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On the singular set, the resolvent and Fermi's Golden Rule for finite volume hyperbolic surfaces

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Consider a finite volume hyperbolic surface. Under perturbation the spectrum of the Laplace operator is unstable but the singular set is stable. We characterize the singular set in terms of the resolvent of the Laplace operator and extend Fermi's Golden Rule to the case of multiple eigenvalues.

Introduction

In this work we are interested in the spectral theory of the Laplace operator acting on non-compact finite volume surfaces obtained as the quotient of hyperbolic 2-space $\mathbb{H}^2$ by discrete subgroups of $PSL(2,\mathbb{R})$. The spectrum consists of a continuous part filling $[\frac{1}{4},\infty)$ and a discrete set of eigenvalues, of which finitely many are less than or equal to $\frac{1}{4}$. Associated with the problem of existence of infinitely many cusp forms (i.e. L^2 eigenfunctions with zero Fourier coefficient) is the problem of stability of the spectrum. The spectrum is unstable under perturbation ([8] p.363). However, the spectrum becomes more manageable when the scattering frequencies are adjoined with certain multiplicities prescribed by the scattering matrix. More precisely, let us denote the scattering matrix by $\Phi(s)$, and $\varphi(s)$ the determinant of the scattering matrix. Then we have the following definition:

Definition. ([9] p.2) The singular set consists of the points of positive multiplicity, if we define the multiplicity of any point s in the complex plane as follows:

(a) If $Re\ s \geq \frac{1}{2}$ but $s \neq \frac{1}{2}$ we define the multiplicity at s to be the dimension of the eigenspace for $s(1-s)$ of Δ on $L^2(\Gamma\backslash\mathbb{H})$. Consequently, this multiplicity is zero unless $Re\ s = \frac{1}{2}$ or $s \in (\frac{1}{2}, 1]$.

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(b) If $Re\ s < \frac{1}{2}$ we define the multiplicity at s to be the multiplicity of the eigenvalue $s(1-s)$ of Δ on $L^2(\Gamma \setminus \mathbb{H})$ plus the order of the pole (or minus the order of the zero) of $\varphi(s)$ at s. Consequently, if $Re\ s < \frac{1}{2}$ and $s \notin \mathbb{R}$ this multiplicity is simply the order of the pole of $\varphi(s)$ at s, since in this case $\varphi(s)$ cannot have a zero and $s(1-s)$ is not an L^2- eigenvalue.

(c) For $s = \frac{1}{2}$ the multiplicity is defined as twice the dimension of cusp forms with eigenvalue $\frac{1}{4}$ plus $\frac{n+tr(\Phi(\frac{1}{2}))}{2}$.

The singular set occurs in a one-sided version of the Selberg trace formula and is essentially the spectrum of the cut-off wave operator B in the Lax-Phillips approach ([5] p.28).

In this work a new characterization of the singular set is provided using the analytic continuation of the resolvent $R(s) = (-\Delta - s(1-s))^{-1}$. We have the following theorem:

Theorem 1. *The multiplicity of a singular point s_0 is (for $s_0 \neq \frac{1}{2}$) the rank of the operator:*

$$P_{s_0} = \oint (2s-1)R(s)\, ds \tag{1.1}$$

where the contour encloses only s_0 among the singular points.

The factor $2s-1$ appears because we are using the variable s instead of the natural $z = s(1-s)$. We remark that $R(s)$ and P_{s_0} do not act on $L^2(\Gamma \setminus \mathbb{H})$ for $Res_0 \leq \frac{1}{2}$ but on certain Banach spaces instead (see section 2). The operator P_{s_0} is not a projection by any means in contrast to the discrete spectral problem. However it is a finite-rank operator. As shown in [9], p.21-23 after the spectrum of B has been identified in terms of the singular set i.e. the spectrum of $B + \frac{1}{2}$ is exactly the singular set with the correct multiplicities, standard perturbation theory can be applied to B when we consider variations of the metric g and one sees that the singular set is stable under deformation.

For a complete analysis of what happens at $s = \frac{1}{2}$ we refer to [7].

The proof of theorem 1 actually provides also a basis for the image of $\oint(2s-1)R(s)ds$ in terms of the various spectral data. See Remark 2 at the end of section 3 for details. The same method used for theorem 1 can be applied to determine the Taylor coefficients of $R(s)$ for all negative terms $(s-s_0)^{-i}$, $i = 1, 2, \ldots$ whenever they exist. In particular the following theorem can be proved, which we state for the case of one cusp:

Theorem 1′. *If $Re\, s_0 < \frac{1}{2}$, the resolvent has a pole of order κ at s_0 exactly when $E(z,s)$ has a pole of order κ at s_0. We set $A(z,s) = (s-s_0)^\kappa E(z,s)$ and*

$$(2s-1)R(s) = \frac{B_{-\kappa}}{(s-s_0)^\kappa} + \cdots + \frac{B_{-1}}{s-s_0} + \cdots \tag{1.2}$$

Then the B_{-i}'s are finite rank operators and, if $\kappa > 1$, a basis for the image of B_{-i}, $i > 1$, is:

$$\frac{d^{\kappa-i-j}}{ds^{\kappa-i-j}} A(z, s_0) \tag{1.3}$$

for $j = 0, 1, \ldots, \kappa - i$.

The proof is essentially the same as the one of theorem 1, if we notice that $r(z, z'; 1-s)$ in (2.8) will never contribute for the B_{-i}, $i > 1$, since it has at most a simple pole. We omit the obvious details.

In the last section we provide an extension of Fermi's Golden rule, which is a formula giving us the rate at which an imbedded eigenvalue leaves the real line $Re\, s = \frac{1}{2}$ to become a resonance. This is treated in [9] p.23-26 for a simple eigenvalue. We treat the case of multiple eigenvalues, which is not at all uncommon. If Γ is a congruence subgroup of level N and $f(z)$ is an eigenfunction of $SL(2,\mathbb{Z})$, then the functions $f(kz)$, $k|N$ are linearly independent eigenfunctions for Γ with the same eigenvalue. Let us assume that m is the multiplicity of an imbedded eigenvalue. We set $\hat{\lambda}$ the weighted mean of eigenvalues: i.e. if the perturbation parameter is ε and $\lambda_1(0) = \lambda_2(0) = \cdots = \lambda_m(0)$ split as $\lambda_1(\varepsilon), \dots, \lambda_m(\varepsilon)$ when the perturbation is switched on, we set:

$$\hat{\lambda}(\varepsilon) = \frac{1}{m}\sum_{k=1}^{m} \lambda_k(\varepsilon) \tag{1.4}$$

and then we have:

Theorem 2. *If ψ_k, $k = 1, \dots, m$ is an orthonormal basis of eigenfunctions of the Laplace operator with eigenvalue $\frac{1}{4} + \sigma^2 > \frac{1}{4}$ then the rate at which the weighted mean of eigenvalues leaves the real line to become a resonance is:*

$$Re\, \hat{\lambda}^{(2)} = Re\, \frac{d^2}{d\varepsilon^2}\hat{\lambda}(\varepsilon)\,|_{\varepsilon=0} = -\frac{1}{4m\sigma^2}\sum_{k=1}^{m}\sum_{i=1}^{n} |(E_i(z, \frac{1}{2} + i\sigma), \dot{\Delta}\psi_k(z))|^2 \tag{1.5}$$

If $Re\, \hat{\lambda}^{(2)} \neq 0$, then at least one of the branches $\lambda_i(\varepsilon)$ becomes a resonance.

