Topology and Groups

Week 3, Monday

1 Preparation

- 2.01 (Topological spaces, continuous maps),
- 2.02 (Bases, metric and product topologies),
- 2.03 (Subspace topology).

2 Discussion: topological spaces

1. What is a topology?
2. (PCQ) Can you think of an infinite collection U_i, $i \in \mathbf{N}$, of open sets in $\mathbf{R}$ such that $\bigcap_{i\in\mathbf{N}} U_i$ is not open?
3. (PCQ) Instead of checking that all finite intersections of open sets are open, we can just check that intersections of two open sets are open. Why is this sufficient?
4. (PCQ) Let X be a set. Can you give a base for the discrete topology on X? What is the smallest base you could give?
5. What is the subspace topology?
6. (PCQ) Show that the subspace topology satisfies the axioms for a topology.

3 Discussion: continuous maps

1. When is a map $F\colon X \to Y$ between two topological spaces *continuous*?
2. Let X be the set $\{0,1\}$, equipped with the discrete topology and let Y be the set $\{0,1\}$, equipped with the indiscrete topology. Which of the following maps is continuous?
   - $f\colon X \to Y$, $f(0) = 0$, $f(1) = 1$.
   - $g\colon Y \to X$, $g(0) = 0$, $g(1) = 1$.
3. Let $X = \{0,1\}$ equipped with the discrete topology and $Y = \{0,1\}$ equipped with the indiscrete topology; let $\mathbf{R}$ be the real line equipped with its usual (metric) topology. Which of the following functions are continuous?
   - $f\colon X \to \mathbf{R}$, $f(0) = 0$, $f(1) = 1$.
   - $g\colon Y \to \mathbf{R}$, $g(0) = 0$, $g(1) = 1$.
   - $p\colon \mathbf{R} \to X$, $p(x) = \begin{cases} 0 & \text{if } x < 0, \\ 1 & \text{if } x \geq 1. \end{cases}$
   - $q\colon \mathbf{R} \to Y$, $q(x) = \begin{cases} 0 & \text{if } x < 0, \\ 1 & \text{if } x \geq 1. \end{cases}$

4 Classwork

In your learning groups, tackle the following questions:

1. Suppose that T is a topology on X. We say that a subset $A \subset X$ is *closed* if $X \setminus A \in T$ (i.e. its complement is open). Let T' be the set of closed sets in X. Prove that T' contains $\emptyset$ and X and that it is closed under taking arbitrary intersections and finite unions. Deduce that we can equally well define topological spaces by specifying their closed sets. Give a characterisation of continuous maps in terms of closed sets instead of open sets.

2. Let X, Y be topological spaces, and suppose that $U, V \subset X$ are closed subsets. If $F\colon U \to Y$ and $G\colon V \to Y$ are continuous maps such that $F|_{U\cap V} = G|_{U\cap V}$ then prove that

$$H\colon X \to Y \qquad H(x) = \begin{cases} F(x) & \text{if } x \in U \\ G(x) & \text{if } x \in V, \end{cases}$$

   is a well-defined continuous map. What if U, V are open?

3. Let X, Y be topological spaces and let $p\colon X \times Y \to X$ and $q\colon X \times Y \to Y$ be the projection maps $p(x, y) = x$, $q(x, y) = y$. Suppose that T is a topology on $X \times Y$ such that p and q are continuous. Show that T contains the product topology.

4. Let $F\colon Z \to X \times Y$ be a map. Show that F is continuous if and only if $F_X := p \circ F$ and $F_Y := q \circ F$ are continuous. (Hint: Write $F^{-1}(U \times V)$ in terms of $(p \circ F)^{-1}(U)$ and $(q \circ F)^{-1}(V)$). Deduce that $\pi_1(X \times Y, (x, y)) \cong \pi_1(X, x) \times \pi_1(Y, y)$.

5 Zariski topology (nonexaminable)

Here is an example of a topology which arises naturally in commutative algebra and is extremely useful for studying *algebraic geometry.*

Let R be a commutative ring. Recall that a subset $I \subset R$ is called an *ideal* if $ri \in I$ for all $r \in R$, $i \in I$ (for example, the subset $n\mathbf{Z} \subset \mathbf{Z}$ of integers divisible by n, is an ideal in the integers). An ideal I is called a *prime ideal* if $I \neq R$ and, whenever $a, b \in R$ and $ab \in I$, we have $a \in I$ or $b \in I$ (for example, $n\mathbf{Z}$ is a prime ideal if and only if n is prime). Let $Spec(R)$ denote the set of prime ideals.

Given an ideal I, let $V(I) = \{P \in Spec(R) \ : \ I \subset P\}$ (what does this mean when $I = n\mathbf{Z}$ and $R = \mathbf{Z}$?). The Zariski topology is the topology for which the *closed sets* are the sets $\{V(I) \ : \ I \text{ an ideal}\}$.

1. What is $V(\{0\})$? What is $V(R)$?
2. Show that if $I \subset J$ then $V(J) \subset V(I)$.
3. Given two ideals I, J, the subset $IJ = \{ab \ : \ a \in I, \ b \in J\}$ is also an ideal. Show that
   - $V(I) \cup V(J) \subset V(IJ)$,
   - if P is a prime ideal containing IJ then either P contains I or P contains J.

   Deduce that $V(I) \cup V(J) = V(IJ)$.
4. Given a collection of ideals I_k, define the ideal $\sum I_k$ to be the smallest ideal containing all the I_k. Show that $V(\sum I_k) = \bigcap V(I_k)$.
5. Deduce that the Zariski topology on $Spec(R)$ is a topology.
6. Which points in $Spec(\mathbf{Z})$ are closed sets?
7. If k is a field, what is $Spec(k)$?
8. Let $R = \mathbf{C}[x]$, the ring of polynomials in one variable. Given a polynomial $f \in R$, when is the ideal $(f) = \{rf \ : \ r \in R\}$ of multiples of f prime? Since $\mathbf{C}[x]$ is a principal ideal domain, any ideal has the form (f) for some $f \in \mathbf{C}[x]$. Can you describe the topological space $Spec(\mathbf{C}[x])$ more concretely?