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PERTURBATION OF SCATTERING POLES FOR HYPERBOLIC SURFACES AND CENTRAL VALUES OF L-SERIES

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1. Introduction. Let $\Gamma\backslash\mathbb{H}$ be a noncompact hyperbolic surface of finite area. In the analysis of the Laplace-Beltrami operator on it, apart from the L^2 spectrum, a crucial role is played by the scattering poles. They appear in the analog of Weyl's law, in the Selberg zeta function, and in the determinant of the Laplace operator; see [18]. More important is the fact that imbedded eigenvalues become scattering poles under perturbation (see [25]). In this work we study variation formulas arising from perturbations of scattering poles.

A scattering pole is a pole of the analytic continuation of the determinant of the scattering matrix $\Phi(s)$ to the left half-plane $\Re s < 1/2$. These poles show among the poles of the analytic continuation of the resolvent on the same half-plane. But, generally, they are not exactly the same. There can be only finitely many exceptions for $0 \le s < 1/2$, where the resolvent has a pole at s_0, corresponding to a cuspidal eigenvalue $s_0(1-s_0) \in [0, 1/4)$, and where $\det\Phi(s)$ has a zero at s_0. We assume that this does not happen, or, alternatively, we look at scattering poles that do not lie on the interval $[0, 1/2)$.

Let $\mathbf{E}(z,s) = (E_1(z,s), \ldots, E_n(z,s))^T$ be the vector of Eisenstein series indexed by the cusps. Let m be the multiplicity of the pole of $\det\Phi(s)$ at s_0. We set $s(\epsilon)$ to be the weighted mean of scattering poles; that is, if $s_1(0) = s_2(0) = \cdots = s_m(0) = s_0$ and if the scattering poles split as $s_1(\epsilon), s_2(\epsilon), \ldots, s_m(\epsilon)$, when the perturbation is switched on, then

$$s(\epsilon) = \frac{1}{m}\sum_{i=1}^{m} s_i(\epsilon).$$

The first theorem concerns the first variation of $s(\epsilon)$ at $\epsilon = 0$:

$$\dot{s} = \frac{d}{d\epsilon}s(0).$$

Throughout this work $d\mu$ denotes the invariant hyperbolic measure $dx\,dy/y^2$.

THEOREM 1.1. *Assume all the Eisenstein series have a pole of order at most* 1 *at* s_0. *For a compactly supported perturbation of the metric, the first variation of the*

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weighted mean of the scattering pole at s_0 *of multiplicity* m *is*

$$\dot{s} = \frac{1}{m(2s_0-1)} \int_{\Gamma\backslash\mathbb{H}} \mathbf{E}(z,1-s_0)^T \operatorname{Res}_{s=s_0} \Phi(s) \dot{\Delta} \mathbf{E}(z,1-s_0)\, d\mu. \tag{1.1}$$

Remark 1.2. Let the space of residues of Eisenstein series at s_0 have dimension $m \le n$. According to Lemma 3.1 the order of the pole of $\det \Phi(s)$ at s_0 is m. We can find an invertible matrix M such that

$$\mathbf{A}(z,s_0) = M \lim_{s\to s_0} (s-s_0)\mathbf{E}(z,s)$$

has the first m elements $A_i(z,s_0)$ linearly independent and $A_i(z,s_0) = 0$ for $i > m$. Extend M to a holomorphic matrix $M(s)$ close to s_0 in any way desired. Let $\tilde{\mathbf{E}}(z,s) = M(s)\mathbf{E}(z,s) = (e_1(z,s),\dots,e_n(z,s))^T$ be a new basis of the Eisenstein series. Set $\mathbf{A}(z,s) = (s-s_0)\tilde{\mathbf{E}}(z,s)$. We have $(\Delta+1/4)A_i(z,s_0) = \lambda_0^2 A_i(z,s_0)$ with $\lambda_0 = s_0 - 1/2$. Set $V = V(s_0) = (v_{ji}) = M(s_0)^{-1}$. It is easy to see that all entries of $\Phi(s)$ have a pole of order at most 1 at s_0 by looking at the zero Fourier coefficient of $E_i(z,s)$ at the j-cusp. We can write (1.1) as

(1.2)
$$\dot{s} = \frac{1}{m(2s_0-1)} \int_{\Gamma\backslash\mathbb{H}} \mathbf{E}(z,1-s_0)^T \operatorname{Res}_{s=s_0} \Phi(s) M(1-s_0)^{-1} \dot{\Delta} \tilde{\mathbf{E}}(z,1-s_0)\, d\mu,$$

if one continues the new basis analytically to the point $1-s_0$. In the important example of congruence subgroups, $M(s)$ is explicitly given as the change of basis matrix from the Eisenstein series indexed by the cusps to the Eisenstein series with characters.

COROLLARY 1. *For a compactly supported perturbation of the metric, the first variation of a scattering pole at* s_0 *with multiplicity* 1 *for a surface with one cusp is*

$$\dot{s} = \frac{1}{2s_0-1} \operatorname{Res}_{s=s_0} \Phi(s) \int_{\Gamma\backslash\mathbb{H}} \Big(\dot{\Delta} E(z,1-s_0)\Big) E(z,1-s_0)\, d\mu.$$

COROLLARY 2. *For a compactly supported conformal perturbation of the metric* $g_\epsilon = e^{\epsilon f} g_0$, *where* $f \in C_c^\infty(\Gamma\backslash\mathbb{H})$, *the first variation of the scattering pole at* s_0 *with multiplicity* 1 *for a surface with one cusp is*

$$\dot{s} = \frac{s_0(1-s_0)}{2s_0-1} \operatorname{Res}_{s=s_0} \Phi(s) \int_{\Gamma\backslash\mathbb{H}} f(z) E(z,1-s_0)^2\, d\mu. \tag{1.3}$$

Since we assume that the scattering pole s_0 has multiplicity m, we are also interested in breaking the degeneracy under perturbation. We have the following theorem.

THEOREM 1.3. *Assume that all the Eisenstein series have a pole of order at most* 1 *at* s_0, *which is a scattering pole of multiplicity* m. *Consider the matrix*

$$\big(a^i_j\big) = \int_{\Gamma\backslash\mathbb{H}} \dot{\Delta} A_i(z,s_0) \sum_{k=1}^{n} E_k(z,1-s_0) v_{kj}\, d\mu, \quad i,j = 1,2,\dots,m.$$

If its eigenvalues are distinct and nonzero, then the degeneracy of the scattering pole at s_0 is removed by the compactly supported perturbation and the branches of scattering poles are distinct at s_0.

Remark 1.4. The restriction that all the Eisenstein series have a pole of order at most 1 at s_0 may seem too strict. However, in all examples where the scattering matrix has been computed, that is, certain congruence subgroups of $\mathrm{SL}(2,\mathbb{Z})$, it is easily verified, assuming that the zeros of the L-series involved are simple.

The simplest hyperbolic surface with a cusp is $\mathrm{SL}(2,\mathbb{Z})\backslash\mathbb{H}$. The scattering matrix for $\mathrm{SL}(2,\mathbb{Z})\backslash\mathbb{H}$ is

$$\phi(s) = \frac{\pi^{2s-1}\Gamma(1-s)\zeta(2-2s)}{\Gamma(s)\zeta(2s)}. \tag{1.4}$$

The function $\phi(s)$ plays a special role for other congruence subgroups, because it appears as a factor of $\det\Phi(s)$.

CONDITION A. *The nontrivial zeros ρ of the Riemann zeta function are simple, and they lie on $\Re s = 1/2$.*

The second part of the condition is the Riemann hypothesis (RH). It implies that the poles of $\phi(s)$ are at $s_0 = \rho/2 = 1/4 + i\gamma/2$, where $\gamma \in \mathbb{R}$. The simplicity of the zeros is also a natural condition to assume. Numerical evidence by Odlyzko [20] suggests that this is indeed true. Montgomery [17] proved that the pair correlation conjecture implies that almost all zeros are simple, and RH implies that at least $2/3$ of the zeros are simple. Unconditionally Conrey [3] proved that at least $2/5$ of the zeros are simple, while under RH and the generalized Lindelöf hypothesis at least $19/27$ of the zeros are simple (see [4]). The following weak form of the Mertens hypothesis implies that all the zeros of the zeta function on the critical line are simple (see [31, Th. 14.29, p. 376]). Let $\mu(n)$ be the Möbius function, and let $M(x) = \sum_{n\le x}\mu(n)$. Then

$$\int_1^X \left(\frac{M(x)}{x}\right)^2 dx = O(\log X).$$

We apply the perturbation results of Theorems 1.1 and 1.3 to the character varieties of $\Gamma_0(q)$, where q is a prime number. The perturbations are generated by holomorphic cusp forms of weight 2 for $\Gamma_0(q)$. Let $g(z)$ be such a form that has real coefficients, is a Hecke eigenform, and has eigenvalue ϵ_q for the Fricke involution W_q ($\epsilon_q = \pm 1$). We have the following theorem.

THEOREM 1.5. *For $\Gamma_0(q)$, q prime, the first variation of the weighted mean of scattering poles at any s_0 is zero.*

(a) *If the central value $L(g,1)$ of the L-series of g vanishes (which happens automatically if $\epsilon_q = 1$), then first-order degenerate perturbation theory does not remove the degeneracy at any s_0.*

(b) *If* $L(g,1)\neq 0$, *then the degeneracy is resolved,* $L(g,1)$ *is a factor of the first variation of the branches of scattering poles at any* s_0, *and the branches are not constant, that is, the scattering poles move.*

In fact, $L(g,1)$ and $L(g,2-2s_0)$ are factors of all the variation formulas, and the second value is on the line of convergence of $L(g,s)$.

Examples of groups that have a newform $g(z)$ with $L(g,1)=0$ are $\Gamma_0(q)$ with $q=37,43,53,61,79,83,89$. The groups with $q=37$ and 89 have two newforms and only one has the central value vanishing. The other groups have only one newform. In all cases the order of vanishing is exactly 1 and $\epsilon_q=1$. Examples of groups with all newforms $g(z)$ having $L(g,1)\neq 0$ are $\Gamma_0(q)$, with $q=11,17,19,67,73$. In fact they all have only one newform. The special value at 1 of the L-series $L(g,s)$ has been studied extensively in relation to the rank of the group of rational points of the elliptic curve $\mathbf{C}/\Lambda_g$, where Λ_g is the period lattice of $g(z)$. The order of vanishing of $L(g,s)$ at $s=1$ is, according to the Birch-Swinnerton-Dyer conjecture, equal to the rank of the elliptic curve. There has been extensive numerical evidence for the conjecture (see [5]), from where we collected the data above.

Theorem 1.5 relates the nonvanishing of $L(g,s)$ at $s=1$ to the movement of the scattering poles away from the position described by the zeros of the Riemann zeta function.

Remark 1.6. The central value of the L-series of a cusp form of weight 2 or 4 also appears in the first variation of resonances at $1/4$ in [22].

Remark 1.7. In Section 4.2 we explain how Theorems 1.1 and 1.3 can be applied to the groups $\Gamma_0(N)$, N composite, and $\Gamma(N)$ to produce similar results as in Theorem 1.5. The values of the L-series involved are central values of twists $L(g,\chi,s)$, where the conductor of χ divides N.

Remark 1.8. One can also study perturbations of the metric in Teichmüller space. In this case the tangent direction is specified by a holomorphic cusp form g of weight 4. One gets similar results to Theorem 1.5. The central value $L(g,2)$ shows up as a factor of the variation formulas. We intend to study this phenomenon elsewhere.