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1. The resolvent kernel

In this section we recall some standard facts about the resolvent and its analytic continuation, which is due to Faddeev [2] (see also [4] Chapter XIV). Let $R(s) = (-\Delta - s(1-s))^{-1}$. The fundamental point pair invariant is:

$$u(z, z') = \frac{|\, z - z' \,|^2}{4yy'} \tag{2.1}$$

for $z, z' \in \mathbb{H}$. We set:

$$\varphi(u, s) = \frac{1}{4\pi}\int_0^1 [t(1-t)]^{s-1}(t+u)^{-s}\, dt \tag{2.2}$$

for $\sigma > 0, u > 0 \quad (s = \sigma + it)$ and $k(z, z'; s) = \varphi(u(z, z'), s)$ and then the kernel $k(z, z'; s)$ is the Green's function for the problem $\Delta h + s(1-s)h = f$ at least for $\sigma > 1$ ([4] p.275).

For a discrete subgroup Γ of $PSL(2,\mathbb{R})$ with $\Gamma \setminus \mathbb{H}$ non compact of finite volume we set:

$$r(z, z'; s) = \frac{1}{2}\sum_{\gamma\in\Gamma} \varphi(u(z, \gamma z'), s) \tag{2.3}$$

for $\sigma > 1$. This is the resolvent kernel. We decompose the fundamental domain F of Γ into

$$F = F_0 \cup \bigcup_{\alpha=1}^{n} F_\alpha$$

where the F_α are isometric to the standard cusp. Let us denote the stabilizer of the j-cusp z_j by Γ_j. It is generated by a single parabolic element S_j and there exists a g_j with $g_j\infty = z_j$. One can choose $g_\alpha \in SL(2, \mathbb{R})$ so that $z \to g_\alpha z$ maps $C = \{z; -\frac{1}{2} \leq Re\ z \leq \frac{1}{2}, Im\ z \geq a\}$ one-to-one onto F_α. Each function f on F has $n+1$ components $f_0(z) = f(z)$ for $z \in F_0$ and $f_\alpha(z) = f(g_\alpha z)$ for $z \in C$. One has the decomposition:

$$L^2(\Gamma \setminus \mathbb{H}) = L^2(F_0) \oplus \bigoplus_{\alpha=1}^{n} L^2(F_\alpha) \tag{2.4}$$

but it turns out that certain Banach spaces $\mathcal{B}_\mu$ play an even more important role in Faddeev's approach. The space $\mathcal{B}_\mu$ consists of complex valued functions $f(z)$ whose components $f_0(z)$ and $f_\alpha(z)$, $\alpha = 1, \ldots, n$ are continuous on F_0 and C respectively with:

$$| f_\alpha(z) | \leq cy^\mu \tag{2.5}$$

for $z \in C$ with the μ-norm:

$$|| f ||_\mu = \max_{z \in F_0} | f_0(z) | + \sum_{\alpha=1}^{n} \max_{z \in C} \frac{| f_\alpha(z) |}{y^\mu} \tag{2.6}$$

Since the Laplace operator is a negative operator, the resolvent $R(z) = (-\Delta - z)^{-1}$ is defined on $\mathbb{C} \setminus [0, \infty)$. However, one can use meromorphic continuation to attach a meaning to the resolvent on a Riemann surface which is a two sheeted covering of the z- plane. Instead of the natural z variable, one introduces $z = s(1-s)$ and then the z plane cut along the ray $[0, \infty)$ corresponds to the right half plane $Re\ s > \frac{1}{2}$ cut along $\frac{1}{2} \leq s \leq 1$. The analytic continuation of the resolvent kernel can have poles only at the following set of points:

- at s_0, if $s_0(1-s_0)$ is an L^2- eigenvalue and $Re\ s_0 \geq \frac{1}{2}$ but $s_0 \neq \frac{1}{2}$, and the pole is simple and:

$$r(z, z'; s) = \frac{1}{s_0(1-s_0) - s(1-s)} \sum_{i=1}^{m} \psi_i(z)\psi_i(z') + r_1(z, z'; s) \tag{2.7}$$

  where $\psi_i, (i = 1, \cdots, m)$ is an orthonormal system of eigenfunctions (chosen to be real) with eigenvalue $s_0(1-s_0)$ and $r_1(z, z'; s)$ is regular close to s_0, (see [4], p.333)
- at $s_0 = \frac{1}{2}$, possibly
- at points s_0 with $Re\ s_0 < \frac{1}{2}$. These points are called resonances and the Eisenstein series and the scattering matrix can have poles at those points only for $Res < \frac{1}{2}$ (see [4] p.338-340).

  For s, $1-s$ non-singular, we have:

$$r(z, z'; s) - r(z, z'; 1-s) = \frac{1}{2s-1} \sum_{\beta=1}^{n} E_\beta(z, s) E_\beta(z', 1-s) \tag{2.8}$$

a proof of which is given for instance in [4] p.344. For a generalization of (2.8) to the case of finitely generated Kleinian groups we refer the reader to [1].

After obtaining the analytic continuation of the resolvent kernel one defines $R(s)$ as follows: Fix $\mu \leq \frac{1}{2}$. Then $R(s) : \mathcal{B}_\mu \to \mathcal{B}_{1-\mu}$ is defined for $Re\, s > \mu$ as the integral operator with kernel $r(z, z'; s)$.

2. Proof of Theorem 1

Let us suppose, for simplicity, that we have only one cusp. Let s_0 be a singular point and define the following operator:

$$P_{s_0} = \frac{1}{2\pi i} \int_\Gamma (2s-1) R(s)\, ds \tag{3.1}$$

where Γ is a circle enclosing s_0 but no other singular points. The operator P_{s_0} has kernel:

$$\frac{1}{2\pi i} \int_\Gamma (2s-1) r(z, z'; s)\, ds \tag{3.2}$$

The proof splits in a number of cases, since the definition of the singular set contains several cases.

Case I: The first case occurs when the singular point s_0 satisfies $Re\, s_0 \geq \frac{1}{2}$ but $s_0 \neq \frac{1}{2}$. Since $\lim_{s \to s_0} \frac{(2s-1)(s-s_0)}{s_0(1-s_0)-s(1-s)} = 1$, it is easy to see that (2.7) implies in this case that P_{s_0} has kernel $\sum_{i=1}^m \psi_i(z)\psi_i(z')$, where the $\psi_i(z)$, $i = 1, \ldots, m$ is an orthonormal basis of eigenfunctions with eigenvalue $s_0(1-s_0)$ (chosen to be real). The theorem for s_0 now becomes obvious.

Before discussing the other cases we remark that for $Re\, s_0 < \frac{1}{2}$ the order of the pole of $E(z, s)$ and $\varphi(s)$ at s_0 is the same: this follows from the functional equation $E(z, s) = \varphi(s)E(z, 1-s)$, since $E(z, 1-s)$ never vanishes identically (unless possibly for $s_0 = \frac{1}{2}$) and if $E(z, 1-s)$ has a pole at s_0, then so does $\varphi(1-s)$, but $\varphi(s)\varphi(1-s) = 1$.

