Topology and Groups

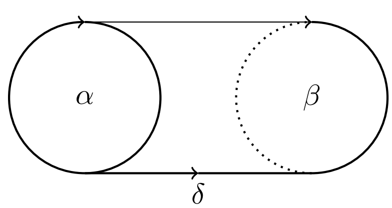
Week 2, Monday

1 Preparation

- 1.04 (Examples, simply-connectedness),
- 1.05 (Basepoint dependence),
- 1.06 (Fundamental theorem of algebra reprise).

2 Discussion

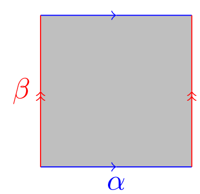
1. What does *simply-connected* mean?
2. If I give you a simply-connected space and a pair of points, how many homotopy classes of paths are there joining these points? How did we prove this? Was the proof correct?
3. What did we prove next in the video? Can you give a 20-second summary of the proof?
4. (PCQ) What about the unit sphere $S^n = \{(x_0, \ldots, x_n) \in \mathbf{R}^{n+1} : \sum_{k=0}^{n} x_k^2 = 1\}$ in higher dimensions? Is it simply-connected? Why does the argument break down for S^1?
5. The picture below shows a tube traced out by a free homotopy between α and β. We know that this means α is based homotopic to $\delta^{-1} \cdot \beta \cdot \delta$. Can you sketch on the picture a family of based loops interpolating between these two?



6. (PCQ) Suppose that X is a topological space and $x \in X$ is a basepoint with $\pi_1(X, x) \cong S_3$, where S_3 is the group of permutations of three objects. How many free homotopy classes of loops are there in X?

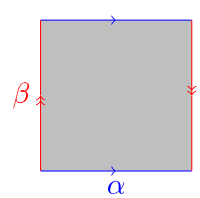
7. (PCQ) Did you believe the final proof of the fundamental theorem of algebra?

8. Consider the torus T^2 considered as a square with its sides identified as in the figure. The blue and red edges of the square become loops in the torus. Let α be the blue loop and β be the red loop (meeting at the point x, the vertex). If I rotate α around the torus in the β-direction, it comes back to itself. What does that imply about the elements α and β in $\pi_1(T^2, x)$?



9. Consider the Klein bottle K, which you can think of as a square with its sides identified as in the figure. The blue and red edges are circles (respectively α and β) after the identifications are made. Which of the following statements is true?

   - α and β commute in $\pi_1(K, x)$.
   - β is conjugate to β^{-1} in $\pi_1(K, x)$.
   - α is conjugate to α^{-1} in $\pi_1(K, x)$.
   - α is conjugate to β in $\pi_1(K, x)$.



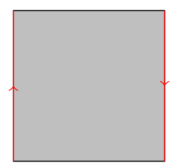
3 Classwork

In your learning groups, think about the following questions. We will need volunteers to present their solutions at the end of the session.

1. We know that if two loops based at x are freely homotopic then their homotopy classes are conjugate in $\pi_1(X, x)$. Is the converse true?

2. Suppose that X is a path-connected space in which two loops are freely homotopic if and only if they are based homotopic. Show that $\pi_1(X, x)$ is abelian.

3. The figure below shows how to glue together a Möbius strip (identifying the two red edges with a twist as indicated by the arrows):



The boundary of a Möbius strip is a circle. The *real projective plane* $\mathbf{RP}^2$ is obtained from a Möbius strip by collapsing this circle down to a single point. Consider the dotted purple loop below. What is the *order* of this loop when considered as an element of the fundamental group of $\mathbf{RP}^2$?