One would like to know which direction (most of) the scattering poles move. One hopes that most will move off the line $\Re s=1/4$ under perturbations in character varieties. Let us assume the character variety has two tangent directions generated by two distinct newforms $f(z)$ and $g(z)$ of $\Gamma_0(q)$, which have nonvanishing central values for their L-series: $L(f,1)L(g,1)\neq 0$. An extra technical assumption is the following condition.

CONDITION B. *There exists a prime* p *such that the corresponding Fourier coefficients* $b(p)$ *and* $a(p)$ *satisfy*

$$a(p)^2\neq b(p)^2. \tag{1.5}$$

This condition is mild. Two cusp forms with equal coefficients (except possibly finitely many of them) are identical. In the case when one of the f and g is a quadratic twist of the other, condition (1.5) fails, but the levels of the forms do not agree. Let the curves of characters generated by f and g be $\chi_{f,\epsilon}$ and $\chi_{g,\epsilon}$, respectively; see (4.1). We prove that, under Conditions A and B and for at least one of the forms f, g, a positive proposition of the scattering poles move off the line prescribed by the Riemann hypothesis. More precisely, we prove the following theorem.

THEOREM 1.9. *For at least one of the two curves $\chi_{f,\epsilon}$, $\chi_{g,\epsilon}$ and for T sufficiently large, we can find a $\delta = \delta(T) > 0$ such that there exists a positive proportion of the scattering poles with $|\Im s_0| \leq T$ that are to the left of $\Re s = 1/4$ and a positive proportion that are to the right of it for characters in the δ-neighborhood of the trivial character and on the curve.*

Remark 1.10. Selberg [29] has constructed a family $\chi(\alpha)$ of characters of $\Gamma_0(4)$ with the following property: for a given vertical line, one can find α such that there are scattering poles for the character $\chi(\alpha)$ to the left of the vertical line, and, in fact, approximately $c \cdot T$ of them have imaginary parts γ satisfying $|\gamma| < T$, for T large. In the case of Theorem 1.9 the number of scattering poles with $|\gamma| < T$ is asymptotic to $c \cdot T \log T$. Extensions of this result can be found in [2].

Remark 1.11. The assumption of RH is technical but clarifies the picture on the side of the scattering theory. If RH fails in the strong sense that there is a positive proportion of zeros of $\zeta(s)$ off $\Re s = 1/2$, then the corresponding scattering poles are off $\Re s = 1/4$ before the deformation and remain off for $\delta(T)$ sufficiently small. If zero proportion of the zeros of $\zeta(s)$ are off $\Re s = 1/2$, the proof in Section 5 needs to be modified. We can work as in [8], where the main theorem on discrete mean values of $\zeta(s)$ and its derivatives is proved without assuming RH.

Remark 1.12. For hyperbolic surfaces and for a character χ that is singular with respect to $\kappa_1 > 0$ cusps, one has

$$\det \Phi(s) = \left(\frac{\sqrt{\pi}\,\Gamma\big(s - 1/2\big)}{\Gamma(s)} \right)^{\kappa_1} a(\chi) b(\chi)^{1-2s} L(s, \chi),$$

where $L(s, \chi)$ is a Dirichlet series with constant term 1. It follows that $\det \Phi(s)$ does not vanish for $\Re s$ sufficiently large, which implies through the functional equation that the scattering poles are contained in a vertical strip $\sigma < \Re s < 1/2$. The constant σ depends on the group and character. Müller [18] raised the question of whether the same is true for a general surface with cusps. Froese and Zworski [7] gave a counterexample, which is a rotational symmetric surface. The motivation of this work was to see whether conformal perturbations of hyperbolic surfaces keep the scattering poles in a vertical strip or not. We study the size of $\dot{s}$ for conformal perturbations of $\mathrm{SL}(2, \mathbf{Z}) \backslash \mathbb{H}$ in [23].

Remark 1.13. Selberg [29] has shown that if we denote the scattering poles by s_0, then

$$\sum_{0\le\Im s_0\le T}\left(\frac{1}{2}-\Re s_0\right)=\frac{\kappa_1}{4\pi}T\log T+O(T),$$

$$\sum_{0\le\Im s_0\le T,\,\Re s_0<1/4}\left(\frac{1}{4}-\Re s_0\right)<\frac{\kappa_1}{2\pi}T\log\log T+c(a,b,\sigma)T+O(\log T),$$

and

$$\sum_{0\le\Im s_0\le T,\,\Re s_0<\sigma}(\sigma-\Re s_0)<c'(a,b,\sigma)T+O(\log T)$$

for $\sigma<1/4$. This shows that the distance of the scattering poles to the unitary axis $\Re s=1/2$ is on the average not greater than $1/4$. All the estimates are uniform in the characters for Γ fixed, as long as the number $a=a(\chi)$ is bounded away from zero. It is not known whether other surfaces with cusps satisfy these estimates.

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2. Preliminary material. We recall some standard facts about the resolvent and its analytic continuation, which is due to Faddeev [6] (see also [15, Ch. XIV]), and define the operators and norms of the Lax-Phillips scattering theory as applied to automorphic functions. Let $R(s)=(-\Delta-s(1-s))^{-1}$ be the resolvent of the Laplace operator. Its kernel is constructed as follows. The fundamental point-pair invariant is $u(z,z')=|z-z'|^2/(4yy')$ for $z,z'\in\mathbb{H}$. We set $\varphi(u,s)=\int_0^1[t(1-t)]^{s-1}(t+u)^{-s}\,dt/(4\pi)$ for $\sigma>0, u>0$, $s=\sigma+it$, and $k(z,z';s)=\varphi(u(z,z'),s)$. The kernel $k(z,z';s)$ is the Green function for the problem $\Delta h+s(1-s)h=f$ at least for $\sigma>1$ (see [15, p. 275]). For a discrete cofinite subgroup Γ of $\mathrm{SL}(2,\mathbb{R})$, we set

$$r(z,z';s)=\frac{1}{2}\sum_{\gamma\in\Gamma}\varphi\big(u(z,\gamma z'),s\big) \tag{2.1}$$

for $\sigma>1$. This is the resolvent kernel. We decompose the fundamental domain F of Γ into

$$F=F_0\cup\bigcup_{j=1}^{n}F_j,$$

where the F_j's are isometric to the standard cusp. Let the cusps be $z_1,z_2,\dots,z_n$. There exists a g_j with $g_j\infty=z_j$. One can choose $g_j\in\mathrm{SL}(2,\mathbb{R})$ so that $z\to g_jz$ maps $C=\{z;-1/2\le\Re z\le 1/2,\ \Im z\ge a\}$ one-to-one onto F_j. Each function f on F

has $n+1$ components $f_0(z) = f(z)$ for $z \in F_0$ and $f_j(z) = f(g_j z)$ for $z \in C$. One has the decomposition

$$L^2(\Gamma \setminus \mathbb{H}) = L^2(F_0) \oplus \bigoplus_{j=1}^{n} L^2(F_j),$$

but it turns out that one has to use weighted spaces to define the analytic continuation of the resolvent. Faddeev [6] introduced the Banach spaces B_μ, which consist of complex-valued functions $f(z)$ whose components $f_0(z)$ and $f_j(z)$, $j = 1, \dots, n$, are continuous on F_0 and C, respectively, with $|f_j(z)| \leq c y^\mu$ for $z \in C$. The μ-norm is

$$\|f\|_\mu = \max_{z \in F_0} \left| f_0(z) \right| + \sum_{j=1}^{n} \max_{z \in C} \frac{\left| f_j(z) \right|}{y^\mu}.$$

One can attach a meaning to the analytic continuation of the resolvent kernel on a Riemann surface that is a 2-sheeted covering of the z-plane. We set $z = s(1-s)$, and then the z-plane cut along the ray $[0, \infty)$ corresponds to the right half-plane $\Re s > 1/2$ cut along $1/2 \leq s \leq 1$. For s, $1-s$ nonsingular, we have the limiting absorption principle

$$r(z, z'; s) - r(z, z'; 1-s) = \frac{1}{2s-1} \sum_{j=1}^{n} E_j(z, s) E_j(z', 1-s), \tag{2.2}$$

a proof of which is given in [15, p. 344]. After obtaining the analytic continuation of the resolvent kernel, one defines $R(s)$ in the following manner. Fix $\mu \leq 1/2$; then $R(s) : B_\mu \to B_{1-\mu}$ is defined for $\Re s > \mu$ as the integral operator with kernel $r(z, z'; s)$. A slightly different approach that works for all surfaces with cusps and where the resolvent is considered as an operator-valued function with values in the bounded operators between weighted L^2-spaces was worked out by Müller [19]. It should be remarked that in (2.2) the Eisenstein series are indexed by the cusps and are defined as

$$E_j(z, s) = \sum_{\sigma \in \Gamma_j \setminus \Gamma} \Big(\Im\big(g_j^{-1} \sigma z\big) \Big)^s,$$

where Γ_j is the stabilizer of the cusp z_j. This way the zero Fourier coefficient of $E_j(z, s)$ at the cusp z_i is $\delta_{ij} y^s + \phi_{ij}(s) y^{1-s}$. The scattering matrix is $\Phi(s) = (\phi_{ij}(s))$.

We set $L = \Delta + (1/4)$, $A = \left(\begin{smallmatrix} 0 & 1 \\ L & 0 \end{smallmatrix}\right)$, and E the energy form for the wave equation $u_{tt} = Lu$, that is,

$$E\big((f_1, f_2)^T\big) = -(f_1, Lf_1)_{L^2} + (f_2, f_2)_{L^2}.$$

Let H_G be the completion of the space of pairs of C^∞ functions with compact support in the norm

$$G\big((f_1, f_2)^T\big) = E\big((f_1, f_2)^T\big) + c\|f_1\|_2^2$$

for c sufficiently large. Let P be the E-orthogonal projection to H, which is the complement of the space $D_+ \oplus D_-$, in H_G, where $D_\pm$ are the spaces of outgoing and incoming data (see [16, p. 121]). The operator P may only change the zero Fourier coefficients of data at each cusp. The operator B is the infinitesimal generator of the semigroup $PU(t)P$, where $U(t)$ is the standard wave operator. We denote by $R_F(z)$ the resolvent of an operator F, that is, $R_F(z) = (F - z)^{-1}$. We have

$$R_B(\lambda) = PR_A(\lambda)P \tag{2.3}$$

for $\Re\lambda$ sufficiently large (see [16, p. 29]). A calculation with matrices gives

$$\begin{aligned} R_A(\lambda) &= \begin{pmatrix} \lambda R_L(\lambda^2) & R_L(\lambda^2) \\ LR_L(\lambda^2) & \lambda R_L(\lambda^2) \end{pmatrix} \\ &= \begin{pmatrix} -\lambda R(\lambda+1/2) & -R(\lambda+1/2) \\ I - \lambda^2 R(\lambda+1/2) & -\lambda R(\lambda+1/2) \end{pmatrix}, \end{aligned} \tag{2.4}$$

since $R_L(\lambda^2) = -R(s)$ and $\lambda = s - 1/2$.

Call f_i the vector $\binom{A_i(z,s_0)}{\lambda_0 A_i(z,s_0)}$. Then $Af_i = \lambda_0 f_i$. The eigenvector of the cutoff wave operator B is Pf_i, since $BPf_i = PAf_i = \lambda_0 Pf_i$.

3. Proof of Theorems 1.1 and 1.3. The idea of the proof of Theorem 1.1 is to use perturbation theory for the cutoff wave operator B, which is an operator with discrete spectrum (see [25, p. 6]). However, since B is not selfadjoint, we choose to use variational formulas that use traces (see [14, p. 90]), instead of energy inner products, as was done in [25, p. 24]. A similar method was used in [21] and [22] to study the variation of the resonances at 1/4 and Fermi's golden rule.

