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Spectral Data for Finite Volume Hyperbolic Surfaces at the Bottom of the Continuous Spectrum

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The spectrum of the Laplace operator on finite area non-compact surfaces becomes stable if one adjoins to the L^2 eigenvalues the scattering frequencies. For the bottom of the continuous spectrum ($\frac{1}{4}$) we need to take into account any non-vanishing Eisenstein series at $s=\frac{1}{2}$. In this work the particular behaviour of the spectrum at $\frac{1}{4}$ is studied with respect to genericity of L^2 eigenvalues and of non-vanishing Eisenstein series at $s=\frac{1}{2}$.

1. INTRODUCTION

In this work we discuss some problems which arise in the spectral theory of the Laplace operator acting on non-compact finite volume surfaces obtained as the quotient of hyperbolic 2-space $\mathbb{H}^2$ by discrete subgroups of $PSL(2, \mathbb{R})$. We shall be interested in the resolvent kernel, the generalized eigenfunctions, known also as Eisenstein series, and the scattering matrix.

The spectrum consists of a continuous part filling $[\frac{1}{4}, \infty)$ and a discrete set of eigenvalues, of which finitely many are less than or equal to $\frac{1}{4}$. Associated with the problem of existence of infinitely many cusp forms (i.e., L^2 eigenfunctions with zero Fourier coefficient) is the problem of stability of the spectrum. The spectrum is unstable under perturbation. In [PS1] a sufficient condition was found which ensures that a cusp form with eigenvalue $\lambda=\frac{1}{4}+r^2$, $r>0$ is dissolved. However, the spectrum becomes more manageable when the scattering frequencies are adjoined with multiplicity equal to the order of the pole of the scattering matrix. In [PS2] it is proved that the augmented spectrum, named the singular set, is real analytic in Teichmüller space. The singular set occurred in a natural way in a one-sided version of the Selberg trace formula and is the actual spectrum of the cut-off wave operator B in the Lax–Phillips scattering theory [LP1]. The multiplicity of $\frac{1}{2}$ is defined to be twice the dimension of cusp forms with eigenvalue $\frac{1}{4}$ plus $(n+\mathrm{Tr}(\Phi(\frac{1}{2})))/2$, which is the dimension of the space

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of Eisenstein series at $s=\frac{1}{2}$. Here $\Phi(s)$ is the scattering matrix. It has been conjectured [PS2] that generically in Teichmüller space this multiplicity is 0. In particular this conjecture contains the question of the genericity of the eigenvalue $\frac{1}{4}$ for finite volume surfaces, raised by Wolpert [W] in connection to the study of limits of eigenvalues and eigenfunction branches when we pinch geodesics. In this work we show:

THEOREM 1. *For a generic (hyperbolic) Riemann surface with cusps with signature $(g, 0, m)$ where $g+m \geqslant 3$ there are no cusp forms with eigenvalue $\lambda=\frac{1}{4}$.*

We also identify the singular part of the Green's function (i.e., the kernel of $(-\Delta - s(1-s))^{-1}$ at $s=\frac{1}{2}$:

THEOREM 2. *The kernel of the resolvent has the following expansion close to $s=\frac{1}{2}$,*

$$r(z, z'; s)=\frac{\sum_{i=1}^{k} \vartheta_i(z)\, \vartheta_i(z')}{(s-\frac{1}{2})^2}+\frac{\sum_{i=1}^{n} E_i(z, \frac{1}{2})\, E_i(z', \frac{1}{2})}{4(s-\frac{1}{2})}+r^{+}(z, z'; s),$$

where $r^{+}(z, z'; s)$ is regular close to $\frac{1}{2}$, $E_i(z, s)$ are the Eisenstein series indexed by the cusps, and the $\vartheta_i(z)$ form an orthonormal basis of the cusp forms with eigenvalue $\frac{1}{4}$ (all taken to be real). The resolvent kernel is regular at $s=\frac{1}{2}$ iff there are no cusp forms with eigenvalues $\frac{1}{4}$ and all Eisenstein series vanish at $s=\frac{1}{2}$.

For the Schrödinger operator the corresponding theorem is discussed in [JK]. Throughout this work a non-vanishing Eisenstein series at $s=\frac{1}{2}$ will also be called resonance at the bottom of the continuous spectrum or even nullvector (the second term comes from the Lax–Phillips scattering theory, because a non-vanishing Eisenstein series $e(z, \frac{1}{2})$ gives a pair $j=\{e(z), 0\}$ for which the energy form degenerates: $E(j, \mathscr{H}_G)=0$; see Sect. 2). With the help of Theorem 2 and the Lax–Phillips scattering theory, we find a variation formula for Eisenstein series at $s=\frac{1}{2}$:

THEOREM 3. *The first variation of a resonance $e(z)$ at the bottom of the continuous spectrum $\frac{1}{4}$ is given by the formula*

$$\dot{\lambda}=\frac{1}{4}\int_{\Gamma\backslash\mathbb{H}} e(z)(\dot{\Delta} e)(z)\, dz$$

provided there are no cusp forms with eigenvalue $\frac{1}{4}$.

This allows us to deduce the following (Theorem 4): if one allows metrics which are hyperbolic outside of a compact set then $\operatorname{Tr}(\Phi(\frac{1}{2}))=-n$ generically so the multiplicity of the point $\frac{1}{2}$ is indeed 0. Theorem 3 applies also to the case of quasiconformal deformation. We get:

THEOREM 5. *The variation of a resonance at the bottom of the continuous spectrum* $\dot{\lambda}(0)$ *is different from* 0 *iff* Re $F(\frac{1}{2}) \neq 0$ *where*

$$F(s) = \frac{(4\pi)^{-s}}{32\pi} \frac{\Gamma(s+\frac{3}{2})^2}{\Gamma(s)} L(s+\tfrac{3}{2})$$

and $L(s)$ *is the Rankin–Selberg convolution of* Q, *the holomorphic cusp form of weight* 4 *giving the direction in the deformation space, and* $E(z, \frac{1}{2})$ *the resonance.*

We note that the non-vanishing of $F(\frac{1}{2})$ is associated with the value of an L-function at the middle of its critical line.

We also treat the degenerate case (Theorem 6). The conditions for dissolving more than one resonance at $\frac{1}{4}$ are the same ones expected for a multiple isolated eigenvalue (see [SCH]).

The examples of $\Gamma = \Gamma^0(25)$, $\Gamma = \Gamma^0(49)$, and $\Gamma = \Gamma^0(81)$ are investigated. In the deformation space of $\Gamma^0(81)\backslash\mathbb{H}$ the scattering matrix at $s = \frac{1}{2}$ is discontinuous (Corollary 7). The surface $\Gamma^0(25)\backslash\mathbb{H}$ has one null-vector and its destruction is associated with the non-vanishing of twisted L-series of holomorphic cusp forms of weight 4 for $\Gamma^0(25)$. The destruction of the two resonances at $\frac{1}{4}$ for $\Gamma^0(49)$ is related to the Birch Swinnerton–Dyer conjecture for elliptic curves (Sect. 9).

2. Scattering Theory and Perturbation Theory for Δ on Automorphic Functions

This section contains introductory material on the Lax–Phillips scattering theory, its application to the stability of the singular set, and the Faddeev method for the analytic continuation of the resolvent kernel. This material is used extensively in Sections 4, 5, 6. The automorphic wave equation has the form:

$$u_{tt} = \Delta u + \tfrac{1}{4}u = (\Delta + \tfrac{1}{4})u. \tag{2.1}$$

Let us denote by L the operator $\Delta + \frac{1}{4}$.

Now the study of (2.1) starts with rewriting it as a first-order system by introducing the time derivative of u as a new variable v: $v = u_t$ so that:

$$v_t = Lu. \tag{2.2}$$

The pair $\{u, u_t\} = f(t)$ is called data. We define $\mathscr{H}_0$ to be the space of all data $f = \{f_1, f_2\}$, where f_1 is in the domain of $|L|^{1/2}$ and $f_2 \in L^2(\Gamma\backslash\mathbb{H})$. We rewrite the wave equation in matrix notation as:

$$f_t = Af, \qquad \text{where} \quad A = \begin{pmatrix} 0 & 1 \\ L & 0 \end{pmatrix}. \tag{2.3}$$

We solve Eq. (2.1) with initial data $f = \{f_1, f_2\}$, i.e.,

$$u(z, 0) = f_1(z) \qquad \text{and} \qquad u_t(z, 0) = f_2(z). \tag{2.4}$$

The data at time t is uniquely determined by its initial data and the operator $U(t)$ is defined as:

$$U(t)\colon \{u(0), u_t(0)\} \to \{u(t), u_t(t)\}. \tag{2.5}$$

The energy of the data f associated with the wave equation is

$$E(f) = -(f_1, Lf_1) + \|f_2\|^2 = -(u, Lu) + \|u_t\|^2. \tag{2.6}$$

In general E is not positive definite. To circumvent this we define a new quadratic form G which is positive definite and is closely related to E as follows:

We decompose the fundamental domain F into

$$F = F_0 \cup \bigcup_{\alpha=1}^{n} F_\alpha, \tag{2.7}$$

where F_0 is compact and F_α is isometric to the standard cusp $C = \{z; -\frac{1}{2} \leqslant \operatorname{Re} z \leqslant \frac{1}{2}, \operatorname{Im} z \geqslant a\}$. Let:

$$K(f) = \int_{F_0} |f_1|^2 \frac{dx\,dy}{y^2}. \tag{2.8}$$

Then we set

$$G(f) = E(f) + cK(f), \tag{2.9}$$

where c is sufficiently large. Lemma 4.3 in [LP1] asserts that the quadratic form $K(f)$ is compact with respect to the quadratic form G. Lemma 5.1 in [LP1] shows that G is positive definite. We now define $\mathscr{H}_G$ as the completion in the G-norm of the space $\mathscr{H}_0$.