Case II: The second case involves a singular point s_0 with $Re\, s_0 < \frac{1}{2}$ but $s_0 \notin \mathbb{R}$. We note that : $\oint (2s-1) r(z, z'; 1-s)\, ds = 0$ since, for $Re\, s < \frac{1}{2}, s \notin \mathbb{R}$, $r(z, z'; 1-s)$ is regular. So, using equation (2.8), we get:

$$\frac{1}{2\pi i} \int_\Gamma (2s-1) r(z, z'; s)\, ds = \frac{1}{2\pi i} \int_\Gamma E(z, s) E(z', 1-s)\, ds \tag{3.3}$$

The Eisenstein series $E(z, s)$ has a pole of finite order at s_0 and the contour integral on the right hand side of (3.3) is the residue of $E(z, s)E(z', 1-s)$ at s_0. Suppose that κ is the order of this pole. Then

$$\begin{aligned} \frac{1}{2\pi i} \int_\Gamma E(z, s) E(z', 1-s)\, ds &= \frac{d^{\kappa-1}}{ds^{\kappa-1}} \Big((s-s_0)^\kappa E(z, s) E(z', 1-s) \Big)\Big|_{s=s_0} \\ &= \sum_{i+j=\kappa-1} \binom{\kappa-1}{i} \frac{d^i}{ds^i} A(z, s_0)(-1)^j \frac{d^j E}{ds^j}(z', 1-s_0) \end{aligned} \tag{3.4}$$

where $A(z, s) = (s-s_0)^\kappa E(z, s)$ is regular at s_0. The integral kernel of this operator is, therefore, of the form:

$$\sum_{l=0}^{N} a_l(z)b_l(z') \tag{3.5}$$

and, for such a kernel, it is obvious that it defines a finite rank operator, of rank $\leq N+1$. In our case $N = \kappa - 1$. We need to know that the functions: $\frac{d^i}{ds^i}A(z, s_0)$, for $i = 0, 1, \dots, \kappa - 1$ are linearly independent. The following lemma proves something stronger which we will use later:

Lemma 3.1. *In the same context as above, the functions*

$$\frac{d^i}{ds^i}A(z, s_0) \tag{3.6}$$

for $i = 0, 1, \dots, \kappa - 1$ have no non-trivial linear combination that is an L^2 eigenfunction.

Proof. Denote $\frac{d^i}{ds^i}A(z,s)$ by $A^{(i)}(z,s)$. For $s \neq s_0$, s non-singular, we have:

$$-\Delta A(z,s) = -(s-s_0)^{\kappa}\Delta E(z,s) = s(1-s)A(z,s) \tag{3.7}$$

$$-\Delta A^{(1)}(z,s) = s(1-s)A^{(1)}(z,s) + (1-2s)A(z,s) \tag{3.8}$$

and an easy induction shows that $\Delta A^{(i)}$ belongs to the linear span of A, $A^{(1)}, \dots, A^{(i)}$. Let us write:

$$-\Delta A^{(i)}(z,s) = \sum_{j\leq i} \mu_j^i(s)A^{(j)}(z,s) \tag{3.9}$$

The following two relations will be proved inductively:

$$\mu_i^i = s(1-s) \tag{3.10}$$

for $i = 0, 1, \dots$ and:

$$\mu_{i-1}^i = \mu_{i-2}^{i-1} + (1-2s) \tag{3.11}$$

for $i = 2, 3, \dots$. Equation (3.9) implies:

$$-\Delta A^{(i+1)}(z,s) = \sum_{j\leq i} \frac{d\mu_j^i(s)}{ds}A^{(j)}(z,s) + \sum_{j\leq i} \mu_j^i(s)A^{(j+1)}(z,s) \tag{3.12}$$

So $A^{(i+1)}(z,s)$ has coefficient $\mu_{i+1}^{i+1}(s) = \mu_i^i(s)$, which is equal to $s(1-s)$ by inductive hypothesis. Also $\mu_i^{i+1}(s)$ is the coefficient of $A^{(i)}$ in the left side of (3.12), while the right side gives:

$$\frac{d\mu_i^i(s)}{ds} + \mu_{i-1}^i(s) = \frac{d}{ds}(s(1-s)) + \mu_{i-1}^i(s) = (1-2s) + \mu_{i-1}^i(s) \tag{3.13}$$

The induction is complete and (3.11) is proved. Formula (3.11) gives a recursive formula for $\mu_{i-1}^i(s)$ which can be solved explicitly to give:

$$\mu_{i-1}^i(s) = i(1-2s) \tag{3.14}$$

Now we can prove the statement of the lemma. Another induction will do it. Let us first remark that $A(z, s_0)$ is not in $L^2(\Gamma\backslash\mathbb{H})$. Since the Eisenstein series have a pole of order κ at s_0, the zero Fourier coefficient of $A(z,s_0)$ is of the form cy^{1-s_0} with $c \neq 0$ and, since $Re\, s_0 < \frac{1}{2}$, this term cannot be in L^2. Suppose that the $A^{(j)}(z,s_0)$'s have no non-trivial linear combination which is an L^2 eigenfunction for $j \leq i$. Also suppose that :

$$A^{(i+1)}(z,s_0) = \sum_{j\leq i} \lambda_j A^{(j)}(z,s_0) + f(z) \tag{3.15}$$

with f an L^2-eigenfunction. We see that $\Delta f = -\lambda f$ and (3.9), (3.15) imply:

$$-\Delta A^{(i+1)}(z,s_0) = \sum_{j\leq i} \lambda_j \sum_{l\leq j} \mu_l^j(s_0) A^{(l)}(z,s_0) - \Delta f(z) \tag{3.16}$$

On the other hand:

$$\begin{aligned}
-\Delta A^{(i+1)}(z,s_0) &= \frac{d}{ds}(-\Delta)A^{(i)}(z,s)|_{s=s_0} = \frac{d}{ds}\sum_{j\leq i} \mu_j^i(s) A^{(j)}(z,s)|_{s=s_0} \\
&= \sum_{j\leq i} \frac{d\mu_j^i(s_0)}{ds} A^{(j)}(z,s_0) + \sum_{j\leq i} \mu_j^i(s_0) A^{(j+1)}(z,s_0) \\
&= \sum_{j\leq i} \left(\frac{d\mu_j^i}{ds}(s_0) + \mu_{j-1}^i(s_0) \right) A^{(j)}(z,s_0) + \mu_i^i(s_0) A^{(i+1)}(z,s_0)
\end{aligned}$$

$$= \sum_{j\leq i} \left(\frac{d\mu_j^i}{ds}(s_0) + \mu_{j-1}^i(s_0) \right) A^{(j)}(z,s_0) + \mu_i^i(s_0) \left(\sum_{j\leq i} \lambda_j A^{(j)}(z,s_0) + f(z) \right) \tag{3.17}$$

In (3.16) $A^{(i)}(z,s_0)$ has coefficient $\lambda_i \mu_i^i(s_0)$ and in (3.17) it has $\lambda_i \mu_i^i(s_0) + \frac{d\mu_i^i}{ds}(s_0) + \mu_{i-1}^i(s_0)$. Moreover the equations (3.16) and (3.17) give a linear combination among the $A, A^{(1)}, \ldots, A^{(i)}$ which is an L^2 eigenfunction, because $-\Delta f$ and $\mu_i^i(s_0) f$ are L^2 eigenfunctions with the same eigenvalue. However, this linear combination is non-trivial:

$$\frac{d\mu_i^i}{ds}(s_0) + \mu_{i-1}^i(s_0) = (1-2s_0) + i(1-2s_0) = (i+1)(1-2s_0) \neq 0 \tag{3.18}$$

since $s_0 \neq \frac{1}{2}$. This finishes the proof of the lemma.