An outgoing eigenfunction of A with eigenvalue λ_0 is a pair $f = (f_1, f_2)^T$ that satisfies the following:

(1) $(A - \lambda_0)^k f = \mathbf{0}$ for some integer k;

(2) f is outgoing; that is, the zero Fourier coefficients $f_1^{(0)}$, $f_2^{(0)}$ of f_1, f_2 satisfy $f_2^{(0)} = -y^{3/2}\partial_y(f_1^{(0)}/y^{1/2})$ in the cusps; and

(3) $A^j f$ minus its zero Fourier coefficient in the cusps lies in H_G for $j = 0, 1, \ldots, k-1$.

LEMMA 3.1. *All the Eisenstein series have a pole of order at most* 1 *at* s_0 *if and only if* $s_0 - 1/2$ *is a semisimple eigenvalue of the cutoff wave operator* B.

Proof. Assume that λ_0 is a semisimple eigenvalue of B. According to [25, Th. 3.1] the semisimplicity of the eigenvalue $s_0 - 1/2$ for B is equivalent to the semisimplicity of the outgoing eigenspace of A at $\lambda_0 = s_0 - 1/2$. We show that any pole of order greater than 1 for any Eisenstein series produces an outgoing eigenfunction $g = (g_1, g_2)^T$ of A with $(A - \lambda_0)^2 g = 0$ but $(A - \lambda_0)g \neq 0$, which contradicts the semisimplicity of the eigenspace for A. If the pole of the Eisenstein series

$E(z,s)$ at s_0 has order $k \geq 2$, set $A(z,s) = (s-s_0)^k E(z,s)$. By differentiation of $\Delta A(z,s) + s(1-s)A(z,s) = 0$ in s, we get $LB(z,s_0) - \lambda_0^2 B(z,s_0) = 2\lambda_0 A(z,s_0)$, where $B(z,s_0) = dA(z,s_0)/ds$. Set $g_1 = B(z,s_0)$ and $g_2 = A(z,s_0) + \lambda_0 B(z,s_0)$. Then $(A-\lambda_0)g = (A(z,s_0), \lambda_0 A(z,s_0))^T \neq \mathbf{0}$. However, $(A-\lambda_0)^2 g = \mathbf{0}$. It is easy to check condition (2) for g.

We now prove the converse. Assume the Eisenstein series have poles of order at most 1 and the space of residues has dimension m as in Remark 1.2. We have at least m eigenvectors for A, the f_i, $i \leq m$: $Af_i = \lambda_0 f_i$. If we prove that the order of the pole of $\det \Phi(s)$ at s_0 is at most m, since by [25, Th. 4.1] the order of the pole is the multiplicity of the outgoing eigenspace of A, then the dimension of the eigenspace is exactly m. Therefore the eigenvalue is semisimple.

We see that $\mathbf{A}(z,s_0) = M(s_0)\operatorname{Res}_{s=s_0}\Phi(s)\mathbf{E}(z,1-s_0)$. Since $A_i(z,s_0) = 0$ for $i > m$ and $E_j(z,1-s_0)$ are linearly independent, the matrix $M(s_0)\operatorname{Res}_{s=s_0}\Phi(s)$ has zero entries on the rows $m+1, m+2, \dots, n$. Set $N(s) = M(s)\Phi(s)$. Its entries on the rows $m+1, m+2, \dots, n$ should be regular, while the other entries have a pole of order at most 1. By multiplying the first m rows by $s-s_0$, we see that

$$(s-s_0)^m \det N(s) = (s-s_0)^m \det M(s) \det \Phi(s)$$

remains bounded close to s_0. However, $\det M(s) \neq 0$, so $(s-s_0)^m \det \Phi(s)$ remains bounded close to s_0. □

The first variation of the weighted mean for a scattering pole, which is also the first variation of the weighted mean for the eigenvalue λ_0 of B, is given by

$$\dot{s} = \frac{1}{m}\operatorname{Tr}(\dot{B}Q), \tag{3.1}$$

where Q is the projection to the eigenspace of B generated by the Pf_i's (see [14, 2.33, p. 90]). If Γ is a contour enclosing only $\lambda_0 = s_0 - 1/2$ among the eigenvalues of B, then, using (2.3) and (2.4) one gets

$$\begin{aligned} Q &= -\frac{1}{2\pi i}\int_\Gamma R_B(\lambda)\,d\lambda = -\frac{1}{2\pi i}P\int_\Gamma R_A(\lambda)\,d\lambda P \\ &= -\frac{1}{2\pi i}P\int_\Gamma \begin{pmatrix} -\lambda R(\lambda+1/2) & -R(\lambda+1/2) \\ I-\lambda^2 R(\lambda+1/2) & -\lambda R(\lambda+1/2)\end{pmatrix} d\lambda P. \end{aligned} \tag{3.2}$$

With the standard inner product on $\mathbf{R}^n$, we have $\sum_{j=1}^n E_j(z,s)E_j(z',1-s) = \mathbf{E}(z',1-s)^T \cdot \mathbf{E}(z,s) = \mathbf{E}(z',1-s)M(s)^{-1}\tilde{\mathbf{E}}(z,s)$.

By (2.2), since $r(z,z',1-s)$ is regular at s_0, the contour integral in (3.2) is an operator with integral kernel

$$-\begin{pmatrix} \mathbf{E}(z',1-s_0)^T V\mathbf{A}(z,s_0)/2 & \mathbf{E}(z',1-s_0)^T V\mathbf{A}(z,s_0)/(2s_0-1) \\ (s_0-1/2)\mathbf{E}(z',1-s_0)^T V\mathbf{A}(z,s_0)/2 & \mathbf{E}(z',1-s_0)^T V\mathbf{A}(z,s_0)/2 \end{pmatrix}.$$

If $g=(g_1,g_2)^T$ is any pair of data supported in some compact set of the surface and with zero Fourier coefficient of g_1 and g_2 vanishing above the cut at $y=a$, then $Pg=g$. Let K be the integral operator with kernel

$$K(z,z')=\begin{pmatrix}1\\2\end{pmatrix}\mathbf{E}(z',1-s_0)^T V\mathbf{A}(z,s_0).$$

Then

$$Qg=P\begin{pmatrix}Kg_1+1/(s_0-1/2)Kg_2\\(s_0-1/2)Kg_1+Kg_2\end{pmatrix}.$$

Notice that Kg_1 and Kg_2 are linear combinations of the $A_j(z,s_0)$'s, $j\le m$, and Qg is a linear combination of the Pf_j's, $j\le m$, as it should be. Since $\dot{B}=\left(\begin{smallmatrix}0&0\\\dot{L}&0\end{smallmatrix}\right)$, we have

$$\dot{B}f_i=\begin{pmatrix}0\\\dot{L}A_i(z,s_0)\end{pmatrix}\tag{3.3}$$

and

$$Q\begin{pmatrix}0\\\dot{L}A_i(z,s_0)\end{pmatrix}=\frac{1}{s_0-1/2}P\begin{pmatrix}K\big(\dot{L}A_i(z,s_0)\big)\\(s_0-1/2)K\big(\dot{L}A_i(z,s_0)\big)\end{pmatrix}.$$

We have

$$K\big(\dot{L}A_i(z,s_0)\big)=\int_{\Gamma\backslash\mathbb{H}}\dot{L}A_i(z',s_0)K(z,z')\,d\mu(z')=\begin{pmatrix}1\\2\end{pmatrix}\mathbf{a}^i\cdot\mathbf{A}(z,s_0),$$

where $\mathbf{a}^i$ is the row vector

$$\mathbf{a}^i=\int\dot{L}A_i(z',s_0)\mathbf{E}(z',1-s_0)^T V\,d\mu(z').\tag{3.4}$$

Then

$$Q\begin{pmatrix}0\\\dot{L}A_i(z,s_0)\end{pmatrix}=\frac{1}{2s_0-1}P\begin{pmatrix}\mathbf{a}^i\cdot\mathbf{A}(z,s_0)\\(s_0-1/2)\mathbf{a}^i\cdot\mathbf{A}(z,s_0)\end{pmatrix}.\tag{3.5}$$

Finally,

$$\dot{B}Q\begin{pmatrix}0\\\dot{L}A_i(z,s_0)\end{pmatrix}=\frac{1}{2s_0-1}\begin{pmatrix}0\\\mathbf{a}^i\cdot\dot{L}\mathbf{A}(z,s_0)\end{pmatrix}$$

for $j\le m$. The operator $\dot{B}Q$ maps H into the space spanned by $\left(\begin{smallmatrix}0\\\dot{L}A_i(z,s_0)\end{smallmatrix}\right)$. The functions $\dot{L}A_i(z,s_0)$, $i\le m$, may not be linearly independent, but, still, $\mathrm{Tr}(\dot{B}Q)=1/(2s_0-1)\sum_{i=1}^m a_i^i$. This follows from elementary linear algebra: assume that only the first k out of the m vectors $\left(\begin{smallmatrix}0\\\dot{L}A_i(z,s_0)\end{smallmatrix}\right)$ are linearly independent and extend them to any basis of the whole space. If

$$\dot{L}A_j(z,s_0)=\sum_{t=1}^k b_{jt}\dot{L}A_t(z,s_0),\quad j=k+1,\dots,m,$$

then

$$(2s_0-1)\dot{B}Q\begin{pmatrix}0\\ \dot{L}A_i(z,s_0)\end{pmatrix}=\sum_{j=1}^{k}a_j^i\begin{pmatrix}0\\ \dot{L}A_j(z,s_0)\end{pmatrix}+\sum_{j=k+1}^{m}a_j^i\sum_{t=1}^{k}b_{jt}\begin{pmatrix}0\\ \dot{L}A_t(z,s_0)\end{pmatrix}.$$

The contribution to $\mathrm{Tr}(\dot{B}Q)$ is $(a_i^i+\sum_{j=k+1}^{m}a_j^i b_{ji})/(2s_0-1)$ and

$$\mathrm{Tr}(\dot{B}Q)=\frac{1}{2s_0-1}\sum_{i=1}^{k}\left(a_i^i+\sum_{j=k+1}^{m}a_j^i b_{ji}\right).$$

However, $\sum_{i=1}^{k}a_j^i b_{ji}=a_j^j$ for $j=k+1,\dots,m$. This is so because (3.5) gives

$$\begin{aligned}(2s_0-1)Q\begin{pmatrix}0\\ \dot{L}A_j(z,s_0)\end{pmatrix}&=(2s_0-1)\sum_{t=1}^{k}b_{jt}Q\begin{pmatrix}0\\ \dot{L}A_t(z,s_0)\end{pmatrix}\\ &=\sum_{t=1}^{k}b_{jt}\sum_{p=1}^{m}a_p^t P\begin{pmatrix}A_p(z,s_0)\\ \lambda_0 A_p(z,s_0)\end{pmatrix}=\sum_{p=1}^{m}a_p^j P\begin{pmatrix}A_p(z,s_0)\\ \lambda_0 A_p(z,s_0)\end{pmatrix}.\end{aligned}$$

Using (3.4), one gets

$$\frac{1}{2s_0-1}a_i^i=\frac{1}{2s_0-1}\sum_{j=1}^{n}\int_{\Gamma\backslash\mathbb{H}}E_j(z',1-s_0)v_{ji}\,\dot{\Delta}A_i(z',s_0)\,d\mu(z'),\tag{3.6}$$

since $\dot{L}=\dot{\Delta}$. Since $A_i(z,s_0)=0$ for $i=m+1,\dots,n$, we can take the same formula as valid for all $i=1,\dots,n$. Therefore,

$$\begin{aligned}(2s_0-1)\,\mathrm{Tr}(\dot{B}Q)&=\sum_{i,j=1}^{n}\int_{\Gamma\backslash\mathbb{H}}E_j(z',1-s_0)v_{ji}\,\dot{\Delta}A_i(z',s_0)\,d\mu(z')\\ &=\int_{\Gamma\backslash\mathbb{H}}\mathbf{E}(z',1-s_0)^T M(s_0)^{-1}\dot{\Delta}\mathbf{A}(z',s_0)\,d\mu(z').\end{aligned}$$

Now we switch back to the original basis of Eisenstein series indexed by the cusps, and we use the functional equation for $\mathbf{E}(z,s)$ to get (1.1).