For the automorphic wave equation with $U(t)$ the family of unitary operators defined above, $\mathscr{D}_\pm$ are defined to be the closures in $\mathscr{H}_G$ of the initial data of incoming and outgoing solutions, respectively,

$$\mathscr{D}_- = \overline{\{f; f_0 = 0, (f_1)_j = y^{1/2}\varphi_j(y), (f_2)_j = y^{3/2}\varphi_j'(y) \text{ for } j \neq 0\}},$$

$$\mathscr{D}_+ = \overline{\{f; f_0 = 0, (f_1)_j = y^{1/2}\varphi_j(y), (f_2)_j = -y^{3/2}\varphi_j'(y) \text{ for } j \neq 0\}},$$

in each cusp (transformed to ∞), where φ_j is a C_0^∞ function on the real line $\mathbb{R}$ which is zero for $y < a$. Here $\varphi_j'(y) = d\varphi_j(y)/dy$.

We denote the zero Fourier coefficient of f with respect to x by $f^{(0)}$ for $y > a$. If $f \in \mathscr{K} = \mathscr{H}_G \ominus (\mathscr{D}_- \oplus \mathscr{D}_+)$, the E-orthogonal complement of $\mathscr{D}_- \oplus \mathscr{D}_+$ then

$$f^{(0)} = \{cy^{1/2}, 0\} \tag{2.10}$$

for $y > a$ in each cusp. We denote the E-orthogonal projection of $\mathscr{H}_G$ onto $\mathscr{K}$ by P and by

$$Z(t) = PU(t)P \tag{2.11}$$

for $t \geqslant 0$. The operators $Z(t)$ form a strongly continuous semigroup of operators on $\mathscr{K}$ with infinitesimal generator denoted by B. The operator B has compact resolvent and has pure point spectrum of finite multiplicity and the resolvent $R_B(\lambda)$ is a meromorphic function in the whole complex plane.

One wants to use perturbation theory for the non-self-adjoint operator B, so it is of importance to analyse its spectrum. This is the main technical issue in [PS2]. Let us denote the scattering matrix by $\Phi(s)$, which appears as a factor in the zero Fourier coefficient of the Eisenstein series, and $\varphi(s)$ the determinant of the scattering matrix. The singular set is defined in [PS2] as follows:

(a) If $\operatorname{Re} s \geqslant \frac{1}{2}$ but $s \neq \frac{1}{2}$ we define the multiplicity at s to be the dimension of the eigenspace for $s(1-s)$ of Δ on $L^2(\Gamma \backslash \mathbb{H})$. Consequently, this multiplicity is zero unless $\operatorname{Re} s = \frac{1}{2}$ or $s \in (\frac{1}{2}, 1]$.

(b) If $\operatorname{Re} s < \frac{1}{2}$ we define the multiplicity at s to be the multiplicity of the eigenvalue $s(1-s)$ of Δ on $L^2(\Gamma \backslash \mathbb{H})$ plus the order of the pole (or minus the order of the zero) of $\varphi(s)$ at s. Consequently, if $\operatorname{Re} s < \frac{1}{2}$ and $s \notin \mathbb{R}$ this multiplicity is simply the order of the pole of $\varphi(s)$ at s, since in this case $\varphi(s)$ cannot have a zero and $s(1-s)$ is not an L^2-eigenvalue.

(c) For $s = \frac{1}{2}$ the multiplicity is defined as twice the dimension of cusp forms with eigenvalue $\frac{1}{4}$ plus $(n + \operatorname{tr}(\Phi(\frac{1}{2})))/2$.

Then we have the following theorem:

THEOREM 2.1 (see Theorems 3.1, 3.2, 4.1 in [PS2]). *The singular set is the spectrum of the operator* $B + \frac{1}{2}I$ *(counting multiplicities).*

The null eigenspace of B is crucial in the rest of this work. It consists of data f with $B^k f = 0$ for some k. Such data are $\{\vartheta, 0\}$, $\{0, \vartheta\}$ where ϑ is a cusp form with eigenvalue $\frac{1}{4}$, i.e., $L\vartheta = 0$, since in this case,

$$B\{\vartheta, 0\} = BP\{\vartheta, 0\} = PA\{\vartheta, 0\} = P\begin{pmatrix} 0 & 1 \\ L & 0 \end{pmatrix}\begin{pmatrix} \vartheta \\ 0 \end{pmatrix} = P\begin{pmatrix} 0 \\ L\vartheta \end{pmatrix} = 0$$

and

$$B\{0, \vartheta\} = BP\{0, \vartheta\} = PA\{0, \vartheta\} = P\begin{pmatrix} 0 & 1 \\ L & 0 \end{pmatrix}\begin{pmatrix} 0 \\ \vartheta \end{pmatrix} = P\{\vartheta, 0\} = \{\vartheta, 0\}$$

so $B^2\{0, \vartheta\} = 0$. However, there may exist other data in the null space of B. If $E(z, \frac{1}{2})$ is a non-zero Eisenstein series at $s = \frac{1}{2}$ (having zero Fourier coefficient of the form $cy^{1/2}$ in one cusp) then $LE(z, \frac{1}{2}) = 0$, so

$$B\begin{pmatrix} E(z, \frac{1}{2}) \\ 0 \end{pmatrix} = PA\begin{pmatrix} E(z, \frac{1}{2}) \\ 0 \end{pmatrix} = P\begin{pmatrix} 0 & 1 \\ L & 0 \end{pmatrix}\begin{pmatrix} E(z, \frac{1}{2}) \\ 0 \end{pmatrix} = P\begin{pmatrix} 0 \\ LE(z, \frac{1}{2}) \end{pmatrix} = 0.$$

But in this case we cannot take into account $\left({}^{\;\;0}_{E(z,1/2)}\right)$ because it does not belong to $\mathscr{H}$. These are the only data in the nullspace of B (see Theorem 3.2 in [PS2]).

The scattering matrix $\Phi(s)$ at $s = \frac{1}{2}$ is real and symmetric with $\Phi^2 = I$, the identity matrix. Consequently, its eigenvalues are ± 1. We diagonalize $\Phi(\frac{1}{2})$ and suppose the basis in which the diagonalization occurs is

$$\mathbf{e}(z) = \begin{pmatrix} e_1(z) \\ e_2(z) \\ \vdots \\ e_n(z) \end{pmatrix} = C\begin{pmatrix} E_1(z, \frac{1}{2}) \\ E_2(z, \frac{1}{2}) \\ \vdots \\ E_n(z, \frac{1}{2}) \end{pmatrix}$$

with

$$C\Phi(\tfrac{1}{2})C^{-1} = \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & -1 & & \\ & & & & \ddots & \\ & & & & & -1 \end{pmatrix}.$$

We note that, if $\Phi(\frac{1}{2})$ represents the scattering matrix in the basis vector $\mathbf{E}(z, \frac{1}{2})$, then in the new basis $\mathbf{e}(z)$ the scattering matrix is represented by $C\Phi(\frac{1}{2})C^{-1}$. Moreover C is an orthogonal matrix. Then the functional equation for the Eisenstein series $\mathbf{E}(z, \frac{1}{2})$: $\mathbf{E}(z, \frac{1}{2}) = \Phi(\frac{1}{2})\,\mathbf{E}(z, \frac{1}{2})$ becomes

$$\begin{pmatrix} e_1(z) \\ e_2(z) \\ \vdots \\ e_n(z) \end{pmatrix} = C\Phi(\tfrac{1}{2})\, C^{-1} C\mathbf{E}(z, \tfrac{1}{2})$$

$$= \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & -1 & & \\ & & & & \ddots & \\ & & & & & -1 \end{pmatrix} \begin{pmatrix} e_1(z) \\ e_2(z) \\ \vdots \\ e_n(z) \end{pmatrix}$$

from which it follows that $e_j(z) = \pm e_j(z)$. Consequently, the only non-zero basis elements for the Eisenstein series at $s = \frac{1}{2}$ correspond to the $+1$ eigenvalues of $\Phi(\frac{1}{2})$. These are $(n + \operatorname{tr}(\Phi(1/2)))/2$ in number.

