x = 1$, $y = 1$ of the nonlinear system is also asymptotically stable.

Let us now compute the linearization at the other equilibrium $x = -1$ and $y = 1$. Again, introduce new functions $u = x + 1$ and $v = y - 1$. We have

$$\begin{cases} u' = x' = 1 - y = -v, \\ v' = x^2 - y^2 = (u-1)^2 - (v+1)^2 = -2u - 2v + u^2 - v^2. \end{cases}$$

so

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -2 & -2 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} + \begin{bmatrix} 0 \\ u^2 - v^2 \end{bmatrix}.$$

The linearization at $x = -1$ and $y = 1$ is the system

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -2 & -2 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix}.$$

The eigenvalues are $-\sqrt{3} - 1$ and $\sqrt{3} - 1$. Since the second eigenvalue is positive, the equilibrium of the linear system is unstable. Therefore, the equilibrium $x = -1$ and $y = 1$ of the nonlinear system is unstable.

The second problem is solved in the same way. For computations, see the handwritten notes from office hours on April 15.