Proof of Corollary 1. It is obvious from Theorem 1.1. □

Proof of Corollary 2. If $g_\epsilon=e^{\epsilon f}g_0$, then $\Delta_\epsilon=e^{-\epsilon f}\Delta_0$ and $\dot{\Delta}=-f\Delta_0$, where Δ_0 is the Laplacian of the unperturbed metric. Since the Eisenstein series $E(z,1-s_0)$ corresponds to the eigenvalue $s_0(1-s_0)$, formula (1.3) follows. □

Proof of Theorem 1.3. Since the eigenvalue λ_0 of B is semisimple, [14, Th. 2.3, p. 93] implies that the eigenvalue branches at λ_0 are continuously differentiable at $\epsilon = 0$. They are of the form

$$\lambda_0 + \epsilon\lambda_i^{(1)} + o(\epsilon), \quad i = 1, 2, \dots, m, \tag{3.7}$$

where $\lambda_i^{(1)}$ are the eigenvalues of $B^{(1)} = Q\dot{B}Q$. If we can solve the eigenvalue problem for $B^{(1)}$ on the image of Q, that is, $B^{(1)}w = \lambda w$, then we remove the degeneracy and, if the eigenvalues $\lambda_i^{(1)}$ are distinct, then the perturbation produces distinct power series at λ_0. Let

$$w = \sum_{i=1}^{m} x_i P f_i$$

be any vector in the image of Q. Then

$$\begin{aligned} B^{(1)}w = Q\dot{B}w &= \sum_{i=1}^{m} x_i Q \begin{pmatrix} 0 \\ \dot{L}A_i(z, s_0) \end{pmatrix} \\ &= \frac{1}{2s_0 - 1} \sum_{i=1}^{m} x_i P \begin{pmatrix} \mathbf{a}^i \cdot \mathbf{A}(z, s_0) \\ \lambda_0 \mathbf{a}^i \cdot \mathbf{A}(z, s_0) \end{pmatrix} = \frac{1}{2s_0 - 1} \sum_{i=1}^{m} x_i \sum_{j=1}^{m} a_j^i P f_j. \end{aligned}$$

Since the Pf_j's are linearly independent, the equation $B^{(1)}w = \lambda w$ gives

$$\frac{1}{2s_0 - 1} \sum_{i=1}^{m} x_i a_j^i = \lambda x_j, \quad j = 1, 2, \dots, m.$$

This is the eigenvalue equation for the matrix (a_j^i), $i, j = 1, \dots, m$, with eigenvalue $\lambda(2s_0 - 1)$. □

4. Character varieties

4.1. General theory of character perturbations. For $\Gamma \setminus \mathbb{H}$, the first homology group is isomorphic to $\Gamma/[\Gamma, \Gamma]$. Its dual group consists of the unitary characters χ of Γ and $A_{\text{cusp}} = \{\chi \mid \chi(p) = 1,\ p \in \Gamma,\ p \text{ parabolic}\}$. The cohomology classes in the first de Rham cohomology which can be represented by forms of compact support have a square integrable harmonic representative (which can be taken to be cuspidal, that is, if $w = w_0\,dy + w_1\,dx$, then $\int_C w_0 = \int_C w_1 = 0$, where C is a path corresponding to a parabolic). Fix $z_0 \in \Gamma \setminus \mathbb{H}$, and let $\pi : \Gamma \to \Gamma/[\Gamma, \Gamma]$ be the natural projection from $\pi_1(\Gamma \setminus \mathbb{H}) \to H_1(\Gamma \setminus \mathbb{H}, \mathbb{R})$. For any cuspidal harmonic square integrable form w, we set

$$\chi_w(\gamma) = \exp\left(2\pi i \int_{\pi(\gamma)} w\right), \tag{4.1}$$

which is a cuspidal character in the connected component of the trivial character in A_{cusp}.

The deformation we consider depends on a real parameter ϵ, and the corresponding spectral problem concerns L^2-functions satisfying

$$h(\gamma z) = \chi_{\epsilon w}(\gamma) h(z).$$

Let us denote the corresponding L^2 space by $L^2(\Gamma \setminus \mathbb{H}, \chi_{\epsilon w})$. We conjugate this space to the fixed space $L^2(\Gamma \setminus \mathbb{H})$ as follows. Set

$$(U_\epsilon h)(z) = \exp\left(2\pi i \int_{z_0}^{z} \epsilon w\right) h(z)$$

so that $U_\epsilon : L^2(\Gamma \setminus \mathbb{H}) \to L^2(\Gamma \setminus \mathbb{H}, \chi_{\epsilon w})$. We set

$$L(\epsilon) = U_\epsilon^{-1} \Delta U_\epsilon,$$

which now acts on $L^2(\Gamma \setminus \mathbb{H})$. We define $\delta(p\,dx + q\,dy) = -y^2(p_x + q_y)$, $\langle p\,dx + q\,dy, f\,dx + g\,dy \rangle = y^2(p\bar{f} + q\bar{g})$, and $|p\,dx + q\,dy|_H^2 = y^2(|p|^2 + |q|^2)$. Then it is easy to see that

$$L(\epsilon)u = \Delta u + 4\pi i \epsilon \langle du, w \rangle - 4\pi^2 \epsilon^2 |w|_H^2 u - 2\pi i \epsilon (\delta w) u.$$

If w is harmonic, the last term vanishes. Let $f(z)$ now be a holomorphic cusp form of weight 2, and let w be the real-valued harmonic form $\Re(f(z)\,dz)$. So $f(z) = w_1 - i w_0$. As usual we define for a function $f(z)$ of weight k and $T \in \mathrm{GL}(2, \mathbb{R})$,

$$(f \mid T)(z) = (\det T)^{k/2} (cz + d)^{-k} f(Tz).$$

Let $f \mid U(z) = \sum_{n>0} a_n e^{2\pi i n z}$ be its expansion at the cusp z_i, where $U = g_i$ in the notation of Section 2. Let $u(z)$ be any of the Eisenstein series $e_k(z, 1 - s_0)$ with Fourier expansion at the z_i cusp of the form

$$u \mid U(z) = A y^{s_0} + B y^{1-s_0} + \sum_{n \neq 0} c_n y^{1/2} K_{1/2 - s_0}(2\pi |n| y) e^{2\pi i n x}. \tag{4.2}$$

Then

$$\dot{\Delta} u(z) = 4\pi i \langle du, w \rangle.$$

The equation (1.2) shows that the variation of the weighted mean of the scattering pole is a linear combination of

$$\int_{\Gamma \setminus \mathbb{H}} E_i(z, 1 - s_0) \dot{\Delta} u(z)\, d\mu$$

and this is $R(1 - s_0)$, where

$$R(s) = \int_{\Gamma \setminus \mathbb{H}} E_i(z, s) \dot{\Delta} u(z)\, d\mu.$$

Since $2w = f(z)\,dz + \overline{f(z)\,dz}$, we have

$$\langle du, w\rangle = y^2\big(u_z\overline{f(z)} + u_{\bar z} f(z)\big).$$

We also set

$$f \mid U(z) - \overline{f \mid U(z)} = \sum_{n\neq 0} b_n e^{-2\pi|n|y} e^{2\pi i n x}$$

so that $b_n = a_n$ for $n > 0$ and $b_n = -\overline{a_{-n}}$ for $n < 0$. Then for $\Re s \gg 0$ we have

$$\begin{aligned} R(s) &= 4\pi i \int_{U^{-1}\Gamma U\backslash\mathbb{H}} \big(\langle du, w\rangle \mid U\big)(z) E_i(U(z), s)\,dx\,dy/y^2 \\ &= 4\pi i \int_{\Gamma_\infty\backslash\mathbb{H}} \big(\langle du, w\rangle \mid U\big)(z) y^s\,dx\,dy/y^2 \\ &= 4\pi i \int_{\Gamma_\infty\backslash\mathbb{H}} \big[(u \mid U)_z \overline{f \mid U(z)} + (u \mid U)_{\bar z} f \mid U(z)\big] y^s\,dx\,dy. \end{aligned}$$

An integration by parts gives

$$\begin{aligned} R(s) &= 2\pi \int_0^\infty \int_0^1 (u \mid U)(z) s y^{s-1}\big[f \mid U(z) - \overline{f \mid U(z)}\big]\,dx\,dy \\ &= 2\pi s \sum_{n\neq 0} \frac{c_n b_{-n}}{(2\pi|n|)^{s+1/2}} \int_0^\infty e^{-y} y^{s-1/2} K_{1/2-s_0}(y)\,dy \qquad (4.3) \\ &= \frac{s}{4^s\pi^{s-1}} \frac{\Gamma(s+1-s_0)\Gamma(s+s_0)}{\Gamma(s+1)} \sum_{n=1}^\infty \frac{-c_n\bar a_n + c_{-n}a_n}{n^{s+1/2}}, \end{aligned}$$

using [11, 6.621.3, p. 733].

In the case that $c_{-n} = c_n$ and the a_n are purely imaginary, the Dirichlet series in (4.3) becomes $2\sum_{n>0} c_n a_n n^{-(s+1/2)}$.

Remark 4.1. The case $c_n = c_{-n}$ is the only case we are interested in because it is relevant to congruence subgroups (see (4.4) below). If the a_n are real, then the Dirichlet series in (4.3) vanishes. This can also be explained in the following manner. The Eisenstein series $E_i(z,s)$ is even in x, and so is u and u_y, while u_x is odd. If the a_n's are real, $w_1 = \Re f$ is even, while $w_0 = -\Im f$ is odd. So $\langle du, w\rangle = y^2(u_x w_1 + u_y w_0)$ is odd and $R(s)$ vanishes.