After the spectrum of B has been identified in terms of the singular set, standard perturbation theory can be applied to B when we consider variations of the metric g. The main interest is in quasiconformal deformations in Teichmüller space. The theory developed above, however, goes through in the case of admissible surfaces [MU] where we consider variations of the metric in a compact set. We consider a one-parameter family of metrics $g(\tau)$ varying real analytically in τ and $B(\tau)$ the associated B operators. The result in both cases can be stated as:

THEOREM 2.2 (Corollary 5.2 in [PS2]). *If $\lambda(0)$ is an eigenvalue of $B(0)$ of multiplicity 1 then $B(\tau)$ has eigenvalues $\lambda(\tau)$ and associated eigenfunctions f_τ varying real analytically in τ for small $|\tau|$. In the case of higher multiplicity of $\lambda(0)$, the eigenvalues decompose into a finite system of real analytic functions having at most algebraic singularities.*

The question of the spectral decomposition of $L^2(\Gamma\backslash\mathbb{H})$ is viewed quite differently by Faddeev [FA] (see also [LA, V]). The source of the functional equation and analytic continuation for the Eisenstein series lies in the resolvent identity

$$R(s) - R(s') = (s(1-s) - s'(1-s'))\, R(s')\, R(s), \tag{2.12}$$

where $R(s) = (-\Delta - s(1-s))^{-1}$. The fundamental point-pair invariant is

$$u(z, z') = \frac{|z-z'|^2}{4yy'} \tag{2.13}$$

for $z, z' \in \mathbb{H}$. We set

$$\varphi(u, s) = \frac{1}{4\pi} \int_0^1 [t(1-t)]^{s-1} (t+u)^{-s} \, dt \tag{2.14}$$

for $\sigma > 0$, $u > 0$ ($s = \sigma + it$), and $k(z, z'; s) = \varphi(u(z, z'), s)$ and then the kernel $k(z, z'; s)$ is the Green's function for the problem $\Delta h + s(1-s)h = f$ at least for $\sigma > 1$.

For a discrete subgroup Γ of $PSL(2, \mathbb{R})$ with $\Gamma \backslash \mathbb{H}$ non-compact of finite volume we set

$$r(z, z'; s) = \tfrac{1}{2} \sum_{\gamma \in \Gamma} \varphi(u(z, \gamma z'), s) \tag{2.15}$$

for $\sigma > 1$.

Let us denote the stabilizer of the j-cusp z_j by Γ_j. It is generated by a single parabolic element S_j and there exists a g_j with $g_j \infty = z_j$. One can choose $g_\alpha \in SL(2, \mathbb{R})$ so that $z \to g_\alpha z$ maps $C = \{z;\ -\tfrac{1}{2} \leqslant \operatorname{Re} z \leqslant \tfrac{1}{2}, \operatorname{Im} z \geqslant a\}$ one-to-one onto F_α. Each function f on F has $n+1$ components $f_0(z) = f(z)$ for $z \in F_0$ and $f_\alpha(z) = f(g_\alpha z)$ for $z \in C$. One has the decomposition

$$L^2(\Gamma \backslash \mathbb{H}) = L^2(F_0) \oplus \bigoplus_{\alpha=1}^{n} L^2(F_\alpha) \tag{2.16}$$

but it turns out that certain Banach spaces $\mathscr{B}_\mu$ play an even more important role in Faddeev's approach. The space $\mathscr{B}_\mu$ consists of complex valued functions $f(z)$ whose components $f_0(z)$ and $f_\alpha(z)$, $\alpha = 1, \ldots, n$ are continuous on F_0 and C, respectively, with

$$|f_\alpha(z)| \leqslant c y^\mu \tag{2.17}$$

for $z \in C$ with the μ-norm:

$$\|f\|_\mu = \max_{z \in F_0} |f_0(z)| + \sum_{\alpha=1}^{n} \max_{z \in C} \frac{|f_\alpha(z)|}{y^\mu}. \tag{2.18}$$

Since the Laplace operator is a negative operator the resolvent $R(z) = (-\Delta - z)^{-1}$ is defined on $\mathbb{C} \backslash [0, \infty)$. However, one can use meromorphic continuation to attach a meaning to the resolvent on a Riemann surface which is a two sheeted covering of the z-plane. Instead of the natural variable z, one introduces $z = s(1-s)$ and then the z-plane cut along the ray $[0, \infty)$ corresponds to the right half plane $\operatorname{Re} s > \tfrac{1}{2}$ cut along $\tfrac{1}{2} \leqslant s \leqslant 1$. The analytic continuation of the resolvent kernel can have poles only at the following set of points:

- at s_0, if $s_0(1-s_0)$ is an L^2-eigenvalue and $\operatorname{Re} s_0 \geqslant \frac{1}{2}$ but $s_0 \neq \frac{1}{2}$, and the pole is simple and

$$r(z, z'; s) = \frac{1}{s_0(1-s_0) - s(1-s)} \sum_{i=1}^{m} \psi_i(z)\, \psi_i(z') + r_1(z, z'; s), \quad (2.19)$$

where ψ_i $(i = 1, ..., m)$ is an orthonormal system of eigenfunctions (chosen to be real) with eigenvalue $s_0(1-s_0)$ and $r_1(z, z'; s)$ is regular close to s_0 (see [LA, p. 333]);

- at $s_0 = \frac{1}{2}$, possibly;
- at points s_0 with $\operatorname{Re} s_0 < \frac{1}{2}$. These points are called resonances and the Eisenstein series and the scattering matrix can have poles at those points only for $\operatorname{Re} s < \frac{1}{2}$ (see [LA, pp. 338–340]).

For s, $1-s$ non-singular, we have

$$r(z, z'; s) - r(z, z'; 1-s) = \frac{1}{2s-1} \sum_{\beta=1}^{n} E_\beta(z, s)\, E_\beta(z', 1-s) \quad (2.20)$$

a proof of which is given for instance in [LA, p. 344].

After obtaining the analytic continuation of the resolvent kernel one defines $R(s)$ as follows: Fix $\mu \leqslant \frac{1}{2}$. Then: $R(s)\colon \mathscr{B}_\mu \to \mathscr{B}_{1-\mu}$ is defined for $\operatorname{Re} s > \mu$ as the integral operator with kernel $r(z, z'; s)$.

3. Cusp Forms with Eigenvalue $\frac{1}{4}$

In this section we prove the following theorem:

Theorem 1. *For a generic (hyperbolic) Riemann surface with cusps with signature $(g, 0, m)$ where $g + m \geqslant 3$ there are no cusp forms with eigenvalue $\lambda = \frac{1}{4}$.*

Recall the numbering of the eigenvalues: $0 = \lambda_0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots$. The proof of the theorem uses the work of Buser [BU] about the construction of compact Riemann surfaces of genus g with $2g-3$ small eigenvalues, and Randol's observation [RD] that if those $2g-3$ eigenvalues are sufficiently small then $\lambda_{2g-2} > \frac{1}{4}$.

Before going into the proof, let us review the argument in [BU, RD]. We dissect the surface M into $2g-2$ 3-holed spheres by a collection of $3g-3$ simple closed geodesics in M having minimal aggregate length. The lowest Dirichlet eigenvalue of each 3-holed sphere can be made as small as one wants by shrinking the boundary curves. Minimax implies then that λ_{2g-3}

can be made as small as wanted. Together with this, through, the Cheeger constant of a 3-holed sphere T becomes >1 and so the first non-zero Neumann eigenvalue $\mu_1(T)$ satisfies the inequality $\mu_1(T)>\frac{1}{4}$. Minimax implies that $\lambda_{2g-2}>\frac{1}{4}$.

Instead of using the Laplace–Beltrami operator on the closure of $H^1(X)$ in $L^2(X)$, which has continuous spectrum $[\frac{1}{4}, \infty)$, we use the pseudo-Laplacian and minimax for it on the appropriate Hilbert space. We recall the construction of the pseudo-Laplacian [CDV, LP1].

Let M be a hyperbolic surface and $H^1(M)$ the first Sobolev space. Let $\mathscr{H}$ be a closed subspace of $H^1(M)$. We denote $\bar{\mathscr{H}}$ the closure of $\mathscr{H}$ in $L^2(M)$. For any such subspace $\mathscr{H}$ one can define a self-adjoint positive operator $\Delta_{\mathscr{H}}$ on $\bar{\mathscr{H}}$. It is the operator associated with the Friedrichs extension to the restriction of the quadratic form

$$q(f)=\int_M |\nabla f|^2 \tag{3.1}$$

to $\mathscr{H}$.

In this case the space $\mathscr{H}$ is parametrized by $a\in\mathbb{R}_+$ and we denote by $\mathscr{H}_a$ and Δ_a the closed subspace and the extension, respectively. Let

$$f_j^0(y)=\int_0^{2\pi} f_j(y, x)\,dx \tag{3.2}$$

be the zero Fourier coefficient of f in the j-cusp. We set $\mathscr{H}_a=\{f\in H^1(M)$: $f_j^0 \mid (a, \infty)=0, j=1, ..., m\}$.

The advantage of Δ_a is that they have compact resolvent and consequently purely discrete spectrum. Any cusp form for Δ_∞ is eigenfunction of Δ_a for all a with the same eigenvalue. The rest of the eigenvalues form a sequence

$$0<\mu_0(a)\leqslant\mu_1(a)\leqslant\cdots \tag{3.3}$$

with $\mu_j(a)=s_j(1-s_j)$ where $\operatorname{Im} s_j\geqslant 0$, $\operatorname{Re} s_j\geqslant\frac{1}{2}$, and, if $s_j\neq\frac{1}{2}$, $a^{s_j}+\varphi(s_j)a^{1-s_j}=0$ or, $s_j=\frac{1}{2}$, $\varphi(\frac{1}{2})=-1$, $\varphi'(\frac{1}{2})=-2\log a$ (assuming one cusp). We note that the Δ_a do not see the resonances at the bottom of the continuous spectrum; i.e., if they exist, they do not create an eigenvalue of Δ_a and if they do not exist, Δ_a may have eigenvalue $\frac{1}{4}$ if $\varphi'(\frac{1}{2})=-2\log a$.