We now know that the image of P_{s_0} is a certain subspace of the space spanned by the κ elements $A^{(i)}(z,s_0)$, for $i = 0, 1, \ldots, \kappa-1$. To conclude the proof that the rank is exactly κ, we need to know that those functions really belong to the image. We are going to present κ functions $f_0(z), f_1(z), \ldots, f_{\kappa-1}(z)$ which are compactly supported and the elements $P_{s_0} f_i(z)$ are κ linearly independent vectors in the span of $A^{(0)}(z)$, $A^{(1)}(z)$,..., $A^{(\kappa-1)}(z)$ for $i = 0, 1, \ldots, \kappa-1$.

We first remark that in this setting the height at which we split the surface into a compact part and a cusp does not play any role: the analytic continuation of the resolvent is independent of it and the same is true for the Eisenstein series, their poles and their singular parts at the poles. Consequently the rank of P_{s_0} is independent of the height we cut at. We choose the functions $f_i(z)$ to be :

$$f_i(z) = \begin{cases} \frac{d^i E}{ds^i}(z, 1-s_0) & \text{for } z \in F_0; \\ 0 & \text{otherwise} \end{cases} \tag{3.19}$$

We need the following lemma about the Macdonald Bessel functions:

Lemma 3.2. *The Macdonald Bessel function $K_s(x)$ together with all its derivatives in s ($\frac{d}{ds}K_s(x)$ etc.) decrease exponentially as $x \to \infty$.*

The lemma is well-known for $K_s(x)$. A proof can be based on the following formula:

$$K_s(x) = \int_0^\infty e^{-x\cosh t}\cosh st\, dt \tag{3.20}$$

(see, for instance, [11], p.181). Then:

$$\frac{d^i}{ds^i}K_s(x) = \begin{cases} \int_0^\infty e^{-x\cosh t}t^i \cosh st\, dt, & \text{for } i \text{ even} \\ \int_0^\infty e^{-x\cosh t}t^i \sinh st\, dt, & \text{for } i \text{ odd} \end{cases} \tag{3.22}$$

For $x > 4$, $x\cosh t \geq \frac{x}{2} + 2\cosh t$ (because for $a, b > 2$ we have $ab \geq a + b$), so:

$$\frac{d^i}{ds^i}K_s(x) \leq \left\{ \begin{array}{ll} e^{-\frac{x}{2}}\int_0^\infty e^{-2\cosh t}t^i \cosh st\, dt, & \text{for } i \text{ even}; \\ e^{-\frac{x}{2}}\int_0^\infty e^{-2\cosh t}t^i \sinh st\, dt, & \text{for } i \text{ odd}; \end{array} \right\} = e^{-\frac{x}{2}}\frac{d^i}{ds^i}K_s(2) \tag{3.22}$$

This completes the proof of the lemma.

We have the following asymptotic expansions for $E(z', 1-s_0)$:

$$E(z', 1-s_0) = {y'}^{1-s_0} + \phi(1-s_0){y'}^{s_0} + O(e^{-y'}) \tag{3.23}$$

and:

$$\frac{d^j E}{ds^j}(z', 1-s_0) = {y'}^{1-s_0}(\ln y')^j + \sum_{a+b=j}\binom{j}{a}\phi^{(a)}(1-s_0)(-1)^b {y'}^{s_0}(\ln y')^b + O(e^{-y'}) \tag{3.24}$$

for $j = 1, \ldots, \kappa - 1$. From this we see that the dominant term, i.e. the term growing more quickly as $y' \to \infty$, is ${y'}^{1-s_0}(\ln y')^j$. Here we note that $Re\, s_0 < \frac{1}{2}$, i.e. $1 - Re\, s_0 > \frac{1}{2}$. Then P_{s_0} maps as follows:

$$f_i \to A^{(0)}(z, s_0)\left(\int_{F_0}({y'}^{1-s_0})^2(\ln y')^{\kappa-1+i} + \text{lower order terms } dz'\right) +$$

$$+\binom{\kappa-1}{1}A^{(1)}(z, s_0)\left(\int_{F_0}({y'}^{1-s_0})^2(\ln y')^{\kappa-2+i} + \text{lower order terms } dz'\right) + \cdots \tag{3.25}$$

for $i = 0, 1, \ldots, \kappa - 1$, where lower order terms means terms of the form $y^a(\ln y)^b$ with $Re\; a \leq 1$.

An integration by parts gives:

$$\int y^a(\ln y)^b\, dy = \frac{y^{a+1}}{a+1}(\ln y)^b - \frac{b}{a+1}\int y^a(\ln y)^{b-1}\, dy \tag{3.26}$$

and, inductively:

$$\int y^a (\ln y)^b \, dy = \sum_{j=0}^{b} \frac{(-1)^j}{(a+1)^{j+1}} P_j^b y^{a+1} (\ln y)^{b-j} \tag{3.27}$$

where P_j^b is the number of permutations of j elements out of b objects. This formula suggests that the main contribution in the integrals in (3.25) comes out of integrating $(y'^{1-s_0})^2 (\ln y')^j$, because all other terms have order of growth at most $y'^{1-s_0+s_0}(\ln y')^{2\kappa-2}$. After performing the integration, the terms $\int (y'^{1-s_0})^2 (\ln y')^b \frac{dy'}{y'^2}$ give contributions whose coefficient of $y^{(1-s_0)\cdot 2-1}$ is:

$$\frac{P_b^b(-1)^b}{[2(1-s_0)-1]^{b+1}} = \frac{b!(-1)^b}{[2(1-s_0)-1]^{b+1}} \tag{3.28}$$

In order to show that the $\kappa \times \kappa$ matrix of the coefficients of $A^{(i)}(z, s_0)$ is non-singular it suffices to show that its determinant has order of growth at least $y^{(1-s_0)\cdot 2-1}$. Since all terms of the form $y^{(1-s_0)\cdot 2-1}(\ln y)^m$ with $m \neq 0$ are of order bigger than $y^{(1-s_0)\cdot 2-1}$, when evaluating the determinant of the $\kappa \times \kappa$ matrix mentioned above, $y^{(1-s_0)\cdot 2-1}$ gets as coefficient the determinant of the matrix:

$$\begin{pmatrix} \frac{(\kappa-1)!(-1)^{\kappa-1}}{[2(1-s_0)-1]^{\kappa}} & \frac{(\kappa-2)!(-1)^{\kappa-2}}{[2(1-s_0)-1]^{\kappa-1}} & \cdots & \frac{0!(-1)^0}{[2(1-s_0)-1]^1} \\ \frac{\kappa!(-1)^{\kappa}}{[2(1-s_0)-1]^{\kappa+1}} & \frac{(\kappa-1)!(-1)^{\kappa-1}}{[2(1-s_0)-1]^{\kappa}} & \cdots & \frac{1!(-1)^1}{[2(1-s_0)-1]^2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{(2(\kappa-1))!(-1)^{2(\kappa-1)}}{[2(1-s_0)-1]^{2(\kappa-1)+1}} & \frac{(2\kappa-3)!(-1)^{2\kappa-3}}{[2(1-s_0)-1]^{2\kappa-3}} & \cdots & \frac{(\kappa-1)!(-1)^{\kappa-1}}{[2(1-s_0)-1]^{\kappa}} \end{pmatrix} \tag{3.29}$$

By taking out a factor $[2(1-s_0)-1]$ from each row and setting $x = \frac{-1}{2(1-s_0)-1}$, we see that the matrix has non-zero determinant iff the determinant of the following matrix is non-zero:

$$\begin{pmatrix} (\kappa-1)!x^{\kappa-1} & (\kappa-2)!x^{\kappa-2} & \cdots & 0!x^0 \\ \kappa!x^{\kappa} & (\kappa-1)!x^{\kappa-1} & \cdots & 1!x^1 \\ \vdots & \vdots & \ddots & \vdots \\ (2(\kappa-1))!x^{2(\kappa-1)} & (2\kappa-3)!x^{2\kappa-3} & \cdots & (\kappa-1)!x^{\kappa-1} \end{pmatrix} \tag{3.30}$$