4.2. Congruence subgroups and central values of L-series. We now assume that $g(z) = -if(z)$ is a Hecke eigenform for some congruence subgroup of level N with Hecke eigenvalues $\alpha_1(p)$ and $\alpha_2(p)$ for T_p and all coefficients $a(n)$ real. Then $(1 - a(p)p^{-s} + p^{1-2s}) = (1-\alpha_1(p)p^{-s})(1-\alpha_2(p)p^{-s})$ for $p \nmid N$. If $p \mid N$, the Euler factor is $(1 - a(p)p^{-s})$. We also assume that for $n > 0$,

$$c_{\pm n} = \sum_{ck=n} \chi_1(c)\bar\chi_2(k)\left(\frac{k}{c}\right)^{1/2-s_0}, \qquad (4.4)$$

where χ_1 and χ_2 are primitive characters modulo q_1 and q_2. It is easy to see that the coefficients c_n are multiplicative. In fact $e_k(z, 1-s_0)$ is also a Hecke eigenform and we easily find the Euler factors for its L-series. It is

$$L(s) = \prod_p \big(1-\chi_1(p)p^{-s-1/2+s_0}\big)^{-1}\big(1-\bar{\chi}_2(p)p^{-s+1/2-s_0}\big)^{-1}.$$

For example, one can explicitly compute $\sum_{k\geq 0} c_{p^k} p^{-ks}$. Now we get the Euler factors of $\sum c_n a(n)n^{-s}$ using [30, Lemma 1]. We get

$$\sum_{n=1}^{\infty} c_n a(n) n^{-s} = \prod_p X_p(s) Y_p(s)^{-1},$$

where

$$X_p(s) = 1-\chi_1(p)\bar{\chi}_2(p)p^{-2s+1}, \quad p \nmid N$$

using $\alpha_1(p)\alpha_2(p) = p$ and $X_p(s) = 1$ for $p \mid N$. Also

$$\begin{aligned} Y_p(s) = &\big(1-\alpha_1(p)\chi_1(p)p^{-s-1/2+s_0}\big)\big(1-\alpha_2(p)\chi_1(p)p^{-s-1/2+s_0}\big) \\ &\cdot\big(1-\alpha_1(p)\bar{\chi}_2(p)p^{-s-s_0+1/2}\big)\big(1-\alpha_2(p)\bar{\chi}_2(p)p^{-s-s_0+1/2}\big). \end{aligned}$$

We denote the twisted L-series of $g(z)$ by a character χ as $L(g, \chi, s)$. Then

(4.5)
$$\sum_{n=1}^{\infty} \frac{c_n a(n)}{n^s} = \prod_{p\nmid N} \big(1-\chi_1(p)\bar{\chi}_2(p)p^{-2s+1}\big) L\big(g, \chi_1, s+1/2-s_0\big) L\big(g, \bar{\chi}_2, s+s_0-1/2\big).$$

We notice that in (4.3) we are interested in the Rankin-Selberg convolution at $s+1/2$ and that in the first variation formula (1.1) we have $s = 1-s_0$. So we set $s = 3/2-s_0$ in (4.5). We get

$$\sum_{n=1}^{\infty} \frac{c_n a(n)}{n^{3/2-s_0}} = L\big(\chi_1\bar{\chi}_2\omega_N, 2-2s_0\big)^{-1} \cdot L\big(g, \chi_1, 2-2s_0\big) \cdot L\big(g, \bar{\chi}_2, 1\big), \tag{4.6}$$

where ω_N is the trivial character modulo N. We notice that the factor $L(g, \bar{\chi}_2, 1)$ shows up irrespective of which scattering pole s_0 we consider, provided it is a pole of $e_k(z, s)$.

For the congruence subgroups $\Gamma(N)$ and $\Gamma^0(N)$, which consist of matrices in $\mathrm{SL}(2, \mathbb{Z})$ which are lower triangular mod N, the discussion above applies. We notice that $\Gamma^0(N)$ is conjugate to $\Gamma_0(N)$, so they have the same spectral theory. According to Huxley [13], the space of nonholomorphic Eisenstein series is spanned by Eisenstein series with characters

$$E_{\chi_1}^{\chi_2}(z, s) = \sum_{(c,d)=1} \frac{\chi_1(c)\chi_2(d)y^s}{|cz+d|^{2s}},$$

where χ_1 and χ_2 are primitive characters modulo q_1 and q_2, respectively, where q_1 and q_2 are appropriate divisors of N and $\chi_1(-1)=\chi_2(-1)$.

(1) For $\Gamma(N)$ we take $E_{\chi_1}^{\chi_2}(m_1 z/m_2, s)$, where $m_1 q_1 \mid N$ and $m_2 q_2 \mid N$.

(2) For $\Gamma^0(N)$ we take $E_{\chi}^{\chi}(z/m, s)$, χ primitive modulo q, and $q \mid m$, $mq \mid N$.

The Fourier expansion of $E_{\chi_1}^{\chi_2}(z,s)$ at ∞ is

$$\frac{1}{L\big(\chi_1\chi_2, 2s\big)}\Bigg(2\delta_0(q_1-1)L(\chi_2,2s)y^s+2\delta_0(q_2-1)\sqrt{\pi}\,\frac{\Gamma\big(s-1/2\big)L\big(\chi_1,2s-1\big)}{\Gamma(s)}y^{1-s}$$
$$+\sum_{n>0}\frac{4\pi^s\tau(\chi_2)}{\Gamma(s)q_2^s}\sum_{ck=n}\chi_1(c)\bar{\chi}_2(k)\left(\frac{k}{c}\right)^{s-1/2}\left(\frac{y}{q_2}\right)^{1/2}$$
$$\times K_{s-1/2}\left(\frac{2\pi ny}{q_2}\right)\cos\left(\frac{2\pi nx}{q_2}\right)\Bigg).$$

The scattering poles appear at s_0, where $2s_0$ is a zero of $L(\chi_1\chi_2, s)$. These zeros are conjecturally simple and different from the zeros of the other L-series. It follows that the coefficients of $e_k(z, 1-s_0)$ are exactly of the form (4.4), up to a factor $2(\pi/q_2)^{1-s_0}\tau(\chi_2)/(\Gamma(1-s_0)L(\chi_1\chi_2, 2-2s_0))$.

Notice that the coefficients c_n in [13] differ from (4.4) by a factor $1/\sqrt{n}$, because Huxley includes in the Bessel function $K_{s-1/2}(u)$ a factor $\sqrt{u}$. This does not affect (4.6).

4.3. The group $\Gamma_0(q)$: Proof of Theorem 1.5. We concentrate now on the Hecke congruence subgroups $\Gamma_0(q)$, where q is a prime number. These groups have only two cusps: at ∞ of width 1 and at 0 of width q. The space of Eisenstein series is spanned by the usual Eisenstein series $E(z,s)$ for $\mathrm{SL}(2,\mathbb{Z})$ and $E(qz,s)$. They are oldforms. Let W_q be the Fricke involution given by the matrix $\left(\begin{smallmatrix}0 & -1\\ q & 0\end{smallmatrix}\right)$. Then $E(z,s)\mid W_q = E(qz,s)$. The Fourier expansion of $E(z,s)$ at ∞ is given by

(4.7)
$$E(z,s)=y^s+\phi(s)y^{1-s}+\frac{2y^{1/2}}{\xi(2s)}\sum_{n=1}^{\infty}n^{s-1/2}\sigma_{1-2s}(n)K_{s-1/2}(2\pi ny)\cos(2\pi nx),$$

where $\xi(s)=\pi^{-s/2}\Gamma(s/2)\zeta(s)$, $\phi(s)=\xi(2s-1)/\xi(2s)$, and $\sigma_\nu(n)=\sum_{d\mid n}d^\nu$. We set A_n to be the coefficients in this expansion. Let T and S be the standard generators of $\mathrm{SL}(2,\mathbb{Z})$ inducing the maps $T(z)=-1/z$ and $S(z)=z+1$ on $\mathbb{H}$. Let $B=\left(\begin{smallmatrix}\sqrt{q} & 0\\ 0 & \sqrt{q}^{-1}\end{smallmatrix}\right)$. Then TB conjugates the stabilizer of zero in $\Gamma_0(q)$ to Γ_∞, the standard parabolic subgroup of $\mathrm{SL}(2,\mathbb{Z})$, and it induces the same map as W_q on $\mathbb{H}$. The expansion of $E(z,s)$ at zero is the expansion of $E\mid(TB)(z)=E(qz,s)$ at infinity.

We have

(4.8)
$$E(qz,s)=(qy)^s+\phi(s)(qy)^{1-s}+\sum_{n=1}^{\infty}A_n(qy)^{1/2}K_{s-1/2}(2\pi nqy)\cos(2\pi nqx).$$

For the Fourier expansion of $E(qz,s)$ at zero, we notice that $E(qz,s)=E\mid B(z)$, which implies that $(E\mid B)\mid(TB)(z)=E\mid T(z)=E(z,s)$. So the Fourier expansion of $E(qz,s)$ at zero is the Fourier expansion of $E(z,s)$ at infinity. Now we can find the matrix $M(s)$. Let $\mathbf{E}(z,s)=(E_\infty(z,s),E_0(z,s))^T$ be the vector of Eisenstein series indexed by the cusps. Set $(E(z,s),E(qz,s))^T=M(s)\mathbf{E}(z,s)$. To find $M(s)$ we look at the zero Fourier coefficients in this matrix equation and recall that the scattering matrix is symmetric to get

$$y^s+\phi(s)y^{1-s}=m_{11}(s)\big(y^s+\phi_{\infty\infty}(s)y^{1-s}\big)+m_{12}(s)\phi_{0\infty}(s)y^{1-s},$$

$$q^sy^s+q^{1-s}\phi(s)y^{1-s}=m_{11}y^{1-s}\phi_{0\infty}(s)+m_{12}(s)\big(y^s+\phi_{00}(s)y^{1-s}\big),$$

$$q^sy^s+q^{1-s}\phi(s)y^{1-s}=m_{21}\big(y^s+\phi_{\infty\infty}(s)y^{1-s}\big)+m_{22}(s)\phi_{0\infty}(s)y^{1-s},$$

$$y^s+\phi(s)y^{1-s}=m_{21}(s)\phi_{0\infty}(s)y^{1-s}+m_{22}(s)\big(y^s+\phi_{00}(s)y^{1-s}\big).$$

This system gives $M(s)=\left(\begin{smallmatrix}1 & q^s\\ q^s & 1\end{smallmatrix}\right)$ and

$$\Phi(s)=\frac{1}{q^{2s}-1}\phi(s)\begin{pmatrix}q-1 & q^s-q^{1-s}\\ q^s-q^{1-s} & q-1\end{pmatrix}$$

as in [12, p. 536]. Consequently $\det\Phi(s)=\phi(s)^2(q^{2-2s}-1)/(q^{2s}-1)$. The scattering poles are at $s_0=\rho/2$, where ρ is a nontrivial zero of $\zeta(s)$, and they have multiplicity 2, assuming Condition A. The zeros of $q^{2s}-1$ do not give scattering poles, since $\phi(s)$ has a factor $1/\zeta(2s)=\prod_p(1-p^{-2s})$.

Now assume that $g(z)$ is a holomorphic cusp form of weight 2 for $\Gamma_0(q)$, which is also an eigenform of the whole Hecke algebra, with eigenvalue ϵ_q for W_q. It follows from [1] that $\epsilon_q=\pm1$, $a(q)=-\epsilon_q$, and $a(nq)=a(n)a(q)$. Let $g(z)=\sum_{n>0}a(n)e^{2\pi inz}$ be the Fourier expansion of $g(z)$ at infinity. For its Fourier expansion at zero, we have

$$g\mid(TB)(z)=\frac{1}{qz^2}g\left(\frac{-1}{qz}\right)=g\mid W_q(z)=\epsilon_q g(z).$$

So the coefficients at zero are $\epsilon_q a(n)$.