Now we consider what happens for finite volume hyperbolic surfaces X of genus g with m cusps. In this case there is a decomposition into $2g-2+m$ 3-holed spheres and m cusps, i.e., pieces isometric to $C=S^1\times(b, \infty)$ with the metric $(dx^2+dy^2)/y^2$.

We will prove that in the dissection of the surface above, if we shrink the boundary geodesics as much as necessary, then Δ_a will have no eigenvalue at $\lambda = \frac{1}{4}$. We notice that

$$\mathscr{H}_a = \bigoplus_{i=1}^{2g-2+m} H^1(T_i) \oplus \bigoplus_{k=1}^{m} (H^1(C_k) \cap \{f : f_k^0 \mid (a, \infty) = 0\}), \quad (3.4)$$

where the T_i's are 3-holed spheres and the C_i's cusps. For the T_i's the argument is the same as in [RD]. Their first non-zero Neumann eigenvalue for Δ is greater that $\frac{1}{4}$, i.e.,

$$\int_{T_i} |\nabla f|^2 > \frac{1}{4} \int_{T_i} f^2 \quad (3.5)$$

for $\int_{T_i} f = 0$, if we shrink the boundary geodesics sufficiently. To apply minimax, i.e., domain monotonicity of eigenvalues with vanishing Neumann data (see [CH, pp. 17–18]) one should study the spectrum of Δ_a on the cusps C_k. The argument is more or less the same as in [CDV]. First consider the Laplace operator acting on $L^2(S^1 \times (a, \infty), dx^2 + dy^2)$ functions with zero Fourier coefficient 0. Note that here we are using the euclidean metric on the cylinder C. In this case

$$f^0(y) = \int_{S^1} f(x, y)\, dx. \quad (3.6)$$

We impose the Neumann condition on the boundary $S^1 \times \{a\}$. The eigenfunctions are

$$\varphi_{\xi, m}(x, y) = \cos \xi\pi(y - a) e^{2\pi i m x}, \qquad m \in \mathbb{Z} - \{0\}, \quad \xi \in \mathbb{R} \quad (3.7)$$

with eigenvalues

$$\lambda_{\xi, m} = \xi^2\pi^2 + 4\pi^2 m^2 \geqslant 4\pi^2 \quad (3.8)$$

since $m \neq 0$. Now consider the same cylinder but with the hyperbolic metric $(dx^2 + dy^2)/y^2$ and the Laplace operator acting on functions with zero Fourier coefficient 0. We use minimax on the space of such functions. We have

$$\frac{\int_C |\nabla f|^2\, dx\, dy}{\int_C |f|^2\, (dx\, dy/y^2)} \geqslant a^2 \frac{\int_C |\nabla f|^2\, dx\, dy}{\int_C |f|^2\, dx\, dy} \quad (3.9)$$

for all f with $f^0 \mid (a, \infty) = 0$, $\int_C |\nabla f|^2\, dx\, dy < \infty$, $\int_C |f|^2\, dx\, dy < \infty$, and then the second quotient is $\geqslant 4\pi^2$. Since such a set of functions is dense in the space of functions with

$$f^0 \mid (a, \infty) = 0, \qquad \int_C |f|^2 \frac{dx\, dy}{y^2} < \infty, \qquad \int_C |\nabla f|^2 < \infty \quad (3.10)$$

the first Neumann eigenvalue for the cusp and for the problem with zero Fourier coefficient identically 0 is greater than $\frac{1}{4}$, provided a is sufficiently large.

Consequently, if we number all the eigenvalues of the pseudo-Laplacian as $\lambda_0 \leqslant \lambda_1 \leqslant \lambda_2 \leqslant \cdots$, we see that minimax implies that $\lambda_{2g-2+m} > \frac{1}{4}$. (Remark: $\lambda_0 > 0$ actually but as $a \to \infty$, $\lambda_0(a) \to 0$.)

However, $\lambda_{2g-3+m} < \varepsilon$ in the above setting, as we get by using domain monotonicity of eigenvalues with vanishing Dirichlet data for the whole surface on one hand and the $2g-2+m$ 3-holed spheres on the other (see [DPRS]). We remark that domain monotonicity of eigenvalues with vanishing Dirichlet data does not require an exhaustion of the surface by the subdomains, in contrast with the case of vanishing Neumann data (see [CH, pp. 17–18]).

Now we prove that generically in the Teichmüller space of Riemann surfaces of signature $(g, 0, m)$ the number $\frac{1}{4}$ is not an eigenvalue. We prove that locally the complement of this set is contained in a proper real analytic subvariety of Teichmüller space. Take any surface $\Gamma_0 \backslash \mathbb{H}$ with $\frac{1}{4}$ in its point spectrum. We can join it with a curve of finite length in the Teichmüller space $T(\Gamma_0)$ with the surface $\Gamma_1 \backslash \mathbb{H}$ constructed above, which does not have $\frac{1}{4}$ in its point spectrum. By compactness we get a finite chain of open sets $U_1, ..., U_k$ covering the curve and $\Gamma_0 \in U_k$, $\Gamma_1 \in U_1$, $U_i \cap U_{i-1} \neq \varnothing$ for $i = 2, ..., k$ and such that we can apply perturbation theory for pseudo-Laplacians Δ_a^Γ, $a \geqslant a(j)$ as described in [PS1] in each U_j (we need this because Lemma 2.2 in [PS1] works in small neighborhoods of Teichmüller space). For each $a \geqslant a(j)$ the set

$$V_j^a = \{\Gamma \in U_j; \tfrac{1}{4} \in \operatorname{spec}(\Delta_a^\Gamma)\}$$

is a real analytic subvariety of U_j: standard perturbation theory for an analytic family of self-adjoint operators with compact resolvents allows us to reduce the eigenvalue problem of Δ_a^Γ to that of an analytic family $\tilde{\Delta}_a^\Gamma$ acting on a finite-dimensional space (see [RS, p. 22]). Then

$$V_j^a = \{\Gamma; \det(\tilde{\Delta}_a^\Gamma - \tfrac{1}{4}) = 0\}$$

which is obviously a real analytic subvariety, since we now work with matrices. If there is an $a \geqslant a(j)$ with V_j^a a proper subvariety, then on U_j the Laplace operator does not have $\frac{1}{4}$ in its point spectrum generically. Otherwise $\frac{1}{4} \in \operatorname{spec}(\Delta_a^\Gamma)$ for all $\Gamma \in U_j$ for each $a \geqslant a(j)$. In this case, since the condition $\varphi'(\frac{1}{2}) = -2 \log a$ cannot hold for more than one a, each $\Gamma \in U_j$ has a cusp form with eigenvalue $\frac{1}{4}$, i.e., the $\frac{1}{4}$-eigenspace of Δ_a^Γ does not contain only truncated Eisenstein series. If $V_k^a = U_k$ for all $a \geqslant a(k)$, then $\frac{1}{4}$ is a cusp form eigenvalue on $U_{k-1} \cap U_k$, so $V_{k-1}^a = U_{k-1}$ for all $a \geqslant a(k-1)$

and so on. But V_1^a is a proper subvariety for some a, which is a contradiction. So V_k^a is proper for some $a \geqslant a(k)$. This completes the proof of the theorem.

4. The singular Part of the Resolvent at the Bottom of the Continuous Spectrum

In this section we prove Theorem 2. The proof proceeds as follows:

Proposition 4.1. *Let*

$$R(s)\colon \mathscr{B}_0 \to \mathscr{B}_{3/4} \tag{4.2}$$

be the operator defined by the kernel $r(z, z'; s)$, *for* s *near* $\frac{1}{2}$. *Then* $s \to R(s)$ *has a pole of order at most* 2 *at* $\frac{1}{2}$.

