Topology and Groups

Week 7, Monday

1 Preparation

- 7.01 (Covering spaces),
- 7.02 (Path-lifting, monodromy).

2 Discussion

1. (PCQ) Are there any 2-to-1 covering spaces of the figure 8 other than the two given in the notes?
2. What are the monodromies for these covering spaces?
3. How many 3-to-1 covers of the figure 8 are there? What are their monodromies?
4. Can you draw a covering space of the figure 8 which:
   - is ∞-to-1?
   - has a fundamental group which is not finitely generated?
   - is simply-connected?
5. (PCQ) Suppose that $p\colon Y \to X$ is a covering map and $x_0, x_1 \in X$. Is it true that there is a bijection $p^{-1}(x_0) \to p^{-1}(x_1)$? Give a proof or a counterexample. What if there is a path connecting x_0 to x_1?

3 Classwork

1. In each case, $p\colon Y \to X$ will be a covering space and $y \in Y$ will be a basepoint. Compute $p_*\pi_1(Y, y) \subset \pi_1(X, p(y))$.

   - $X = Y = S^1$, $p(e^{ix}) = e^{inx}$.
   - $X = 8$, Y is any of its connected double covers, and y is one of the two preimages of the cross-point. (Do it for all such Y, y).
   - $X = 8$, Y is any of its connected triple covers, and y is one of the three preimages of the cross-point. (Do it for all such Y, y).

   Does your answer depend on which preimage of the cross-point you choose as basepoint?

2. In the examples above with $X = 8$, which of the subgroups you computed was normal? What do you notice about these covering spaces?

3. Let X be the figure 8 and let $G = \pi_1(X)$.

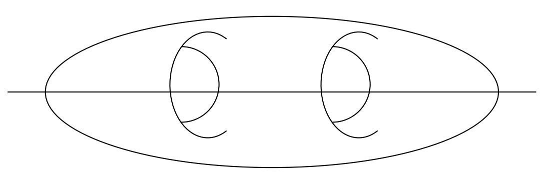
   - The *commutator subgroup* $[G, G]$ is defined to be the subgroup generated by all commutators $aba^{-1}b^{-1}$ with $a, b \in G$. Can you write down a covering space $p\colon Y \to X$ and a basepoint $y \in Y$ such that $p_*\pi_1(Y, y) = [G, G]$?
   - What does the example you just found have to do with the 2-torus?
   - Given *any* subgroup $H \subset G$ is there a covering space $p\colon Y \to X$ such that $p_*\pi_1(Y, y) = \pi_1(X, p(y))$?

4 Assessed project 2

A *branched cover* of (orientable) surfaces $F\colon Y \to X$ is a map with the following properties:

- There exists a finite set $R \subset Y$ (*ramification locus*), a finite set $B \subset X$ (*branch locus*), and an integer d such that
  - $B = F(R)$,
  - $F|_{Y \setminus R}\colon Y \setminus R \to X \setminus B$ is a covering map of degree d.
- For each point $y \in R$, there is an open neighbourhood U of y and an open neighbourhood V of $F(y) \in B$ such that
  - There are (orientation-preserving) homeomorphisms $u\colon U \to D^2$ and $v\colon V \to D^2$ where $D^2 \subset \mathbf{C}$ is the unit disc.
  - Viewed in the coordinate charts u and v[1], $F|_U\colon U \to V$ is the map $z \mapsto z^{n_y}$ for some $n_y \in \{1, 2, \ldots\}$, called the *ramification index* of F at y. Note that if $n = 1$ then this is just an ordinary covering map.

Examples: The archetypal example of a branched cover is the map $F\colon \mathbf{C} \to \mathbf{C}$, $F(z) = z^n$. Another example would be the following. Take a surface of genus 2 and consider the $\mathbf{Z}/2$-action on it where the nontrivial element rotates by 180 degrees around the axis shown. The quotient map is a 2-to-1 branched cover of the sphere with 6 ramification points, each having index 2. (A double branched cover of the sphere is also called a *hyperelliptic* cover).



[1] In other words $(v \circ F \circ u^{-1})(z) = z^n$.

1. (Riemann-Hurwitz formula) Let $\chi(X)$ and $\chi(Y)$ denote the Euler characteristics of X and Y. Show that

$$\chi(Y) = d\chi(X) - \sum_{y \in R}(n_y - 1).$$

(Hint: Consider suitable CW structures on X and Y.)

2. Show that a hyperelliptic cover $F\colon Y \to S^2$ has $2g+2$ ramification points, where g is the genus of Y.

3. Let $p(x)$ be a polynomial of degree d with no repeated roots. Consider the subset $Y = \{(x,y) \in \mathbf{C}^2 \ : \ y^2 = p(x)\}$ and let $F\colon Y \to \mathbf{C}$ be the map $F(x,y) = x$

   - Find the Euler characteristic of Y.
   - For large r, what is the preimage under F of the circle of radius r in $\mathbf{C}$? (Hint: there should be a difference when d is odd or even.)
   - Given that a surface with genus g and b boundary circles has Euler characteristic $2 - 2g - b$, what is the genus of Y?
   - What is the preimage of an arc connecting two of the roots of p?
   - In the case $p(x) = x^3 - x$, sketch the surface Y along with the preimages of the arcs in the real axis connecting -1 to 0, 0 to 1 and 1 to ∞.
   - Sketch the *real* curve $\{(x,y) \in \mathbf{R}^2 \ : \ y^2 = x^3 - x\}$. What does this picture have to do with your drawing from the previous part?