We are interested in the Rankin-Selberg convolutions of $E(z,1-s_0)$ and $E(qz,1-s_0)$ with $g(z)$ expanded at both cusps, infinity and zero. The convolution of $E(qz,1-$

s_0) with $g(z)$ at zero is ϵ_q times the convolution of $E(z,1-s_0)$ with $g(z)$ at infinity. The convolution of $E(z,1-s_0)$ with $g(z)$ at zero is ϵ_q times the convolution of $E(qz,1-s_0)$ with $g(z)$ at infinity. For the Rankin-Selberg convolution R of $E(z,1-s_0)$ with $g(z)$ at infinity, we have $\chi_1=\chi_2=1$, $c_n=n^{1/2-s_0}\sigma_{2s_0-1}(n)$, and $A_n=2c_n/\xi(2-2s_0)$. Using (4.6) we get at $s=3/2-s_0$,

$$R=\frac{2\pi^{-s_0+1}}{\Gamma(1-s_0)\zeta(2-2s_0)^2\big(1-q^{2-2s_0}\big)}L(g,2-2s_0)L(g,1).$$

Consequently, by (4.3),

$$R(1-s_0)=\int_{\Gamma\backslash\mathbb{H}}E_\infty(z,1-s_0)\dot{\Delta}E(z,1-s_0)\,d\mu=\frac{2i(1-s_0)\Gamma(2-2s_0)}{4^{1-s_0}\pi^{-s_0}\Gamma(2-s_0)}R. \tag{4.9}$$

From (4.8) the Fourier coefficients B_m of $E(qz,1-s_0)$ are $B_m=0$, if $q\nmid m$ and $B_{nq}=A_nq^{1/2}$. The Rankin-Selberg convolution of $E(qz,1-s_0)$ with $g(z)$ at infinity is

$$\sum_{m=1}^{\infty}\frac{B_ma(m)}{m^s}=-\epsilon_qq^{1/2-s}\sum_{n=1}^{\infty}\frac{A_na(n)}{n^s}=-\epsilon_qq^{1/2-s}R. \tag{4.10}$$

Using (1.2) we get that the first variation of the weighted mean of the scattering poles at s_0 is

$$\dot{s}=A(1+\epsilon_q)\big(q^{2-2s_0}-1+\epsilon_q\big(q-q^{2s_0-1}\big)\big)L(g,2-2s_0)L(g,1), \tag{4.11}$$

where

$$A=\frac{4^{-s_0}\pi^{2s_0}i(1-s_0)\Gamma(2-2s_0)}{2q^{2s_0-1}\big(1-q^{2-2s_0}\big)^2\Gamma(s_0)\Gamma(2-s_0)\zeta(2-2s_0)\zeta'(2s_0)m(2s_0-1)}.$$

We notice that if $\epsilon_q=1$, then the functional equation for $g(z)$, which has a sign $-\epsilon_q$, forces $L(g,1)=0$. In both cases $\epsilon_q=\pm1$, we have $\dot{s}=0$. This proves the first statement in Theorem 1.5.

To apply Theorem 1.3 we notice that $(A_1(z,s_0),A_2(z,s_0))=\operatorname{Res}_{s=s_0}\phi(s)(E(z,1-s_0),E(qz,1-s_0))$. Also

$$V=M(s_0)^{-1}=\frac{1}{1-q^{2s_0}}\begin{pmatrix}1 & -q^{s_0}\\ -q^{s_0} & 1\end{pmatrix}.$$

Using (4.9) and (4.10) we get

$$\begin{pmatrix}a_1^1 & a_2^1\\ a_1^2 & a_2^2\end{pmatrix}=\operatorname{Res}_{s=s_0}\phi(s)\frac{R(1-s_0)}{1-q^{2s_0}}\begin{pmatrix}1+q^{2s_0-1} & -q^{s_0}-q^{s_0-1}\\ \big(-q^{s_0}-q^{s_0-1}\big)\epsilon_q & \big(1+q^{2s_0-1}\big)\epsilon_q\end{pmatrix}. \tag{4.12}$$

If $L(g,1)=0$, then $R=0$ and we cannot solve the eigenvalue problem for the zero matrix (a_i^j). If $\epsilon_q=-1$, the matrix in (4.12) has eigenvalue equation $\lambda^2=(1+q^{2s_0-1})^2-(q^{s_0}+q^{s_0-1})^2$. If $\lambda=0$ is an eigenvalue, then $q^{s_0}=\pm 1$ or $q^{s_0-1}=\pm 1$. However, since $2s_0$ is a zero of $\zeta(s)$, this is impossible ($|q^{s_0}|=q^{1/4}$, $|q^{s_0-1}|=q^{-3/4}$). Also the eigenvalues are distinct and have sum zero. Since $\Re(2-2s_0)=3/2$, the values $L(g,2-2s_0)$ are at the edge of the critical strip; using the argument of de la Vallée Poussin and Hadamard proving that $\zeta(1+it)\neq 0$ (see [26]), we see that this value is nonzero. This completes the proof of Theorem 1.5.

5. Proof of Theorem 1.9. The idea is to prove that, for a positive proportion of the scattering poles, the $\lambda_i^{(1)}$'s in (3.7) for one of the two directions are not imaginary. For that we should prove that the quotient of the ones corresponding to f and g is not real for a positive proportion of the scattering poles. From (4.9) and (4.12) it is clear that, when we take the quotient, we are left with $L(g,1)L(g,3/2-i\gamma)/(L(f,1)L(f,3/2-i\gamma))$ and all the other factors involving the zeta, gamma functions, and q cancel because they are the same irrespective of the tangent direction. The values $L(f,1)$ and $L(g,1)$ are real for newforms with real coefficients. This reduces the issue to proving that, for a positive proportion of γ's, the quotient $L(f,3/2+i\gamma)/L(g,3/2+i\gamma)$ is not real. This is equivalent to the nonvanishing of $L(g,3/2+i\gamma)L(f,3/2-i\gamma)-L(g,3/2-i\gamma)L(f,3/2+i\gamma)$. We prove weighted mean value results for these values. Let the Hecke eigenvalues for $f(z)$ be $\beta_1(p)$ and $\beta_2(p)$, and let the Hecke eigenvalues for $g(z)$ be $\alpha_1(p)$ and $\alpha_2(p)$. We introduce weights

$$B(s,P)=\prod_{p\leq P}\big(1-\alpha_1(p)p^{-s}\big)\big(1-\alpha_2(p)p^{-s}\big)\big(1-\beta_1(p)p^{-s}\big)\big(1-\beta_2(p)p^{-s}\big)$$

for P a suitable prime. Let us denote by $*$ the operation of Rankin-Selberg convolution on two Dirichlet series, and let $N(T)$ be the number of zeros $\rho=1/2+i\gamma$ of $\zeta(s)$ with $0<\gamma\leq T$.

PROPOSITION 1. *Under the RH,*

$$\sum_{0<\gamma\leq T} B\big(3/2+i\gamma,P\big)L\big(g,3/2+i\gamma\big)L\big(f,3/2-i\gamma\big) \sim B(\cdot,P)L(g,\cdot)*L(f,\cdot)(3)N(T) \tag{5.1}$$

and, by symmetry,

$$\sum_{0<\gamma\leq T} B\big(3/2+i\gamma,P\big)L\big(f,3/2+i\gamma\big)L\big(g,3/2-i\gamma\big) \sim B(\cdot,P)L(f,\cdot)*L(g,\cdot)(3)N(T). \tag{5.2}$$

PROPOSITION 2. *Let*

$$A(\gamma)=B\big(3/2+i\gamma,P\big)\Big(L\big(g,3/2+i\gamma\big)L\big(f,3/2-i\gamma\big)-L\big(f,3/2+i\gamma\big)L\big(g,3/2-i\gamma\big)\Big).$$

Under RH,

$$\sum_{0<\gamma\le T} |A(\gamma)|^2 \ll N(T). \tag{5.3}$$

Remark 5.1. The discrete mean value results in Propositions 1 and 2 correspond to mean value theorems for $\zeta(s)^4$ and $\zeta(s)^8$, respectively, on the line of convergence $\Re s = 1$. It is well known that on this line one has mean value theorems for all powers of $\zeta(s)$ (see [31, p. 148, 7.7.1]).

PROPOSITION 3. *Under RH and (1.5) there exists a P such that the two limits in (5.1) and (5.2) are different, that is,*

$$\sum_{0<\gamma\le T} A(\gamma) \sim C\cdot N(T), \quad C\neq 0. \tag{5.4}$$

In fact, we need the weights $B(s,P)$ here. Without them the mean values in (5.1) and (5.2) are equal. The rest of the proof of Theorem 1.9 is easy. By the Cauchy-Schwarz inequality, (5.3), and (5.4), we have

$$\sum_{0<\gamma\le T,\, A(\gamma)\neq 0} 1 \ge \frac{\left|\sum_{0<\gamma\le T} A(\gamma)\right|^2}{\sum_{0<\gamma\le T} |A(\gamma)|^2} \gg \frac{|C|^2 N(T)^2}{N(T)} = |C|^2 N(T).$$

This proves that a positive proportion of the $A(\gamma)$'s are nonzero; in particular, for the same γ's, the quotient $L(g,3/2+i\gamma)/L(f,3/2+i\gamma)$ is not real.

Proof of Proposition 1. The idea is to imitate the method used by Gonek [9] to prove discrete mean value formulas for the zeta function. More precisely, [9, Th. 1] states

$$\begin{aligned}\sum_{0<\gamma\le T} x^\rho = &-\frac{T}{2\pi}\Lambda(x) + O\big(x\log(2xT)\log\log(3x)\big)\\ &+ O\left(\log x \min\left(T, \frac{x}{\langle x\rangle}\right)\right) + O\left(\log(2T)\min\left(T, \frac{1}{\log x}\right)\right),\end{aligned} \tag{5.5}$$

where $x, T > 1$, and $\langle x\rangle$ denotes the distance from x to the nearest prime power other than x itself. This is a uniform version of a theorem by Landau (see [9]). We use the approximate functional equation for $L(f,s)$ and $L(g,s)$, as stated in [10, Kor. 2, p. 333]. If $s = 3/2+it$, we have

$$L(g,s) = \sum_{n\le |t|q/2\pi} a_n n^{-s} + \chi(s) \sum_{n\le |t|q/2\pi} a_n n^{s-2} + O\big(|t|^{-1/2+\epsilon}\big) \tag{5.6}$$

with $\chi(s) = (2\pi/q)^{2s-2}\Gamma(2-s)/\Gamma(s)$. The weights $B(s,P)$ are given by a Dirichlet polynomial of fixed length depending on P, say, $B(s,P) = \sum_{n\le R} c_n n^{-s}$. Let

$$B(s,P)L(g,s) = \sum_{n=1}^{\infty} \frac{d_n}{n^s}$$

and

$$B(s,P)\sum_{m\le|t|q/2\pi}\frac{a_m}{m^s}=\sum_{n\le|t|qR/2\pi}\frac{d'_n}{n^s}.$$

Clearly, if $n\le|t|q/2\pi$, we have $d'_n=d_n$. Moreover, d_n and d'_n grow no more quickly than a_n, that is, a_n, d_n, $d'_n\ll n^{1/2+\epsilon}$ by the Ramanujan conjecture. The approximate functional equation for $L(f,s)$ gives

$$L(f,s)=\sum_{n\le|t|qR/2\pi}b_nn^{-s}+\chi(s)\sum_{n\le|t|q/(2\pi R)}b_nn^{s-2}+O\big(|t|^{-1/2+\epsilon}\big). \tag{5.7}$$

The main term in

$$\sum_{0<\gamma\le T}B\big(3/2+i\gamma,P\big)L\big(g,3/2+i\gamma\big)L\big(f,3/2-i\gamma\big) \tag{5.8}$$

comes from

$$\begin{aligned}&\sum_{0<\gamma\le T}\sum_{n\le\gamma qR/2\pi}d'_nn^{-3/2-i\gamma}\cdot\sum_{n\le\gamma qR/2\pi}b_nn^{-3/2+i\gamma}\\&\quad=\sum_{0<\gamma\le T}\left(\sum_{n\le\gamma qR/2\pi}\frac{d'_nb_n}{n^3}+\sum_{n\ne m}^{\gamma qR/2\pi}\frac{d'_mb_n}{(nm)^{3/2}}\left(\frac{n}{m}\right)^{i\gamma}\right)=Z_1+Z_2.\end{aligned}$$