Proof. The proof is basically the same as Lemma 4, pp. 334–335 in [LA]. The idea is that

$$\lim_{s \to 1/2} \frac{(s-1/2)^m}{s(1-s)-1/4} = 0 \tag{4.3}$$

if $m > 2$. For simplicity let us assume we have one cusp. Let $\{s_n\}$ be a sequence of non-singular points with $\operatorname{Re} s_n > \frac{1}{2}$, converging to $\frac{1}{2}$ and such that $\operatorname{Re}(s_n(1-s_n)-\frac{1}{4})=0$. For example one can choose $y_n \to 0$ and $x_n = (1+2\,|y_n|)/2$, where $y_n \neq 0$ and $s_n = x_n + iy_n$ is non-singular. Let

$$R_{-m} = \lim_{s \to 1/2} (s-\tfrac{1}{2})^m R(s), \tag{4.4}$$

where m is a positive integer. We have to show that $R_{-m}=0$ if $m>2$. Let $f \in \mathscr{B}_0$. It suffices to prove that $R_{-m}f=0$. Suppose $R_{-m}f \neq 0$. Let

$$g_n = (s_n - \tfrac{1}{2})^m R(s_n) f. \tag{4.5}$$

Then $g_n \to R_{-m}f$ in $\mathscr{B}_{3/4}$. Since $R_{-m}f(z)$ is not identically 0 in the cusp, we have

$$|R_{-m}f(z)\,y^{-3/4}| \geqslant c_1 > 0 \tag{4.6}$$

for some constant c_1 and all y in some open set. We also have

$$|[g_n(z) - R_{-m}f(z)]\,y^{-3/4}| < \varepsilon \tag{4.7}$$

uniformly for y is some open set, for all n sufficiently large, by the definition of the norm in $\mathscr{B}_{3/4}$. Consequently, for large n we get the inequality

$$|g_n(z)\, y^{-3/4}| \geqslant c_2 > 0 \tag{4.8}$$

and for some new constant c_3 and all y in some open set,

$$|g_n(z)| \geqslant c_3 > 0. \tag{4.9}$$

Since $\mathscr{B}_0 \subset H = L^2(\Gamma\backslash\mathbb{H})$ and $\operatorname{Re} s_n > \frac{1}{2}$ it follows that $R(s_n) f \in H$ and so $g_n \in H$. The inequality obtained above shows that for a new positive constant c_4 and for all sufficiently large n we have

$$\| g_n \|_2^2 \geqslant c_4. \tag{4.10}$$

The resolvent inequality in the Hilbert space H asserts that

$$|R(s)_H| \ll \frac{1}{d(\lambda_s, \text{spectrum of } \Delta)}, \tag{4.11}$$

where d stands for the distance function in the complex plane and $\lambda_s = s(1-s)$. In our case the distance of $s_n(1-s_n)$ from the spectrum is $|s_n(1-s_n) - \frac{1}{4}|$, since $\operatorname{Re} s_n(1-s_n) = \frac{1}{4}$. So we have

$$\| g_n \|_2 \leqslant |s_n - \tfrac{1}{2}|^m \, |R(s_n)_H| \, \| f \|_2 \ll \frac{|s_n - \frac{1}{2}|^m \, \| f \|_2}{|s_n(1-s_n) - \frac{1}{4}|} \tag{4.12}$$

which is a contradiction to $\| g_n \|_2^2 \geqslant c_4$, if $m > 2$. This completes the proof.

We can now prove that the kernel $r(z, z'; s)$ has a pole of order at most 2 at $\frac{1}{2}$. From Lemma 2, pp. 331–332 in [LA] it has a pole of finite order at $s = \frac{1}{2}$. Suppose it has a pole of order $l \geqslant 3$. Let C be a small circle around $\frac{1}{2}$ which does not enclose other singular points and let $f \in \mathscr{B}_0$. Then by the previous proposition we have

$$\int_C \int_{\Gamma\backslash\mathbb{H}} (s - \tfrac{1}{2})^{l-1} \, r(z, z'; s) \, f(z') \, dz' \, ds = 0. \tag{4.13}$$

Using the estimate in Lemma 3, p. 332 in [LA] we can interchange the order of integration and, since f can be any smooth function with compact support in $\Gamma\backslash\mathbb{H}$, we deduce that

$$\int_C (s - \tfrac{1}{2})^{l-1} \, r(z, z'; s) \, ds = 0 \tag{4.14}$$

for almost all pairs (z, z') and consequently for all (z, z') with $z \neq z'$ and z, z' not on the boundaries of the sets F_i on which we have divided the fundamental domain of Γ in $\mathbb{H}$. This is never possible, since the integral has to be the leading coefficient of the singularity. Hence the hypothesis $l \geqslant 3$ is false and l was originally $\leqslant 2$.

We can now find the singular part of $r(z, z'; s)$ at $s = \frac{1}{2}$. Let

$$r(z, z'; s) = \frac{A_{-2}}{(s-\frac{1}{2})^2} + \frac{A_{-1}}{s-\frac{1}{2}} + r^+(z, z'; s), \tag{4.15}$$

where $r^+(z, z'; s)$ is regular near $s = \frac{1}{2}$.

(1) We first find A_{-1}. We have

$$r(z, z'; 1-s) = \frac{A_{-2}}{(s-\frac{1}{2})^2} + \frac{A_{-1}}{1-s-\frac{1}{2}} + r^+(z, z'; 1-s) \tag{4.16}$$

and we notice that $r^+(z, z'; 1-s)$ is regular at $s = \frac{1}{2}$. Equations (4.15), (4.16), (2.20) imply that

$$\frac{A_{-1}}{s-\frac{1}{2}} - \frac{A_{-1}}{\frac{1}{2}-s} + r^+(z, z', s) - r^+(z, z', 1-s) = \frac{1}{2(s-\frac{1}{2})} \sum_{i=1}^{n} E_i(z, s)\, E_i(z', 1-s)$$

so

$$\frac{2A_{-1}}{s-\frac{1}{2}} + r^+(z, z', s) - r^+(z, z'; 1-s) = \frac{1}{s-\frac{1}{2}} \frac{1}{2} \sum_{i=1}^{n} E_i(z, s)\, E_i(z', 1-s). \tag{4.17}$$

The residue of the right-hand side is

$$\frac{1}{2\pi i} \int_C \frac{1}{s-\frac{1}{2}} \frac{1}{2} \sum_{i=1}^{n} E_i(z, s)\, E_i(z', 1-s)\, ds \tag{4.18}$$

and, since the pole is of order at most 1, we get

$$2A_{-1} = \lim_{s \to 1/2} \sum_{i=1}^{n} \tfrac{1}{2} E_i(z, s)\, E_i(z', 1-s) = \tfrac{1}{2} \sum_{i=1}^{n} E_i(z, \tfrac{1}{2})\, E_i(z', \tfrac{1}{2})$$

$$A_{-1} = \tfrac{1}{4} \sum_{i=1}^{n} E_i(z, \tfrac{1}{2})\, E_i(z', \tfrac{1}{2}). \tag{4.19}$$

We see that this term is 0 iff all the Eisenstein series at $s = \frac{1}{2}$ vanish. It defines an operator of finite rank, but it is not a bounded operator in L^2, unless all $E_i(z, \frac{1}{2})$ vanish. The rank of the operator is the number of linearly independent (non-vanishing) Eisenstein series at $s = \frac{1}{2}$.

(2) It is a general fact that if $R(z)=(A-z)^{-1}$ is the resolvent of a self-adjoint operator A then

$$\lim_{z\to\mu}(z-\mu)\,R(z)=-P_{\{\mu\}} \tag{4.20}$$

for $\operatorname{Im} z\neq 0$ where $P_{\{\mu\}}$ is the L^2-projection to the μ-eigenspace of A, even when μ is an embedded eigenvalue. To apply this in our case we recall that $R(s)=-(\Delta+s(1-s))^{-1}$ so we change variables and instead of expanding around $s=\frac{1}{2}$, we expand around $\mu=\frac{1}{4}$. We have $s(1-s)-\frac{1}{4}=-(s-\frac{1}{2})^2$

$$\begin{aligned}R(s)&=(-\Delta-s(1-s))^{-1}=\frac{A_{-2}}{(s-\frac{1}{2})^2}+\frac{A_{-1}}{s-\frac{1}{2}}+R^+(s)\\&=-\frac{A_{-2}}{\mu-\frac{1}{4}}+\frac{A_{-1}}{\sqrt{\frac{1}{4}-\mu}}+R^+(\mu),\end{aligned} \tag{4.21}$$

where the last term involves non-negative powers of $\sqrt{\frac{1}{4}-\mu}$. From this it is obvious that A_{-2} is the L^2 projection to the eigenspace of cusp forms with eigenvalue $\frac{1}{4}$ and if $\vartheta_i(z)$, $i=1, ..., k$, form an orthonormal basis for this eigenspace, all taken to be real valued, we have that the kernel of the resolvent has the following expansion:

$$r(z,z';s)=\frac{\sum_{i=1}^{k}\vartheta_i(z)\,\vartheta_i(z')}{(s-\frac{1}{2})^2}+\frac{\sum_{i=1}^{n}E_i(z,\frac{1}{2})\,E_i(z',\frac{1}{2})}{4(s-\frac{1}{2})}+r^+(z,z';s). \tag{4.22}$$

5. First Variation for Resonances at the Bottom of the Continuous Spectrum

In this section we compute the first variation of the 0 eigenvalue for the cut-off wave operator B. Here $\lambda=s-\frac{1}{2}$. We assume that there are no cusp forms with eigenvalue $\frac{1}{4}$ but there exists a single nullvector (resonance) $E(z,\frac{1}{2})$. This way $\lambda=0$ is an eigenvalue of multiplicity 1 for B with eigenvector

$$P\begin{pmatrix}E(z,\frac{1}{2})\\0\end{pmatrix}=\begin{pmatrix}E(z,\frac{1}{2})\\0\end{pmatrix}, \tag{5.1}$$

where P is the projection onto $\mathscr{K}=\mathscr{H}_G\ominus(\mathscr{D}_+\oplus\mathscr{D}_-)$. We subject our metric to a variation inside the compact set F_0. The first variation is given by

$$\dot\lambda(0)=\operatorname{Tr}(\dot B Q) \tag{5.2}$$

(formula 2.33, p. 90 in [K]), where $\dot{B}$ is the infinitesimal variation of the family of operators B_τ, depending on the real parameter $\tau \in (-\varepsilon, \varepsilon)$ and

$$Q = -\frac{1}{2\pi i} \int_C R_B(z)\, dz$$

and the contour C encloses only 0 among the eigenvalues of B. Let P be the (E- and G-) orthogonal projection of $\mathscr{H}_G$ onto $\mathscr{K}$, which is given in particular by: $Pf = f$ except for the zero Fourier coefficient in the cusp which is changed as follows,

$$(Pf)^0 = \left\{ f_1^{(0)}(a) \left(\frac{y}{a}\right)^{1/2}, 0 \right\}, \tag{5.3}$$

for $y > a$. Let ω be a constant such that the semigroup $Z(t)$ satisfies:

$$\|Z(t)\|_G \leqslant c \cdot e^{\omega t}. \tag{5.4}$$

The relation $R_B(\lambda) = PR_A(\lambda)P$ holds for $\operatorname{Re} \lambda > \omega$ and also for $\operatorname{Re} \lambda > 0$, as one can see by analytic continuation. It must be remarked that here the resolvents are always taken in the space $\mathscr{H}_G$ or $\mathscr{K}$.

