Topology and Groups

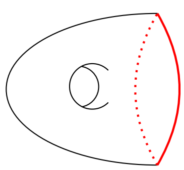
Week 3, Thursday

1 Preparation

- 3.01 (quotient topology),
- 4.01 (CW complexes).

2 Discussion

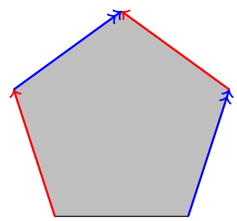
1. (PCQ) Let X be the space in the figure below and let A be the red subset. What is the topological space X/A?

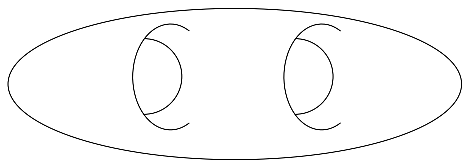


2. Let X be a set equipped with the discrete topology and let $\sim$ be an equivalence relation on X. Which of the following statements is true?
   - $X/\sim$ inherits the discrete topology.
   - $X/\sim$ inherits the indiscrete topology.
   - the topology on $X/\sim$ is neither discrete nor indiscrete.
   - we need to know more about $\sim$ before we can say more about the topology on $X/\sim$.
3. (PCQ) Consider the figure 8 with the two loops labelled a, b. Attach a 2-cell e to this using an attaching map $\varphi\colon \partial e \to 8$ which is a loop representing the homotopy class $ba^{-1}ba$. What space do you get?

3 Classwork

1. If you glue the sides of the pentagon as indicated below (in pairs, leaving one untouched) what space do you get? Can you draw a polygon with side identifications that will give you a *genus 2 surface* (pictured below)?





2. Find a cell structure:
   - on the 2-torus which has two 0-cells, four 1-cells and two 2-cells.
   - on the 2-sphere which has one 0-cell, two 1-cells and three 2-cells.
   - on the Möbius strip with two 0-cells, three 1-cells and one 2-cell.
   - on the solid torus (i.e. the doughnut together with all its jam) with one 0-cell, two 1-cells, two 2-cells and one 3-cell.

3. The Euler characteristic of a CW complex is defined to be the alternating sum $a_0 - a_1 + a_2 - \cdots$ where a_k is the number of k-cells (assuming this sum converges). An amazing fact (proved using *homology theory*) is that it depends only on the homeomorphism type (in fact only on the *homotopy type*) of the CW complex. If we have a CW structure on the 2-torus with m 0-cells and n 1-cells, how many 2-cells must it have? Which values of m and n can you realise?

4 Mini-lecture: Projective spaces

The space of real lines in $\mathbf{R}^3$ passing through the origin is called the real projective plane $\mathbf{RP}^2$. To specify such a line, it suffices to specify a nonzero point p in $\mathbf{R}^3$ (then you just draw the corresponding line from the origin to p). Moreover, if $p = (x, y, z)$ then $\lambda p = (\lambda x, \lambda y, \lambda z)$ gives the same straight line. We define *homogeneous coordinates* on $\mathbf{RP}^2$ to be triples of numbers $[x : y : z]$ specified up to scale (not all equal to zero). For example, $[1 : 0 : 0]$ is the x-axis and $[1 : 1 : 0]$ is the line $x = y$ in the $z = 0$ plane. (Another way of saying this is to define $\mathbf{RP}^2$ as the quotient space of $\mathbf{R}^3 \setminus \{(0, 0, 0)\}$ by the equivalence relation $(x, y, z) \sim (\lambda x, \lambda y, \lambda z)$ for some $\lambda \in \mathbf{R} \setminus \{0\}$.

Away from $z = 0$, we can rescale $[x : y : z]$ by $1/z$ and get $[x : y : z] = [x/z : y/z : 1]$. This means that the space of lines *not contained in the* $(z = 0)$*-plane* is parametrised by two numbers $x/z, y/z$ (and any two numbers specify a unique such line).

When $z = 0$, we have coordinates $[x : y : 0]$. Away from $y = 0$ we can rescale by $1/y$ and we get $[x : y : 0] = [x/y : 1 : 0]$. So the space of lines *contained in the* $(z = 0)$*-plane but not contained in the* $(y = 0)$*-plane* is parametrised by a single number x/y (and any number specifies a unique such line).

When $y = z = 0$, we have a unique line: $[1 : 0 : 0]$, the x-axis.

I claim that this defines for us a cell structure on $\mathbf{RP}^2$.

1. Why?
2. How many cells of each dimension does it have?
3. What are the attaching maps?
4. How would this generalise to $\mathbf{RP}^n$, the space of lines in $\mathbf{R}^{n+1}$?
5. What if we were to work with complex numbers and complex lines?