We have

$$\begin{aligned}Z_1&=\sum_{0<\gamma\le T}\left(\sum_{n=1}^{\infty}\frac{d_nb_n}{n^3}-\sum_{n>\gamma qR/2\pi}\frac{d_nb_n}{n^3}+\sum_{n\le\gamma qR/2\pi}\frac{(d'_n-d_n)b_n}{n^3}\right)\\&=N(T)B(\cdot,P)L(g,\cdot)*L(f,\cdot)(3)+C_1+C_2,\end{aligned}$$

where

$$C_1\ll\sum_{0<\gamma\le T}\sum_{n>\gamma}n^{-2+\epsilon}\ll\sum_{0<\gamma\le T}\gamma^{-1+\epsilon}=o\big(N(T)\big)$$

and

$$C_2\ll\sum_{0<\gamma\le T}\sum_{n>\gamma q/2\pi}\frac{\big(d'_n-d_n\big)b_n}{n^3}\ll\sum_{0<\gamma\le T}\sum_{n>\gamma}n^{-2+\epsilon}=o\big(N(T)\big).$$

Using (5.5) we get

$$
\begin{aligned}
Z_2 =& \sum_{n\le TqR/2\pi} \sum_{m<n} \sum_{2\pi n/(qR)\le\gamma\le T} \left[\frac{d'_m b_n}{n^2 m} \left(\frac{n}{m}\right)^{\rho} + \frac{d'_n b_m}{n^2 m} \overline{\left(\frac{n}{m}\right)^{\rho}} \right] \\
=& -\frac{T}{2\pi} \sum_{n\le TqR/2\pi} \sum_{m<n} \frac{d'_m b_n + d'_n b_m}{n^2 m} \Lambda\left(\frac{n}{m}\right) \\
&+ O\left(\sum_{n\le TqR/2\pi} \sum_{m<n} \frac{d'_m b_n + d'_n b_m}{nm} \Lambda\left(\frac{n}{m}\right) \right) \\
&+ O\left(\sum_{n\le TqR/2\pi} \sum_{m<n} \frac{d'_m b_n + d'_n b_m}{nm^2} \log\left(\frac{2nT}{m}\right) \log\log\left(\frac{3n}{m}\right) \right) \\
&+ O\left(\sum_{n\le TqR/2\pi} \sum_{m<n} \frac{d'_m b_n + d'_n b_m}{n^2 m} \log\left(\frac{n}{m}\right) \min\left(T, \frac{n/m}{\langle n/m\rangle}\right) \right) \\
&+ O\left(\log(2T) \sum_{n\le TqR/2\pi} \sum_{m<n} \frac{d'_m b_n + d'_n b_m}{n^2 m} \min\left(T, \frac{1}{\log(n/m)}\right) \right) \\
=& Z_{21} + Z_{22} + Z_{23} + Z_{24} + Z_{25}.
\end{aligned}
$$

To estimate Z_{21} and Z_{22}, we set $n = km$. We get

$$
Z_{21} \ll \frac{T}{2\pi} \sum_{k\le TqR/2\pi} \sum_{m<TqR/(2\pi k)} \frac{\Lambda(k)}{k^{3/2-\epsilon} m^{2-\epsilon}} = O(T),
$$

since $\Lambda(k) \ll k^{\epsilon}$ and the sums are partial sums of two convergent series. Moreover,

$$
Z_{22} \ll \sum_{k\le TqR/2\pi} \sum_{m<TqR/(2\pi k)} \frac{\Lambda(k)}{k^{1/2-\epsilon} m^{1-\epsilon}} \ll \sum_{k\ll T} \frac{\Lambda(k)}{k^{1/2-\epsilon}} \left(\frac{T}{k}\right)^{\epsilon} \ll T^{1/2+\epsilon}.
$$

Similarly, one easily gets $Z_{23} \ll T^{1/2+\epsilon} \log T \log\log T$. For Z_{24} we set $n = lm + r$, where $-m/2 < r \le m/2$. This implies that

$$
\left\langle l + \frac{r}{m} \right\rangle = \begin{cases} \dfrac{|r|}{m}, & \text{if } l \text{ is a prime power and } r \neq 0, \\ \ge \dfrac{1}{2}, & \text{otherwise.} \end{cases}
$$

Together with $n/m \le n \le cT$, $c = qR/(2\pi)$, this gives

$$Z_{24} \ll \log T \sum_{n \le cT} \sum_{m<n} \frac{1}{m^{3/2-\epsilon} n^{1/2-\epsilon} \langle n/m \rangle}$$

$$\ll \log T \sum_{m \le cT} \sum_{l \le \lfloor cT/m \rfloor + 1} \sum_{-m/2 < r \le m/2} \frac{1}{m^{3/2-\epsilon} \big(lm+r\big)^{1/2-\epsilon} \big\langle l + r/m \big\rangle}$$

$$\ll \log T \sum_{m \le cT} \sum_{l \le \lfloor cT/m \rfloor + 1} \left(\Lambda(l) \frac{m \log m}{m^{3/2-\epsilon} (lm)^{1/2-\epsilon}} + \frac{m}{m^{3/2-\epsilon} (lm)^{1/2-\epsilon}} \right)$$

$$\ll \log T \sum_{m \le cT} \frac{\log m}{m^{1-2\epsilon}} \sum_{l \ll cT/m} \frac{l^{\epsilon}}{l^{1/2-\epsilon}}$$

$$\ll T^{1/2+\epsilon} \log T \sum_{m \le cT} \frac{\log m}{m^{3/2-\epsilon}} \ll T^{1/2+\epsilon} \log T.$$

For Z_{25} we set $m = n - r$, $1 \le r \le n-1$, so that $\log(n/m) > r/n$. This gives

$$Z_{25} \ll \log T \sum_{n \le cT} \sum_{r \le n-1} \frac{n/r}{n^{3/2-\epsilon} (n-r)^{1/2-\epsilon}} \ll \log T \sum_{n \le cT} \frac{1}{n^{1/2-\epsilon}} \sum_{r \le n-1} \frac{1}{r}$$

$$\ll \log T \sum_{n \le cT} \frac{\log n}{n^{1/2-\epsilon}} \ll T^{1/2+\epsilon} \log T.$$

The analysis above depends only on the order of growth of d_n' and b_n, so we can get the bounds

$$\sum_{\gamma \le T} \Bigg| \sum_{n \le \gamma q R/2\pi} d_n' n^{-3/2-i\gamma} \Bigg|^2 \ll N(T), \tag{5.9}$$

$$\sum_{\gamma \le T} \Bigg| \sum_{n \le \gamma q/(2\pi R)} b_n n^{-3/2+i\gamma} \Bigg|^2 \ll N(T). \tag{5.10}$$

Moreover, we can repeat the argument with trivial modifications to estimate

$$\sum_{\gamma \le T} \Bigg| \sum_{n \le \gamma q/2\pi} a_n n^{-1/2+i\gamma} \Bigg|^2 \ll T^{1+\epsilon} N(T), \tag{5.11}$$

$$\sum_{\gamma \le T} \Bigg| \sum_{n \le \gamma q/(2\pi R)} b_n n^{-1/2-i\gamma} \Bigg|^2 \ll T^{1+\epsilon} N(T). \tag{5.12}$$

We also need to analyze the order of growth of the derivative of $|\chi(3/2+i\gamma)|^2$. We have

$$\frac{d}{d\gamma}\Big|\chi\big(3/2+i\gamma\big)\Big|^2 = i(2\pi/q)^2\Big|\chi\big(3/2+i\gamma\big)\Big|^2 \cdot\Big(\psi\big(1/2-i\gamma\big)+\psi\big(1/2+i\gamma\big)-\psi\big(3/2+i\gamma\big)-\psi\big(3/2-i\gamma\big)\Big),$$

where $\psi(z)=\Gamma'(z)/\Gamma(z)$. We can differentiate the asymptotics of $\log\Gamma(z)$ (see [11, 8.344, p. 949]), to get

$$\psi(z)=\log z-\frac{1}{2z}+O(z^{-2}).$$

We also use the asymptotic formula

$$\Big|\Gamma(x+iy)\Big| \sim \sqrt{2\pi}\,e^{-\pi|y|/2}|y|^{x-1/2}, \quad |y|\longrightarrow\infty,\ x,y\in\mathbb{R} \tag{5.13}$$

(see [11, p. 945, 8.328.1]), to get

$$\frac{d}{d\gamma}\Big|\chi\big(3/2\pm i\gamma\big)\Big|^2 \ll \gamma^{-3}. \tag{5.14}$$

We use summation by parts, (5.14), (5.11), and (5.12) to estimate

$$\sum_{\gamma\le T}\Big|\chi\big(3/2+i\gamma\big)\Big|^2\Bigg|\sum_{n\le R}c_n n^{-3/2-i\gamma}\Bigg|^2\Bigg|\sum_{n\le\gamma q/2\pi}a_n n^{-1/2+i\gamma}\Bigg|^2 \ll T^{\epsilon}=o\big(N(T)\big),$$

$$\sum_{\gamma\le T}\Big|\chi\big(3/2-i\gamma\big)\Big|^2\Bigg|\sum_{n\le\gamma q/(2\pi R)}b_n n^{-1/2-i\gamma}\Bigg|^2 \ll T^{\epsilon}=o\big(N(T)\big).$$

Finally we use the Cauchy-Schwarz inequality, (5.9), (5.10), and the two equations above to estimate all other product terms in (5.8) as $o(N(T))$. □

LEMMA 5.2. *For $0\le a<3/2$, there exists a constant $c>0$ such that*

$$\sum_{n\le x}\frac{|a_n|}{n^a} \ll \frac{x^{3/2-a}}{\log^c x} \tag{5.15}$$

and

$$\sum_{n\le x}\frac{|a_n|\log n}{n^a} \ll x^{3/2-a}\log^{1-c}x. \tag{5.16}$$

Similar estimates hold for b_n.

Proof. Set $A(x) = \sum_{n \le x} |a'_n|$, where $a_n = a'_n \sqrt{n}$. Rankin's estimate [27] implies that $A(x) \ll x(\log x)^{-c}$ for some $c > 0$. We have

$$\sum_{n \le x} \frac{|a_n|}{n^a} = \sum_{n \le x} \frac{|a'_n|}{n^{a-1/2}} = \frac{A(x)}{x^{a-1/2}} + (a-1/2) \int_1^x A(t) t^{-a-1/2} \, dt$$

using partial summation. Notice that $\int_2^x t^{1/2-a} \log^{-c} t \, dt \ll x^{3/2-a} \log^{-c} x$. The second estimate is proved similarly. □

Proof of Proposition 2. Since $B(3/2 + i\gamma, P)$ is a finite Dirichlet polynomial, it is bounded independently of T. To estimate $\sum_{\gamma \le T} |A(\gamma)|^2$, it suffices to estimate

$$\sum_{\gamma \le T} \Big| L\big(g, 3/2 + i\gamma\big) \Big|^2 \Big| L\big(f, 3/2 + i\gamma\big) \Big|^2 \ll N(T). \tag{5.17}$$

As in the proof of Proposition 1, we use the approximate functional equation on $\Re s = 3/2$ for $L(g, s)$ and $L(f, s)$ (see (5.6), (5.7)),

$$\begin{aligned} L(f, s) &= \sum_{n \le |t| q/2\pi} b_n n^{-s} + \chi(s) \sum_{n \le |t| q/2\pi} b_n n^{s-2} + O\big(|t|^{-1/2+\epsilon}\big) \\ &= W_1 + \chi(s) W_2 + O\big(|t|^{-1/2+\epsilon}\big), \\ L(g, s) &= Y_1 + \chi(s) Y_2 + O\big(|t|^{-1/2+\epsilon}\big). \end{aligned}$$