We are going to determine the action of $R_A(\lambda)$ and $\int_C R_B(\lambda)\, d\lambda$ on pairs $\binom{f_1}{f_2}$ which are smooth and supported in F_0, the compact part of the surface. Then

$$\begin{aligned} R_A(\lambda) &= (A - \lambda)^{-1} = \begin{pmatrix} -\lambda & 1 \\ L & -\lambda \end{pmatrix}^{-1} \\ &= \begin{pmatrix} \lambda(L-\lambda^2)^{-1} & (L-\lambda^2)^{-1} \\ L(L-\lambda^2)^{-1} & \lambda(L-\lambda^2)^{-1} \end{pmatrix} = \begin{pmatrix} \lambda R_L(\lambda^2) & R_L(\lambda^2) \\ LR_L(\lambda^2) & \lambda R_L(\lambda^2) \end{pmatrix} \end{aligned} \tag{5.5}$$

and

$$R_L(\lambda^2) = (L - \lambda^2)^{-1} = (\Delta + \tfrac{1}{4} - \lambda^2)^{-1} = (\Delta + s(1-s))^{-1} = -R(s) \tag{5.6}$$

with $s = \lambda + \frac{1}{2}$.

Formula (5.5) makes sense as long as $R_L(\lambda^2)$ makes sense and we consider it as $R_L(\lambda)\colon L^2(\Gamma\backslash\mathbb{H}) \to L^2(\Gamma\backslash\mathbb{H})$. For instance, the above calculation cannot be applied to find $R_A(\lambda)\binom{E(z,1/2)}{0}$, because $E(z, \frac{1}{2}) \notin L^2$ in our case.

The operator $R(s)$ has a pole of order 2 at $\frac{1}{2}$ in general but, if there are no cusp forms with eigenvalue $\frac{1}{4}$, it has pole of order 1 and its kernel has singular part

$$\frac{1}{4} \frac{E(z, \frac{1}{2})\, E(z', \frac{1}{2})}{s - \frac{1}{2}} \tag{5.7}$$

(see Theorem 2) so $R_L(\lambda^2)$ has singular part at 0 represented by the kernel:

$$-\frac{1}{4}\frac{E(z,\frac{1}{2})\,E(z',\frac{1}{2})}{\lambda}. \tag{5.8}$$

Let Γ_r be the half circle $\gamma_r(t)=re^{it}$ for $t\in[-\pi/2,\pi/2]$. Then we have

$$\frac{1}{2\pi i}\int_C R_B(\lambda)\,d\lambda=\lim_{r\to 0}\frac{2}{2\pi i}\int_{\Gamma_r} R_B(\lambda)\,d\lambda=\operatorname{res}_0 R_B(\lambda). \tag{5.9}$$

So

$$\begin{aligned}-\frac{1}{2\pi i}\int_C R_B(\lambda)\,d\lambda&=-\lim_{r\to 0}\frac{2}{2\pi i}\int_{\Gamma_r} R_B(\lambda)\,d\lambda=-\lim_{r\to 0}\frac{2}{2\pi i}P\int_{\Gamma_r} R_A(\lambda)\,d\lambda\,P\\&=-\lim_{r\to 0}\frac{2}{2\pi i}P\int_{\Gamma_r}\begin{pmatrix}\lambda R_L(\lambda^2) & R_L(\lambda^2)\\ I+\lambda^2R_L(\lambda^2) & \lambda R_L(\lambda^2)\end{pmatrix}d\lambda\,P\end{aligned} \tag{5.10}$$

since $LR_L(\lambda^2)=I+\lambda^2R_L(\lambda^2)$. Consequently

$$-\lim_{r\to 0}\frac{2}{2\pi i}\int_{\Gamma_r}\begin{pmatrix}\lambda R_L(\lambda^2) & R_L(\lambda^2)\\ I+\lambda^2R_L(\lambda^2) & \lambda R_L(\lambda^2)\end{pmatrix}d\lambda \tag{5.11}$$

ends up having kernel

$$\begin{pmatrix}0 & \frac{1}{4}E(z,\frac{1}{2})\,E(z',\frac{1}{2})\\ 0 & 0\end{pmatrix}. \tag{5.12}$$

This kernel is the kernel of Q for smooth pairs $\binom{f_1}{f_2}$ supported in the compact part of the surface F_0 or, more generally, for smooth pairs in $\mathscr{H}_G$ with zero Fourier coefficient vanishing in the cusp. We note that P leaves those pairs untouched, since their zero Fourier coefficient is 0 in the cusp. For such a pair integration with the kernel given above gives a certain multiple of the null vector $\binom{E(z,1/2)}{0}$ and this is left untouched by P because it has the right kind of zero Fourier coefficient.

We are now in a position to calculate $\operatorname{Tr}(\dot{B}Q)$. The operator Q obviously maps $\mathscr{H}$ onto the space generated by the null vector $\binom{E(z,1/2)}{0}$, the only eigenvector of B at 0, and

$$\dot{B}=\begin{pmatrix}0 & 0\\ \dot{L} & 0\end{pmatrix} \tag{5.13}$$

in the domain F_0 and it is the 0 operator in the cusp. We have

$$\dot{B}\begin{pmatrix}E(z,\frac{1}{2})\\ 0\end{pmatrix}=\begin{pmatrix}0 & 0\\ \dot{L} & 0\end{pmatrix}\begin{pmatrix}E(z,\frac{1}{2})\\ 0\end{pmatrix}=\begin{pmatrix}0\\ \dot{L}(0)\,E(z,\frac{1}{2})\end{pmatrix} \tag{5.14}$$

and the last pair is supported in the compact part of the surface and is smooth. So everything in $\mathscr{K}$ is mapped by $\dot{B}Q$ onto a multiple of:

$$\begin{pmatrix} 0 \\ \dot{L}(0)\, E(z, \tfrac{1}{2}) \end{pmatrix}. \tag{5.15}$$

So, in order to find the trace, we can pick a basis containing $\left(\begin{smallmatrix} 0 \\ \dot{L}(0)\, E(z, 1/2) \end{smallmatrix}\right)$ and we need to find where it is mapped by $\dot{B}Q$. It is going to be a certain multiple of itself and the multiplication coefficient is going to be the trace,

$$\begin{aligned} Q\begin{pmatrix} 0 \\ \dot{L}(0)\, E(z, \tfrac{1}{2}) \end{pmatrix} &= \int_{\Gamma\backslash\mathbb{H}} \begin{pmatrix} 0 & \tfrac{1}{4}E(z, \tfrac{1}{2})\, E(z', \tfrac{1}{2}) \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ \dot{L}(0)\, E(z', \tfrac{1}{2}) \end{pmatrix} dz' \\ &= \begin{pmatrix} \tfrac{1}{4}E(z, \tfrac{1}{2}) \int_{\Gamma\backslash\mathbb{H}} E(z', \tfrac{1}{2})(\dot{L}(0)\, E(z', \tfrac{1}{2}))\, dz' \\ 0 \end{pmatrix} \end{aligned} \tag{5.16}$$

so we get

$$\operatorname{Tr}(\dot{B}Q) = \tfrac{1}{4} \int_{\Gamma\backslash\mathbb{H}} E(z', \tfrac{1}{2})(\dot{L}(0) E(z', \tfrac{1}{2}))\, dz'. \tag{5.17}$$

So we have proved:

THEOREM 3. *The first variation of a resonance $e(z)$ at the bottom of the continuous spectrum $\frac{1}{4}$ is given by the formula*

$$\dot{\lambda} = \tfrac{1}{4} \int_{\Gamma\backslash\mathbb{H}} e(z)(\dot{\Delta} e)(z)\, dz \tag{5.18}$$

provided there are no cusp forms with eigenvalue $\frac{1}{4}$.