The expansion of this determinant gives only terms of degree $\kappa(\kappa-1)$ in x, so the determinant is non-zero iff the following Hankel matrix is non-singular for $n = \kappa - 1$:

$$\begin{pmatrix} 0! & 1! & 2! & \cdots & n! \\ 1! & 2! & 3! & \cdots & (n+1)! \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ n! & (n+1)! & (n+2)! & \cdots & (2n)! \end{pmatrix} \tag{3.31}$$

We evaluate the determinant of this matrix as follows: From the j column we take out a factor of $(j-1)!$, and, then from the i row we take out a factor of $(i-1)!$, so the determinant becomes:

$$\prod_{i' \leq n} (i'-1)! \prod_{j' \leq n} (j'-1)! \left| \left(\frac{(i+j-2)!}{(i-1)!(j-1)!} \right)_{i,j=1,\cdots,n+1} \right| \tag{3.32}$$

We have:

$$\left|\left(\binom{i+j-2}{i-1}\right)_{i,j=1,\cdots,n+1}\right| = 1 \tag{3.33}$$

(see, for instance, [6], page 679). This is enough to ensure the claim about the rank of $\oint(2s-1)R(s)\,ds$, in the case that $Re\, s < \frac{1}{2}$ and $s \notin \mathbb{R}$.

Case III: If $Re\, s_0 < \frac{1}{2}$ and s_0 is real and the Eisenstein series have a pole at s_0, the following thing can happen:

$$\int_\Gamma (2s-1)r(z,z';1-s)\,ds \tag{3.34}$$

may be a non-zero kernel, if the Laplace operator has an L^2 eigenspace at $s_0(1-s_0)$, and in this case this is the kernel of the L^2 projection on the $s_0(1-s_0)$ eigenspace i.e.

$$\sum_{i=1}^{k} \vartheta_i(z)\vartheta_i(z') \tag{3.35}$$

where the $\vartheta_i(z)$ form an orthonormal basis for this eigenspace. The functions $A^{(i)}(z,s_0)$'s and the $\vartheta_i(z)$'s are linearly independent as follows from the lemma 3.1 above. Here we take as test functions to apply P_{s_0} the $f_0, f_1, \ldots, f_{\kappa-1}$ defined above and:

$$\tilde{\vartheta}_i(z) = \begin{cases} \vartheta_i(z) & \text{for } z \in F_0; \\ 0 & \text{otherwise} \end{cases} \tag{3.36}$$

for $i = 1, 2, \ldots k$. The result is a $(\kappa + k) \times (\kappa + k)$ matrix which we hope is non-singular. This matrix can be written in the form:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{3.37}$$

where A is a $\kappa \times \kappa$ matrix, B is $\kappa \times k$, C is $k \times \kappa$ and D is $k \times k$. They depend on the height we separate the fundamental domain and we study their asymptotic behaviour as $y \to \infty$. The matrix A is the matrix studied previously i.e. its determinant has order of growth at least $y^{(1-s_0)\cdot 2-1}$. Also:

$$D = ((\tilde{\vartheta}_i, \vartheta_j)_{L^2})_{i,j=1,\ldots,k} \tag{3.38}$$

and obviously tends to the identity matrix I. For example we have:

$$f_j \to A_0(z,s_0)\int_{F_0}(-1)^{\kappa-1}\frac{d^{\kappa-1}}{ds^{\kappa-1}}E(z',1-s_0)f_j\,dz' + \cdots + \vartheta_1(z)\int_{F_0}\vartheta_1 f_j\,dz' + \cdots \tag{3.39}$$

$$\tilde{\vartheta}_r \to A_0(z,s_0)\int_{F_0}(-1)^{\kappa-1}\frac{d^{\kappa-1}}{ds^{\kappa-1}}E(z',1-s_0)\tilde{\vartheta}_r\,dz' + \cdots + \vartheta_1(z)\int_{F_0}\vartheta_1\tilde{\vartheta}_r\,dz' + \cdots \tag{3.40}$$

So B and C contain elements of the form:

$$\int_{F_0}(-1)^j\frac{d^j}{ds^j}E(z',1-s_0)\vartheta(z')\,dz' \tag{3.41}$$

The zero Fourier coefficient of $\vartheta(z)$ is of the form $c \cdot y^{s_0}$, where $Re\ s_0 < \frac{1}{2}$. The constant c can be zero. Then the main term of the integral, taking into account the asymptotic behaviour of $\frac{d^j}{ds^j}E(z', 1-s_0)$ comes out of integrating:

$$\int y^{1-s_0}(\ln y)^j c y^{s_0} y^{-2} dy = c \int y^{-1}(\ln y)^j dy = \frac{c}{j+1}(\ln y)^{j+1} \tag{3.42}$$

When evaluating a determinant every term contains exactly one element from each row and each column. Consequently, the only terms in the determinant of $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ that have order of growth $y^{(1-s_0)\cdot 2-1}$ can be those coming from $detA$. Note that $\lim_{y\to\infty} D = I$ by (3.38). And we know from the previous case that $detA \neq 0$. So $y^{(1-s_0)\cdot 2-1}$ has again nonzero coefficient in the determinant and the matrix is non-singular iff the cut is made high enough.

Case IV: The next case to consider is $Res_0 < \frac{1}{2}$ and $\varphi(s)$ has a zero at s_0. Then we expect the multiplicity to be the dimension of the L^2 eigenspace for $s_0(1-s_0)$ minus the order of the zero. In this case $\varphi(s)$ would have a pole of order 1 at $1-s_0 \in (\frac{1}{2}, 1]$. The residue of the Eisenstein series $E(z,s)$ at $1-s_0$ will be an L^2 eigenfunction which is non-cuspidal. Set:

$$u(z) = \lim_{s\to 1-s_0}(s-(1-s_0))E(z,s) = \lim_{t\to s_0}(s_0-t)E(z,1-t) \tag{3.43}$$

and

$$c_0 = \lim_{s\to 1-s_0}(s-(1-s_0))\varphi(s) \tag{3.44}$$

Then the Maaß-Selberg relation implies: $\int |u(z)|^2 dz = c_0$ (see [10] p.652).

The operator $P_{s_0} = \oint (2s-1)R(s)ds$ has a kernel in which there are two contributions again: one from $\oint(2s-1)r(z,z';1-s)ds$ and one from $\oint E(z,s)E(z',1-s)ds$. The resolvent kernel $r(z,z';1-s)$ represents the ordinary resolvent kernel, as $1-s$ has real part $> \frac{1}{2}$. So the contribution of $\oint(2s-1)r(z,z';1-s)ds$ equals:

$$\sum_{i=1}^{k} \vartheta_i(z)\vartheta_i(z') \tag{3.45}$$

where ϑ_i, $i = 1,2,\ldots,k$ is an orthonormal basis for the L^2 eigenspace of $s_0(1-s_0)$. We can take $\vartheta_1(z) = \frac{u(z)}{\sqrt{c_0}}$. We now look at the other contribution:

$$\oint E(z,s)E(z',1-s)ds = \oint \varphi(s)E(z,1-s)E(z',1-s)ds \tag{3.46}$$

and the integrand has a pole of order 1, so we get:

$$\lim_{s\to s_0}(s-s_0)\varphi(s)E(z,1-s)E(z',1-s) = -\lim_{s\to s_0}\frac{\varphi(s)}{s_0-s}u(z)u(z') = -\vartheta_1(z)\vartheta_1(z') \tag{3.47}$$

since, if $\varphi(s) = \frac{c_0}{s-(1-s_0)} + \cdots$, then $\varphi(1-s) = \frac{c_0}{s_0-s} + \cdots$ and:

$$\lim_{s\to s_0}\frac{\varphi(s)}{s_0-s}=\lim_{s\to s_0}\frac{1}{\varphi(1-s)(s_0-s)}=\frac{1}{c_0} \tag{3.48}$$

Now that the term $\vartheta_1(z)\vartheta_1(z')$ cancels, the remaining kernel, which is $\sum_{i=2}^{k}\vartheta_i(z)\vartheta_i(z')$, gives an operator of rank $k-1$.