We have

$$\sum_{0 < \gamma \le T} Y_1 \overline{Y_1} W_1 \overline{W_1} = \sum_{0 < \gamma \le T} \sum_{m,n,\mu,\nu \le \gamma q/2\pi} \frac{a_m b_n a_\mu b_\nu}{(mn\mu\nu)^{3/2}} \left(\frac{\mu\nu}{mn}\right)^{i\gamma}. \tag{5.18}$$

The main term comes again from the contribution of the diagonal terms, and the number of solutions to $mn = \mu\nu = r$ is less than or equal to $d(r)^2$, where $d(r)$ is the divisor function. The main term can be estimated as

$$\sum_{\gamma \le T} \sum_{mn = \mu\nu}^{\gamma q/2\pi} \frac{a_m b_n a_\mu b_\nu}{(mn)^3} \ll \sum_{\gamma \le T} \sum_{r=1}^{\infty} \frac{d(r)^2 r^{1+2\epsilon}}{r^3} \ll N(T), \tag{5.19}$$

since the inner series converges. We set $\mu\nu = r$ and $mn = s$. We can treat the case $s < r$ and $s > r$ separately. The range of the following sums is subject to the restriction $m, n, \mu, \nu \le Tq/(2\pi)$. For $s < r$ the other terms in (5.18) contribute

$$Z_2 = \sum_{r \le (Tq/2\pi)^2} \sum_{s < r} \sum_{m | s, \, \mu | r} \frac{a_m b_{s/m} a_\mu b_{r/\mu}}{r^{3/2} s^{3/2}} \sum_{K \le \gamma \le T} \left(\frac{r}{s}\right)^{i\gamma}, \tag{5.20}$$

where $K = \min(T, (2\pi/q)\max(m, s/m, \mu, r/\mu))$. We apply (5.5) to Z_2:

$$\begin{aligned} Z_2 &= \sum_{r \ll T^2} \sum_{s<r} \sum_{m|s,\, \mu|r} \frac{a_m b_{s/m} a_\mu b_{r/\mu}}{r^2 s} \left(\sum_{0<\gamma\le T} \left(\frac{r}{s}\right)^\rho - \sum_{\gamma<K} \left(\frac{r}{s}\right)^\rho \right) \\ &= \sum_{r \ll T^2} \sum_{s<r} \sum_{m|s,\, \mu|r} \frac{a_m b_{s/m} a_\mu b_{r/\mu}}{r^2 s} \frac{K-T}{2\pi} \Lambda\left(\frac{r}{s}\right) \\ &\quad + O\left(\sum_{r \ll T^2} \sum_{s<r} \sum_{m|s,\, \mu|r} \frac{a_m b_{s/m} a_\mu b_{r/\mu}}{rs^2} \log\left(\frac{2Tr}{s}\right) \log\log\left(\frac{3r}{s}\right) \right) \\ &\quad + O\left(\sum_{r \ll T^2} \sum_{s<r} \sum_{m|s,\, \mu|r} \frac{a_m b_{s/m} a_\mu b_{r/\mu}}{r^2 s} \log\left(\frac{r}{s}\right) \min\left(T, \frac{r/s}{\langle r/s \rangle}\right) \right) \\ &\quad + O\left(\sum_{r \ll T^2} \sum_{s<r} \sum_{m|s,\, \mu|r} \frac{a_m b_{s/m} a_\mu b_{r/\mu}}{r^2 s} \log(2T) \min\left(T, \frac{1}{\log(r/s)}\right) \right) \\ &= Z_{21,2} + Z_{23} + Z_{24} + Z_{25}. \end{aligned}$$

For $Z_{21,2}$ we set $r = sk$ and notice that $d(r) \ll r^\epsilon$, $d(s) \ll s^\epsilon$, and $K \le T$ to get

$$Z_{21,2} \ll T \sum_{k \ll T^2} \sum_{s \ll T^2/k} \frac{\Lambda(k) s^{1/2+\epsilon} (sk)^{1/2+\epsilon}}{s^3 k^2} \ll T,$$

since both series converge. Using (5.15), we get

$$\sum_{\mu,\nu \le Tq/2\pi} \frac{a_\mu b_\nu}{\mu\nu} \ll \frac{T}{\log^{2c} T}.$$

We have

$$\begin{aligned} Z_{23} &\ll \log T \log\log T \sum_{m,n,\mu,\nu} \frac{a_n b_m a_\mu b_\nu}{\mu\nu m^2 n^2} \\ &\ll \log T \log\log T \frac{T}{\log^{2c} T} = o\big(N(T)\big), \end{aligned}$$

since the series $\sum a_m m^{-2}$ converges.

For Z_{24} we set $r = ls + t$, $-s/2 < t \le s/2$, and we distinguish two cases as in Proposition 1. Case 1 occurs when l is a prime power and $t \ne 0$, and case 2 happens otherwise. The contribution from case 2 is

$$\ll \log T \sum_{m,n,\mu,\nu} \frac{a_m b_n a_\mu b_\nu}{\mu\nu m^2 n^2} \ll T \log^{1-2c} T,$$

as for the estimate of Z_{23}. In case 1 we distinguish two subcases depending on whether T is larger than $(r/s)/\langle r/s\rangle$ or not. If T is larger, $T > \mu\nu/|t|$, which implies $T > l$. The contribution $Z_{24,1}$ of these terms is

$$\begin{aligned} Z_{24,1} &\ll \sum_{m,n,\mu,\nu} \sum_{T>l} \frac{a_m b_n a_\mu b_\nu}{mn\mu\nu|t|} \log\big(l + t/(mn)\big) \\ &\ll \sum_{m,n} \frac{a_m b_n}{mn} \sum_{0\neq|t|<mn/2} \sum_{l\leq T} \frac{\log l}{(lmn+t)^{1/2-\epsilon}|t|} \\ &\ll \sum_{m,n} \frac{a_m b_n}{(mn)^{3/2-\epsilon}} \log(mn) \sum_{l\leq T} \frac{\log l}{(l-1/2)^{1/2-\epsilon}} \\ &\ll T^{1/2+\epsilon} \sum_{m,n\leq Tq/2\pi} \frac{a_m a_n \log(mn)}{(mn)^{3/2-\epsilon}} \ll T^{1/2+3\epsilon} \log^{2-2c} T, \end{aligned}$$

using (5.16) and $\log(mn) \leq \log n \log m$ for $\log n, \log m \geq 2$.

If T is smaller than $(r/s)/\langle r/s\rangle$, we have $T \leq \mu\nu/|t|$ and this implies $l > T|t|/(2mn)$. Let the contribution of these terms be $Z_{24,2}$. We first analyze the summation over μ, ν. We see that

$$\begin{aligned} \sum_{\mu\nu\geq T|t|} \frac{a_\mu b_\nu}{\mu^2\nu^2} &\ll \sum_{0\neq|t|\leq mn/2} \sum_{l>T|t|/(2mn)} \frac{1}{(lmn+t)^{3/2-\epsilon}} \\ &\ll \frac{1}{(mn)^{3/2-\epsilon}} \sum_{0\neq|t|\leq mn/2} \left(\frac{T|t|}{2mn}\right)^{-1/2+\epsilon} \\ &\ll T^{-1/2+\epsilon} \frac{1}{(mn)^{1-\epsilon}} \sum_{0\neq|t|\leq mn/2} |t|^{-1/2+\epsilon} \ll (Tmn)^{-1/2+\epsilon}. \end{aligned}$$

The summation over m, n now gives

$$\begin{aligned} Z_{24,2} &\ll T \log T \sum_{m,n\leq Tq/2\pi} \frac{a_m b_n}{(mn)^{3/2-\epsilon}} T^{-1/2+\epsilon} \\ &\ll T^{1/2+\epsilon} \log T \big(T^\epsilon/\log^c T\big)^2 = o\big(N(T)\big). \end{aligned}$$

This takes care of Z_{24}. Using again the Ramanujan conjecture for a_n, b_μ, we get

$$\begin{aligned} Z_{25} &\ll \log T \sum_{m,n,\mu,\nu \le Tq/2\pi, s<r} \frac{b_n a_m a_\mu b_\nu}{\mu^2 \nu^2 mn \log(\mu\nu/(mn))} \\ &\ll T^\epsilon \log T \sum_{s<r\le (Tq/2\pi)^2} \frac{d(r)d(s)}{r^{3/2} s^{1/2} \log(r/s)} \\ &\ll T^\epsilon \log T \sum_{s<r\le (Tq/2\pi)^2} \frac{1}{sr\log(r/s)} \ll T^\epsilon \log T \left(\log T^2\right)^2, \end{aligned}$$

where in the last line we used [31, Lemma 7.2, p. 139], with the obvious modifications for $\sigma = 1$.

The rest of the proof follows as in the proof of Proposition 1. In fact one can easily see that we can not only get upper bounds for $\sum_{\gamma \le T} |A(\gamma)|^2$, but we can also identify the main term in the asymptotics of it. □

Proof of Proposition 3. It is clear that the product of two Dirichlet series with multiplicative coefficients also has multiplicative coefficients and this is also true for their Rankin-Selberg convolution. This allows us to work the Euler factors separately for each prime. Since

$$\begin{aligned} B(s,P)L(g,s) &= \prod_{p>P} \left[\left(1-\alpha_1(p)p^{-s}\right)\left(1-\alpha_2(p)p^{-s}\right)\right]^{-1} \\ &\quad \times \prod_{p\le P} \left(1-\beta_1(p)p^{-s}\right)\left(1-\beta_2(p)p^{-s}\right), \end{aligned} \tag{5.21}$$

the convolution $B(s,P)L(g,s) * L(f,s)$ has the same Euler factors for $p > P$ as $L(g,s) * L(f,s)$. The same is true for the Euler factors with $p > P$ for $B(s,P)L(f,s) * L(g,s)$. So when we subtract (5.2) from (5.1) to get (5.4) we get a factor

$$\prod_{p>P} \left(1-\alpha_1(p)\alpha_2(p)\beta_1(p)\beta_2(p)p^{-2s}\right) \prod_{1\le i,\, j\le 2} \left(1-\alpha_i(p)\beta_j(p)p^{-s}\right)^{-1}$$

using [30, Lemma 1]. The value of this at $s = 3$ is nonzero, since 3 is in the domain of convergence. To show that the asymptotics in (5.4) have $C \ne 0$ for some P, let us assume that for all P prime the difference in the other Euler factors with $p \le P$ in (5.1) and (5.2) is zero at $s = 3$. We analyze these Euler factors. Fix $p \le P$. For $B(s,P)L(g,s) * L(f,s)$ we get

$$\left(1-b(p)p^{-s}+p^{1-2s}\right) * \sum_{k=0}^{\infty} b\left(p^k\right)p^{-ks} = 1-b(p)^2 p^{-s} + pb\left(p^2\right)p^{-2s}$$

while for $B(s,P)L(f,s)*L(g,s)$ we get

$$\left(1-a(p)p^{-s}+p^{1-2s}\right)*\sum_{k=0}^{\infty}a(p^k)p^{-ks}=1-a(p)^2p^{-s}+pa(p^2)p^{-2s}.$$

The Hecke relations give $b(p^2)=b(p)^2-p$ and $a(p^2)=a(p)^2-p$. If for all P we have

$$\prod_{p\le P}\left(1+b(p)^2\left(p^{-5}-p^{-3}\right)-p^{-4}\right)=\prod_{p\le P}\left(1+a(p)^2\left(p^{-5}-p^{-3}\right)-p^{-4}\right),$$

we get equality for the individual terms, by considering successive primes P and by dividing the corresponding relations. This gives $b(p)^2=a(p)^2$ for all p, contradicting the assumption (1.5). The case $p=q$ is even simpler. □

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