Remark. One can consider also the following case: there exists a resonance $e(z)$ at $\frac{1}{4}$ and there are k linearly independent cusp forms with eigenvalue $\frac{1}{4}$. We can assume an orthonormal basis for them is given by: $\vartheta_1(z), ..., \vartheta_k(z)$, taken to be real valued. Since there are degeneracies, i.e., the 0 eigenspace of B contains the data $\left(\begin{smallmatrix} e(z) \\ 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} \vartheta_1 \\ 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 \\ \vartheta_1 \end{smallmatrix}\right), ..., \left(\begin{smallmatrix} \vartheta_k \\ 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 \\ \vartheta_k \end{smallmatrix}\right)$, we define the weighted mean of the eigenvalues (p. 86 in [K]) to be

$$\hat{\lambda}(\tau) = \frac{1}{2k+1} \operatorname{Tr}(B(\tau)\, Q(\tau)). \tag{5.19}$$

The weighted mean varies in a real analytic way and its first variation $\hat{\lambda}^{(1)}$ is given by

$$\hat{\lambda}^{(1)} = \frac{1}{2k+1} \operatorname{Tr}(\dot{B}Q) \tag{5.20}$$

[K, p. 90]. Following the same argument as before, we find that Q has kernel represented by the matrix

$$\begin{pmatrix} \sum_{i=1}^{k} \vartheta_i(z)\,\vartheta_i(z') & \frac{1}{4}e(z)\,e(z') \\ 0 & \sum_{i=1}^{k} \vartheta_i(z)\,\vartheta_i(z') \end{pmatrix} \tag{5.21}$$

at least for compactly supported data. We have

$$\dot{B}\begin{pmatrix} e \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \dot{L}e \end{pmatrix}, \qquad \dot{B}\begin{pmatrix} \vartheta_i \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \dot{L}\vartheta_i \end{pmatrix}, \qquad \dot{B}\begin{pmatrix} 0 \\ \vartheta_i \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{5.22}$$

Everything is mapped by $\dot{B}Q$ in a linear combination of $\binom{0}{\dot{L}e}$, $\binom{0}{\dot{L}\vartheta_i}$ and

$$\dot{B}Q\begin{pmatrix} 0 \\ \dot{L}e \end{pmatrix} = \dot{B}\begin{pmatrix} \frac{1}{4}(\int e\dot{L}e)e \\ \sum_{i=1}^{k} (\int \vartheta_i \dot{L}e)\vartheta_i \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{1}{4}(\int e\dot{L}e)\,\dot{L}e \end{pmatrix} \tag{5.23}$$

$$\dot{B}Q\begin{pmatrix} 0 \\ \dot{L}\vartheta_j \end{pmatrix} = \dot{B}\begin{pmatrix} \frac{1}{4}(\int e\dot{L}\vartheta_j)e \\ \sum_{i=1}^{k} (\int \vartheta_i \dot{L}\vartheta_j)\vartheta_i \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{1}{4}(\int e\dot{L}\vartheta_j)\,\dot{L}e \end{pmatrix}. \tag{5.24}$$

We can choose a basis containing $\binom{0}{\dot{L}e}$ and $\binom{0}{\dot{L}\vartheta_j}$, $j = 1, ..., k$ and then

$$\hat{\lambda}^{(1)} = \frac{1}{2k+1} \cdot \frac{1}{4} \int_{\Gamma\backslash\mathbb{H}} e(z)(\dot{L}e)(z)\,dz. \tag{5.25}$$

We note that this number depends only on the resonance. Since the calculation goes through, even if there is no resonance, i.e., $e(z) \equiv 0$ but there exist cusp forms at $\frac{1}{4}$, we get that in this case the weighted mean of the eigenvalues has first variation 0.

6. Applications of the Variation Formula

Consequences of Theorem 3 above are drawn for the case of quasiconformal deformations (i.e., in Teichmüller space) and for deformations in the space of admissible surfaces [MU].

For the second case we note that the Lax–Phillips theory is valid in this setting as noticed by Müller [MU] and that Faddeev's method works too. Consequently, the fundamental identity (2.20) between Eisenstein series and the analytic continuation of the resolvent is still valid and we get the same singular part at the point $s = \frac{1}{2}$. Consider now a family of metrics $g_\varepsilon = e^{\varepsilon f} g$ in the conformal class of g where f is compactly supported. Using Theorem 7 in [CDV] we can assume there are no cusp forms with eigenvalue $\frac{1}{4}$ for a generic such f. Then

$$\Delta_\varepsilon = e^{-\varepsilon f} \Delta_g \tag{6.1}$$

and

$$\dot{\Delta}_\varepsilon = -fe^{-\varepsilon f}\Delta_g = -f\Delta_\varepsilon$$
$$\dot{\Delta} = -f\Delta_g. \tag{6.2}$$

In this case the variation of a resonance at the bottom of the continuous spectrum is

$$\dot{\lambda} = \tfrac{1}{16}\int_F (e(z))^2 f(z)\,dz \tag{6.3}$$

(as follows from (5.18)) and from this formula it follows that $\dot{\lambda}$ can be made different from 0 for almost all functions f which are compactly supported; i.e.:

THEOREM 4. *By a generic perturbation (for a dense G_δ set of functions in $C_0^\infty(K)$ for K compact and fixed) one can destroy a nullvector, and, consequently, in the space of admissible surfaces resonances at the bottom of the continuous spectrum do not exist generically. The multiplicity of $s=\frac{1}{2}$ is generically zero in this setting.*

For the case of hyperbolic surfaces we work as in [PS1]. Lemma 2.2 in [PS1] allows us to reduce a quasiconformal deformation given by a family of metrics $\hat{g}(\varepsilon)$ into one given by a family of metrics $\varphi_\varepsilon^*(\hat{g}(\varepsilon))$ with the properties (here $\varphi_\varepsilon\colon M\to M$ are diffeomorphisms):

(i) $\varphi_\varepsilon^*(\hat{g}(\varepsilon)) = \hat{g}(0)$ for $y>a$,

(ii) $\varphi_\varepsilon^*(\hat{g}(\varepsilon)) = \hat{g}(\varepsilon)$ for $y<a/2$,

(iii) $\varphi_0 =$ identity, $|\partial\varphi_\varepsilon/\partial\varepsilon| = O(1)$ uniformly in the collar $a/2<y<a$.

So the variation is in fact a variation in a compact set. Then we note that Lemma 2.3 in [PS1] is still valid, although the proof changes as follows: In [PS1] one studies a cusp form u with eigenvalue $\frac{1}{4}+r^2$. The first variation of the Laplace operators Δ_ε corresponding to a quasiconformal deformation is denoted by $\hat{L}$ and the first variation of the Laplacians $\tilde{\Delta}_\varepsilon$ corresponding to the metrics $\tilde{g}(\varepsilon) = \varphi_\varepsilon^*(\hat{g}(\varepsilon))$ is denoted by $\tilde{L}$. In our case u is not a cusp form but an Eisenstein series so it does not decrease rapidly in the cusps. However, $\hat{L}u$ decreases quickly in the cusps, since the coefficients of the differential operator $\hat{L}$ involve ϕ, ψ with $\phi+i\psi=Q$, a holomorphic cusp form of weight 4. It is well known that such cusp forms decrease rapidly. The operator $\tilde{L}u-\hat{L}u$ also decreases exponentially in the cusps and the lemma is verified. Now the calculation on pages 355–356 in [PS1] is applied and, as a consequence, we get the following theorem:

THEOREM 5. *The variation of a resonance at the bottom of the continuous spectrum* $\dot{\lambda}(0)$ *is* $\neq 0$ *iff* $\operatorname{Re} F(\frac{1}{2}) \neq 0$ *where*

$$F(s) = \frac{(4\pi)^{-s}}{32\pi} \frac{\Gamma(s+\frac{3}{2})^2}{\Gamma(s)} L\left(s+\frac{3}{2}\right) \tag{6.4}$$

and $L(s)$ *is the Rankin–Selberg convolution of* Q*, the holomorphic cusp form of weight* 4 *giving the direction in the deformation space, and* $E(z, \frac{1}{2})$ *the resonance.*

7. DEGENERATE CASE FOR RESONANCES AT THE BOTTOM OF THE CONTINUOUS SPECTRUM

In this section we prove the following theorem:

THEOREM 6. *If there are n nullvectors* $e_1, e_2, ..., e_n$ *and* $\frac{1}{4}$ *is not an* L^2 *eigenvalue then the degeneracy is removed if the following matrix has distinct eigenvalues*:

$$((e_i, \dot{\Delta} e_j)_{L^2})_{i,j=1,...,n}.$$

If none of the eigenvalues is 0 *then under the perturbation all nullvectors are destroyed. In case the variation is in Teichmüller space the matrix has entries which are Rankin–Selberg convolutions of the* e_i*'s and the holomorphic cusp form* Q *of weight* 4 (*expanded at the various cusps*) *evaluated at the middle of their critical lines.*