Case V: The last case is the case $s_0\in(0,\frac{1}{2}]$ and $\varphi(s)$ has no pole or zero at s_0. Then all the eigenfunctions at $s_0(1-s_0)$ are cusp forms with eigenvalue less than $\frac{1}{4}$. The only term contributing is $\oint r(z,z',1-s)ds$, since $E(z,s)E(z',1-s)$ is regular at s_0 and the result follows.

Remark 1: For $Re\, s_0\geq\frac{1}{2}$, $s_0\neq\frac{1}{2}$ formula (2.7) identifies the residue of the resolvent as follows: if

$$R(s)=\frac{A_{-1}}{s-s_0}+A_0+A_1(s-s_0)+\cdots \tag{3.49}$$

with corresponding kernels satisfying:

$$r(z,z';s)=\frac{A_{-1}(z,z')}{s-s_0}+A_0(z,z')+A_1(z,z')(s-s_0)+\cdots \tag{3.50}$$

close to s_0, then:

$$A_{-1}(z,z')=\frac{1}{2s_0-1}\sum_{j=1}^{m}\psi_j(z)\psi_j(z') \tag{3.51}$$

For applications (see the next section) one is also interested in $A_0(z,z')$ if $s_0=\frac{1}{2}+i\sigma$ is an embedded eigenvalue. For $s_0=\frac{1}{2}+i\sigma$ we write the corresponding expansion close to $1-s_0=\frac{1}{2}-i\sigma$ for $R(s)$ and $r(z,z';s)$:

$$R(s)=\frac{B_{-1}}{s-(1-s_0)}+B_0+B_1(s-(1-s_0))+\cdots$$

with corresponding kernels satisfying:

$$r(z,z';s)=\frac{B_{-1}(z,z')}{s-(1-s_0)}+B_0(z,z')+B_1(z,z')(s-(1-s_0))+\cdots \tag{3.52}$$

Using (3.50), (3.52) and (2.8) we get:

$$r(z,z';s)=\frac{A_{-1}(z,z')}{1-s-s_0}+A_0(z,z')+\cdots+\frac{1}{2s-1}\sum_{i=1}^{n}E_i(z,s)E_i(z',1-s) \tag{3.53}$$

$$\begin{aligned}B_0(z,z')&=\int_{\Gamma'}\frac{r(z,z';s)}{s-(1-s_0)}ds=A_0(z,z')+\int_{\Gamma'}\frac{\sum_{i=1}^{n}E_i(z,s)E_i(z',1-s)}{[s-(1-s_0)](2s-1)}ds\\&=A_0(z,z')+\lim_{s\to1-s_0}\frac{\sum_{i=1}^{n}E_i(z,s)E_i(z',1-s)}{2s-1}\\&=A_0(z,z')+\frac{\sum_{i=1}^{n}E_i(z,1-s_0)E_i(z',s_0)}{1-2s_0}\end{aligned} \tag{3.54}$$

Remark 2: The proof of theorem 1 also identifies a basis for the image of $\oint(2s-1)R(s)ds$ as follows:

In *Case II* we can take $A(z,s_0)$, $\frac{d}{ds}A(z,s_0),\ldots,\frac{d^{\kappa-1}}{ds^{\kappa-1}}A(z,s_0)$.

In *Case III* we take those in *II* and a basis for the eigenfunctions with eigenvalue $s_0(1-s_0)$.

In *Case IV* we only take a basis of eigenfunctions for $s_0(1-s_0)$ but exclude the residue of Eisenstein series at s_0.

In *Cases I* or *V* we take a basis of eigenfunctions for $s_0(1-s_0)$.

3. Proof of Theorem 2

The idea is (see [9] p.21) to use perturbation theory for the cut-off wave operator B which has discrete spectrum. However, since B is not self-adjoint, we choose to use variational formulas that use traces (see [3] p.79) instead of energy inner products, as was done in [9] p.24. The heart of the proof is to reduce the problem of computing $Re\,\hat{\lambda}^{(2)}$ into a comparison between the limits $\lim_{\varepsilon\to0^+}(\Delta+\frac{1}{4}+\sigma^2+i\varepsilon)^{-1}$ and $\lim_{\varepsilon\to0^-}(\Delta+\frac{1}{4}+\sigma^2+i\varepsilon)^{-1}$, where $\frac{1}{4}+\sigma^2$ is an imbedded eigenvalue. This comparison is provided by formula (3.54).

We set $L=\Delta+\frac{1}{4}$, $A=\begin{pmatrix}0&1\\L&0\end{pmatrix}$, E the energy form for the wave equation $u_{tt}=Lu$:

$$E\left(\begin{pmatrix}f_1\\f_2\end{pmatrix}\right)=-(f_1,Lf_1)_{L^2}+(f_2,f_2)_{L^2}$$

$\mathcal{H}_G$ the completion of the space of pairs of C^∞ functions with compact support in the norm:

$$G\left(\begin{pmatrix}f_1\\f_2\end{pmatrix}\right)=E\left(\begin{pmatrix}f_1\\f_2\end{pmatrix}\right)+c\|f_1\|_2^2$$

for c sufficiently large, P the E-orthogonal projection to the complement of the space $\mathcal{D}_+\oplus\mathcal{D}_-$ in $\mathcal{H}_G$, where $\mathcal{D}_\pm$ are the spaces of outgoing and incoming data (see [5] p.121). The operator P may only change the zero Fourier coefficients of data at each cusp. The operator B is the infinitesimal generator of the semigroup $PU(t)P$ where $U(t)$ is the standard wave operator. We denote by $R_F(z)$ the resolvent of an operator F i.e.: $R_F(z)=(F-z)^{-1}$. We have:

$$R_B(\lambda)=PR_A(\lambda)P \tag{4.1}$$

for $Re\,\lambda$ sufficiently large ([5] p.29). A calculation with matrices gives:

$$\begin{aligned}R_A(\lambda)&=\begin{pmatrix}\lambda R_L(\lambda^2)&R_L(\lambda^2)\\LR_L(\lambda^2)&\lambda R_L(\lambda^2)\end{pmatrix}\\&=\begin{pmatrix}-\lambda R(\lambda+\frac{1}{2})&-R(\lambda+\frac{1}{2})\\I-\lambda^2R(\lambda+\frac{1}{2})&-\lambda R(\lambda+\frac{1}{2})\end{pmatrix}\end{aligned}\tag{4.2}$$

since $R_L(\lambda^2)=-R(s)$ and $\lambda=s-\frac{1}{2}$. The operator $R(\lambda)$ has an analytic continuation to the whole plane, although not as an L^2 operator. Since we will consider imbedded eigenvalues corresponding to singular points on $Re\,s=\frac{1}{2}$, it is enough to consider $R(s):\mathcal{B}_0\to\mathcal{B}_1$, $Re\,s>0$. The operator $R_A(\lambda)$ has an analytic continuation by (4.2) and by analytic continuation (4.1) remains valid whenever both sides make sense. One should note that there are data with zero

Fourier coefficient of the form $\{cy^{\frac{1}{2}}, 0\}$, which are left untouched by P and at which we cannot apply $R_A(\lambda)$.

