Topology and Groups

Week 2, Thursday

1 Preparation

- 1.07 (Induced maps),
- 1.08 (Brouwer's fixed point theorem).

2 Discussion

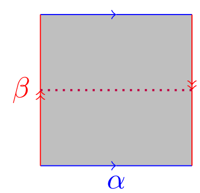
1. Let $F\colon S^1 \to S^1$ be the map $F(e^{i\theta}) = e^{in\theta}$. What is the induced map F_* on $\pi_1(S^1) \cong \mathbf{Z}$?
   - $x \mapsto x + n$?
   - $x \mapsto nx$?
   - $x \mapsto x^n$?
   - $x \mapsto x/n$?
2. (PCQ) Brouwer's fixed point theorem tells us that continuous maps between 2-discs have fixed points. Is the same true for maps between 2-dimensional annuli? (An annulus is $S^1 \times [0, 1]$).
3. (PCQ) Brouwer's fixed point theorem also holds for maps $F\colon D^n \to D^n$ where D^n is the n-dimensional disc; can the proof we gave be adapted to cover this case, or are new ideas required?

3 Classwork

1. Let X, Y be topological spaces and let $F\colon X \to Y$ be a continuous map. Consider the map $\gamma \mapsto F \circ \gamma$ (from loops based at x to loops based at $F(x)$). Show that this descends to give a well-defined homomorphism $F_*\colon \pi_1(X, x) \to \pi_1(Y, F(x))$, and that if Z is a third topological space and $G\colon Y \to Z$ another continuous map, then $G_* \circ F_* = (G \circ F)_*$.

2. Let X be a space and $A \subset X$ be a subset. A map $r\colon X \to A$ is called a *retract* if $r(a) = a$ for all $a \in A$. Let $i\colon A \to X$ be the inclusion map. If there is a retract $X \to A$, show that i_* is injective. What does this have to do with Brouwer's fixed point theorem?

3. The Klein bottle K is obtained by gluing the opposite sides of a square as indicated in the figure below. Projection onto the x-axis defines a continuous map $p\colon K \to S^1$. If γ is the dotted purple loop, what is $p_*\gamma$? What is the order of γ in $\pi_1(K)$?



4. Show that $F_*\colon \pi_1(X) \to \pi_1(Y)$ is trivial in any of the following cases:
   - $\pi_1(X)$ is simple[1] and $\pi_1(X) \not\cong \pi_1(Y)$.
   - $\pi_1(X)$ is finite and $\pi_1(Y) \cong \mathbf{Z}$.

5. Given a 2-by-2 integer matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, consider the map $S^1 \times S^1 \to S^1 \times S^1$ given by

$$F\begin{pmatrix} e^{i\theta} \\ e^{i\phi} \end{pmatrix} = \begin{pmatrix} e^{i(a\theta + b\phi)} \\ e^{i(c\theta + d\phi)} \end{pmatrix}.$$

   Given that $\pi_1(S^1 \times S^1) \cong \mathbf{Z} \times \mathbf{Z}$ (where (m, n) corresponds to the loop $\begin{pmatrix} e^{imt} \\ e^{int} \end{pmatrix}$), what is the induced map F_*?

[1] Recall that a group G is called *simple* if the only normal subgroups of G are G and the trivial subgroup.

4 Periodic orbits and fixed points (nonexaminable)

A *dynamical system* means a system of differential equations like $(\dot{x}(t), \dot{y}(t)) = (-y(t), x(t))$. If you think of the solution as a path $(x(t), y(t))$ (in this case in the plane) then the differential equation tells you that this path is everywhere tangent to the vector field which points in the $(-y, x)$ at the point (x, y).

1. In this example, sketch the vector field and some solution curves.
2. A *periodic orbit* is a solution $(x(t), y(t))$ such that $(x(T), y(T)) = (x(0), y(0))$ for some $T > 0$. Did the previous example have any periodic orbits?
3. Now imagine that you have a vector field on the solid torus. $S^1 \times D^2$. Suppose that the field always has a component in the S^1-direction and that the system has a solution for arbitrarily long times. Show that there exists a periodic orbit. [Hint: Use Brouwer's fixed point theorem.]