We determine the projection Q for B at $i\sigma$ corresponding to an imbedded eigenvalue $\frac{1}{4}+\sigma^2$ ($s_0 = \frac{1}{2}+i\sigma$):

$$\begin{aligned} Q &= -\frac{1}{2\pi i}\int_\Gamma R_B(\lambda)\,d\lambda = -\frac{1}{2\pi i}P\int_\Gamma R_A(\lambda)\,d\lambda\; P \\ &= -\frac{1}{2\pi i}P\int_\Gamma \begin{pmatrix} -\lambda R(\lambda+\frac{1}{2}) & -R(\lambda+\frac{1}{2}) \\ I-\lambda^2 R(\lambda+\frac{1}{2}) & -\lambda R(\lambda+\frac{1}{2}) \end{pmatrix} d\lambda\; P \end{aligned} \tag{4.3}$$

where the contour Γ encloses only $i\sigma$ among the singular points of the resolvent of B. By formula (3.51) the contour integral gives us an operator with kernel:

$$\begin{pmatrix} \frac{1}{2}\sum_{j=1}^m \psi_j(z)\psi_j(z') & \frac{1}{2i\sigma}\sum_{j=1}^m \psi_j(z)\psi_j(z') \\ \frac{-\sigma}{2i}\sum_{j=1}^m \psi_j(z)\psi_j(z') & \frac{1}{2}\sum_{j=1}^m \psi_j(z)\psi_j(z') \end{pmatrix} \tag{4.4}$$

Also we note that, since the $\psi_j(z)$'s are cusp forms, the operator P on the left will not change the outcome of the previous operators. If $g = \binom{g_1}{g_2}$ is any pair which is supported in some compact set and the zero Fourier coefficients of g_1, g_2 vanish above the cut a then: $Pg = g$. So:

$$\begin{aligned} Qg &= \begin{pmatrix} \frac{1}{2}\sum_{j=1}^m \psi_j(z)\int \psi_j(z')g_1(z')dz' + \frac{1}{2i\sigma}\psi_j(z)\int \psi_j(z')g_2(z')dz' \\ \frac{-\sigma}{2i}\sum_{j=1}^m \psi_j(z)\int \psi_j(z')g_1(z')dz' + \frac{1}{2}\sum_{j=1}^m \psi_j(z)\int \psi_j(z')g_2(z')dz' \end{pmatrix} \\ &= \sum_{j=1}^m \left[\tfrac{1}{2}(g_1,\psi_j) - \tfrac{i}{2\sigma}(g_2,\psi_j)\right]\begin{pmatrix} \psi_j(z) \\ i\sigma\psi_j(z)\end{pmatrix} = \sum_{j=1}^m E\left(\begin{pmatrix} g_1 \\ g_2 \end{pmatrix}, \begin{pmatrix} \frac{\psi_j}{\sqrt{2}\sigma} \\ \frac{i\sigma\psi_j}{\sqrt{2}\sigma}\end{pmatrix}\right)\begin{pmatrix} \frac{\psi_j}{\sqrt{2}\sigma} \\ \frac{i\sigma\psi_j}{\sqrt{2}\sigma}\end{pmatrix} \end{aligned} \tag{4.5}$$

where E represents the energy inner product. Since $E(\binom{\psi_j}{i\sigma\psi_j}) = -(\psi_j, L\psi_j) + (i\sigma\psi_j, i\sigma\psi_j) = 2\sigma^2$ we see that Q is an E-orthogonal projection (see [9] p.24).

The second variation for the weighted mean of eigenvalues $\hat{\lambda}$ is given by: (see [3], page 79)

$$\hat{\lambda}^{(2)} = \frac{1}{m}Tr(\ddot{B}Q - 2\dot{B}S\dot{B}Q) \tag{4.6}$$

where $\ddot{B}$ is the second variation of B and S is the constant term in the Laurent expansion of the resolvent of B around the singular point $i\sigma$. We have:

$$\begin{aligned} \ddot{B}Q\begin{pmatrix} g_1 \\ g_2\end{pmatrix} &= \sum_{j=1}^m \ddot{B}\left(\frac{1}{2\sigma^2}E\left(\begin{pmatrix} g_1 \\ g_2\end{pmatrix}, \begin{pmatrix} \psi_j \\ i\sigma\psi_j\end{pmatrix}\right)\begin{pmatrix} \psi_j(z) \\ i\sigma\psi_j(z)\end{pmatrix}\right) \\ &= \frac{1}{2\sigma^2}\sum_{j=1}^m E\left(\begin{pmatrix} g_1 \\ g_2\end{pmatrix}, \begin{pmatrix} \psi_j \\ i\sigma\psi_j\end{pmatrix}\right)\begin{pmatrix} 0 & 0 \\ \ddot{L} & 0\end{pmatrix}\begin{pmatrix} \psi_j \\ i\sigma\psi_j\end{pmatrix} \\ &= \frac{1}{2\sigma^2}\sum_{j=1}^m E\left(\begin{pmatrix} g_1 \\ g_2\end{pmatrix}, \begin{pmatrix} \psi_j \\ i\sigma\psi_j\end{pmatrix}\right)\begin{pmatrix} 0 \\ \ddot{L}\psi_j(z)\end{pmatrix} \end{aligned} \tag{4.7}$$

We want to compute the trace of $\ddot{B}Q$, so, by choosing any basis containing all the data $\binom{0}{\ddot{L}\psi_j}$, $j = 1, 2, \ldots, m$ we find that:

$$\ddot{B}Q\begin{pmatrix}0\\ \dot{L}\psi_k\end{pmatrix} = \sum_{j=1}^{m}\tfrac{1}{2\sigma^2}E\left(\left(\begin{smallmatrix}0\\ \dot{L}\psi_k\end{smallmatrix}\right),\left(\begin{smallmatrix}\psi_j\\ i\sigma\psi_j\end{smallmatrix}\right)\right)\begin{pmatrix}0\\ \ddot{L}\psi_j\end{pmatrix} = \sum_{j=1}^{m}\tfrac{-i\sigma}{2\sigma^2}(\dot{L}\psi_k,\psi_j)\begin{pmatrix}0\\ \ddot{L}\psi_j\end{pmatrix} \tag{4.8}$$

So:

$$Tr(\ddot{B}Q) = \sum_{k=1}^{m}\frac{-i}{2\sigma}(\ddot{L}\psi_k,\psi_k) \tag{4.9}$$

which is purely imaginary and we can ignore it when we take real parts. We now try to evaluate $Tr(\dot{B}S\dot{B}Q) = Tr(S\dot{B}Q\dot{B})$. We have:

$$\dot{B}\begin{pmatrix}g_1\\ g_2\end{pmatrix} = \begin{pmatrix}0\\ \dot{L}g_1\end{pmatrix} \tag{4.10}$$

$$Q\begin{pmatrix}0\\ \dot{L}g_1\end{pmatrix} = \tfrac{1}{2\sigma^2}\sum_{j=1}^{m}E\left(\begin{pmatrix}0\\ \dot{L}g_1\end{pmatrix},\begin{pmatrix}\psi_j\\ i\sigma\psi_j\end{pmatrix}\right)\begin{pmatrix}\psi_j\\ i\sigma\psi_j\end{pmatrix} = \tfrac{1}{2\sigma^2}\sum_{j=1}^{m}(\dot{L}g_1, i\sigma\psi_j)\begin{pmatrix}\psi_j\\ i\sigma\psi_j\end{pmatrix} \tag{4.11}$$

$$\dot{B}Q\dot{B}\begin{pmatrix}g_1\\ g_2\end{pmatrix} = \frac{1}{2\sigma^2}\sum_{j=1}^{m}(\dot{L}g_1, i\sigma\psi_j)\begin{pmatrix}0\\ \dot{L}\psi_j\end{pmatrix} \tag{4.12}$$

Suppose that $S\left(\begin{smallmatrix}0\\ \dot{L}\psi_j\end{smallmatrix}\right) = \left(\begin{smallmatrix}w_{1,j}\\ w_{2,j}\end{smallmatrix}\right) = w_j$, a certain pair of data. Then:

$$S\dot{B}Q\dot{B}\begin{pmatrix}g_1\\ g_2\end{pmatrix} = \frac{1}{2\sigma^2}\sum_{j=1}^{m}(\dot{L}g_1, i\sigma\psi_j)w_j \tag{4.13}$$

and

$$S\dot{B}Q\dot{B}w_k = \frac{1}{2\sigma^2}\sum_{j=1}^{m}(\dot{L}w_{1,k}, i\sigma\psi_j)w_j \tag{4.14}$$

Using (4.6), (4.9), (4.14) we get:

$$Re\,\hat{\lambda}^{(2)} = -\frac{2}{m}Re\,Tr(S\dot{B}Q\dot{B}) = -\frac{1}{m\sigma^2}Re\sum_{k=1}^{m}(\dot{L}w_{1,k}, i\sigma\psi_k) \tag{4.15}$$

We use contour integrals to relate the operator S with the operator A_0 of (3.49):

$$\begin{aligned} S &= \int_\Gamma \frac{R_B(\lambda)}{\lambda - i\sigma}d\lambda = P\int_\Gamma \frac{R_A(\lambda)}{\lambda - i\sigma}d\lambda P \\ &= P\int_\Gamma \begin{pmatrix}\frac{\lambda R_L(\lambda^2)}{\lambda - i\sigma} & \frac{R_L(\lambda^2)}{\lambda - i\sigma}\\ \frac{I+\lambda^2 R_L(\lambda^2)}{\lambda - i\sigma} & \frac{\lambda R_L(\lambda^2)}{\lambda - i\sigma}\end{pmatrix} d\lambda\, P \\ &= P\begin{pmatrix}-i\sigma A_0 & -A_0\\ I+\sigma^2 A_0 & -i\sigma A_0\end{pmatrix}P \end{aligned} \tag{4.16}$$

so:

$$w_k = S\begin{pmatrix}0\\ \dot{L}\psi_k\end{pmatrix} = P\begin{pmatrix}-A_0\dot{L}\psi_k\\ -i\sigma A_0\dot{L}\psi_k\end{pmatrix} \tag{4.17}$$

We remark that the same calculation can be repeated for the eigenvalue $-i\sigma$ of the operator B and that the real part of the variation has to be the same, because of the nature of the spectral problem for Δ, i.e. $\frac{1}{2}+i\sigma$ and $\frac{1}{2}-i\sigma$ give the same eigenvalue for Δ. The corresponding operator $\tilde{S}$ will look like:

$$\tilde{S} = P\begin{pmatrix} i\sigma B_0 & -B_0 \\ I+\sigma^2 B_0 & i\sigma B_0 \end{pmatrix} P \tag{4.18}$$

$$\tilde{S}\begin{pmatrix} 0 \\ \dot{L}\psi_k \end{pmatrix} = P\begin{pmatrix} -B_0\dot{L}\psi_k \\ i\sigma B_0\dot{L}\psi_k \end{pmatrix} = \tilde{w}_k \tag{4.19}$$

and

$$Re\,\hat{\lambda}^{(2)} = -\frac{1}{m\sigma^2}\sum_{k=1}^{m} Re\,(\dot{L}\tilde{w}_{1,k}, -i\sigma\psi_k) \tag{4.20}$$

Noticing that $\dot{L}$ is self adjoint and that $\dot{L}f$ is compactly supported so we can ignore the effect of P on the pairs $\binom{-A_0\dot{L}\psi_k}{-i\sigma A_0\dot{L}\psi_k}$ and $\binom{-B_0\dot{L}\psi_k}{i\sigma B_0\dot{L}\psi_k}$ when taking inner products with $\dot{L}\psi_k$, we get:

$$\begin{aligned} Re\,\hat{\lambda}^{(2)} &= \frac{1}{2}\left(-\frac{1}{m\sigma^2}\right)\sum_{k=1}^{m} Re\,\left[(\dot{L}w_{1,k}, i\sigma\psi_k) + (\dot{L}\tilde{w}_{1,k}, -i\sigma\psi_k)\right] \\ &= -\frac{1}{2m\sigma^2}\sum_{k=1}^{m} Re\,\left[(w_{1,k}, i\sigma\dot{L}\psi_k) - (\tilde{w}_{1,k}, i\sigma\dot{L}\psi_k)\right] \\ &= -\frac{1}{2m\sigma^2}\sum_{k=1}^{m} Re\,\left[(-A_0\dot{L}\psi_k, i\sigma\dot{L}\psi_k) - (-B_0\dot{L}\psi_k, i\sigma\dot{L}\psi_k)\right] \\ &= -\frac{1}{2m\sigma^2}\sum_{k=1}^{m} Re\,((B_0-A_0)\dot{L}\psi_k, i\sigma\dot{L}\psi_k) \\ &= Re\left[\frac{i}{2m\sigma}\sum_{k=1}^{m}((B_0-A_0)\dot{L}\psi_k, \dot{L}\psi_k)\right] \end{aligned} \tag{4.21}$$

From relation (3.54) we get:

$$(B_0-A_0)\dot{L}\psi_k = \sum_{i=1}^{n}\frac{E_i(z,1-s_0)}{-2i\sigma}\int_F E_i(z',s_0)(\dot{L}\psi_k)(z')dz' \tag{4.22}$$

and finally, using (4.21), (4.22):

$$\begin{aligned} Re\,\hat{\lambda}^{(2)} &= -\frac{1}{4m\sigma^2}\sum_{k=1}^{m}\sum_{i=1}^{n}\int_F E_i(z,1-s_0)\dot{L}\psi_k(z)dz \cdot \int_F E_i(z',s_0)\dot{L}\psi_k(z')dz' \\ &= -\frac{1}{4m\sigma^2}\sum_{k=1}^{m}\sum_{i=1}^{n}\int_F \overline{E_i(z,s_0)}\dot{L}\psi_k(z)dz \cdot \int_F E_i(z',s_0)\dot{L}\psi_k(z')dz' \\ &= -\frac{1}{4m\sigma^2}\sum_{k=1}^{m}\sum_{i=1}^{n}|(E_i(z,s_0), \dot{L}\psi_k(z))|^2 \end{aligned}$$

$$\cdot \tag{4.23}$$

Remark: The difference in the constant we find compared to [9] is due to the fact that we take $||\psi_k||_{L^2} = 1$ while in [9] we have $E(\psi_k) = 1$, so in the case $m = 1$ our answer agrees with formula (5.29) in [9].

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