Proof. For simplicity we assume the degeneracy comes out of only two nullvectors. The analysis in [PS2] shows that $\frac{1}{2}$ is in this case a semisimple eigenvalue of $B+\frac{1}{2}$ (see also Sect. 1). Even though $\frac{1}{2}$ may be an exceptional point (in fact a branch point), which would imply that, under perturbation of the metric, the eigenvalue 0 of B produces branches of an analytic function, given by Puiseux series for $|\varepsilon|$ small, Theorem 2.3 (p. 93) in [K] implies, because of the semisimplicity, that the eigenvalues are in fact continuously differentiable at $\varepsilon = 0$. They are of the form (see [K])

$$0 + \varepsilon\lambda_1^{(1)} + o(\varepsilon) \tag{7.1}$$

$$0 + \varepsilon\lambda_2^{(1)} + o(\varepsilon), \tag{7.2}$$

where $\lambda_1^{(1)}, \lambda_2^{(1)}$ are the eigenvalues of

$$\tilde{B}^{(1)} = Q\dot{B}Q. \tag{7.3}$$

The operator $Q = Q(0)$ is the projection to the space of the nullvectors. If the eigenvalues are distinct, there is no further splitting. In case we can solve the eigenvalue problem for $\tilde{B}^{(1)}$ on the image of Q (i.e., the vector space spanned by the two nullvectors) then we remove the degeneracy of the 0 eigenvalue for the operator B. Let us assume that $e_1(z, \frac{1}{2}) = e_1(z)$, $e_2(z, \frac{1}{2}) = e_2(z)$ are two nullvectors, which are linearly independent and span the space of Eisenstein series at $s = \frac{1}{2}$. We try to solve

$$Q\dot{B}Qw = \lambda w, \tag{7.4}$$

where

$$w = \alpha \begin{pmatrix} e_1(z) \\ 0 \end{pmatrix} + \beta \begin{pmatrix} e_2(z) \\ 0 \end{pmatrix}. \tag{7.5}$$

The operator Q leaves $\binom{e_1(z)}{0}$ and $\binom{e_2(z)}{0}$ unaffected, so

$$\alpha Q\dot{B} \begin{pmatrix} e_1(z) \\ 0 \end{pmatrix} + \beta Q\dot{B} \begin{pmatrix} e_2(z) \\ 0 \end{pmatrix} = \lambda \left(\alpha \begin{pmatrix} e_1(z) \\ 0 \end{pmatrix} + \beta \begin{pmatrix} e_2(z) \\ 0 \end{pmatrix} \right). \tag{7.6}$$

Since

$$\dot{B} = \begin{pmatrix} 0 & 0 \\ \dot{L} & 0 \end{pmatrix}$$

we have

$$\dot{B} \begin{pmatrix} e_i(z) \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \dot{L}e_i(z) \end{pmatrix} \tag{7.7}$$

for $i = 1, 2$. As in Section 5, one sees that on compactly supported data Q acts with kernel

$$\begin{pmatrix} 0 & \frac{1}{4}(e_1(z)\, e_1(z') + e_2(z)\, e_2(z')) \\ 0 & 0 \end{pmatrix}. \tag{7.8}$$

Since $e_1(z)$ and $e_2(z)$ are linearly independent, Eq. (7.6) implies:

$$\alpha \left(\tfrac{1}{4} \int_{\Gamma \backslash \mathbb{H}} e_1(z')\, \dot{L}e_1(z')\, dz' - \lambda \right) + \beta \tfrac{1}{4} \int_{\Gamma \backslash \mathbb{H}} e_1(z')\, \dot{L}e_2(z')\, dz' = 0 \tag{7.9}$$

$$\alpha \tfrac{1}{4} \int_{\Gamma \backslash \mathbb{H}} e_2(z')\, \dot{L}e_1(z')\, dz' + \beta \left(\tfrac{1}{4} \int_{\Gamma \backslash \mathbb{H}} e_2(z')\, \dot{L}e_2(z')\, dz' - \lambda \right) = 0. \tag{7.10}$$

This is a homogeneous system in α, β and we are looking for non-trivial solutions. So the determinant has to be 0, i.e.,

$$\begin{vmatrix} \frac{1}{4} \int_{\Gamma \backslash \mathbb{H}} e_1(z')\, \dot{L}e_1(z')\, dz' - \lambda & \frac{1}{4} \int_{\Gamma \backslash \mathbb{H}} e_1(z')\, \dot{L}e_2(z')\, dz' \\ \frac{1}{4} \int_{\Gamma \backslash \mathbb{H}} e_2(z')\, \dot{L}e_1(z')\, dz' & \frac{1}{4} \int_{\Gamma \backslash \mathbb{H}} e_2(z')\, \dot{L}e_2(z')\, dz' - \lambda \end{vmatrix} = 0. \tag{7.11}$$

Since $\dot{L}$ is self-adjoint

$$\int_{\Gamma\backslash\mathbb{H}} e_1(z')\,\dot{L}e_2(z')\,dz' = \int_{\Gamma\backslash\mathbb{H}} e_2(z')\,\dot{L}e_1(z')\,dz'. \tag{7.12}$$

One removes the degeneracy if the two solutions of the quadratic equation (7.11) are distinct. The two solutions are not distinct only in the case that the discriminant D is 0,

$$\begin{aligned} 16D = {} & \left(\int_{\Gamma\backslash\mathbb{H}} e_1(z')\,\dot{L}e_1(z')\,dz' + \int_{\Gamma\backslash\mathbb{H}} e_2(z')\,\dot{L}e_2(z')\,dz'\right)^2 \\ & -4\left(\int_{\Gamma\backslash\mathbb{H}} e_1(z')\,\dot{L}e_1(z')\,dz' \cdot \int_{\Gamma\backslash\mathbb{H}} e_2(z')\,\dot{L}e_2(z')\,dz'\right. \\ & \left. -\left(\int_{\Gamma\backslash\mathbb{H}} e_1(z')\,\dot{L}e_2(z')\,dz'\right)^2\right) \\ = {} & \left(\int_{\Gamma\backslash\mathbb{H}} e_1(z')\,\dot{L}e_1(z') - \int_{\Gamma\backslash\mathbb{H}} e_2(z')\,\dot{L}e_2(z')\,dz'\right)^2 \\ & +4\left(\int_{\Gamma\backslash\mathbb{H}} e_1(z')\,\dot{L}e_2(z')\,dz'\right)^2. \end{aligned}$$

So the discriminant is 0 iff

$$\begin{aligned} & \int_{\Gamma\backslash\mathbb{H}} e_1(z')\,\dot{L}e_1(z')\,dz' = \int_{\Gamma\backslash\mathbb{H}} e_2(z')\,\dot{L}e_2(z')\,dz' \\ & \int_{\Gamma\backslash\mathbb{H}} e_1(z')\,\dot{L}e_2(z')\,dz' = 0. \end{aligned} \tag{7.13}$$

If either is not satisfied, there are two distinct eigenvalues of $\tilde{B}^{(1)}$ on the image of Q and if none is 0 the two nullvectors are moved off the point 0, i.e., they cease to be nullvectors under perturbation. We have

$$\lambda \neq 0 \Leftrightarrow$$

$$\int_{\Gamma\backslash\mathbb{H}} e_1(z')\,\dot{L}e_1(z')\,dz' \cdot \int_{\Gamma\backslash\mathbb{H}} e_2(z')\,\dot{L}e_2(z')\,dz' \neq \left(\int_{\Gamma\backslash\mathbb{H}} e_1(z')\,\dot{L}e_2(z')\,dz'\right)^2. \tag{7.14}$$

The conditions (7.13) are exactly the same for the degenerate case of energy levels of the Schrödinger equation (see, for instance, [SCH, p. 249]).

8. Applications of Number Theory: Nullvectors for Congruence Subgroups

In this section p is a prime number.

Lemma 8.1. *No principal congruence subgroup of $SL(2, \mathbb{R})$ has one or two nullvectors.*

Proof. Following Huxley [HU1], we have that the number of cusps for $\Gamma(N)$, $N \geqslant 3$ is $K = \frac{1}{2} N^2 \prod_{p \mid N} (1 - 1/p^2)$ and the trace of the scattering operator at $s = \frac{1}{2}$ is $-K_0$ where $K_0 = N \prod_{p \mid N} (1 + 1/p)$. The number of nullvectors is $(K - K_0)/2$.

If $K - K_0 = 2$ or 4 and $N = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$, where $p_1, ..., p_k$ are primes and $a_1, ..., a_k$ positive integers, then

$$p_1^{a_1 - 1} \cdots p_k^{a_k - 1} (p_1 + 1) \cdots (p_k + 1) = 1 \quad \text{or} \quad 2 \quad \text{or} \quad 4 \tag{8.1}$$

and this can happen only for $N = 3$. But then $K = K_0 = 4$.

Lemma 8.2. *The only congruence group of the form $\Gamma^0(N)$ with only one nullvector is $\Gamma^0(25)$ and the only ones with two nullvectors are: $\Gamma^0(49)$, $\Gamma^0(256)$, $\Gamma^0(81)$, and all $\Gamma^0(25p)$, where p is a prime different from 5.*

Proof. We follow Huxley's notation again. For $\Gamma^0(N)$, $K = f_0(N)$, where $f_0(N)$ is a multiplicative function with

$$\begin{aligned} f_0(p^{2a+1}) &= 2p^a \\ f_0(p^{2a}) &= p^a + p^{a-1} \end{aligned} \tag{8.2}$$

and $K_0 = f_2(N)$ is a multiplicative function given by

$$f_2(p^a) = 2a \qquad \text{for} \quad p \neq 2 \tag{8.3}$$

$$f_2(2^a) = \begin{cases} a + 1 & \text{for} \quad a = 1, 2, 3; \\ 2a - 2 & \text{for} \quad a = 4, 5; \\ 4a - 12 & \text{for} \quad a \geqslant 6. \end{cases} \tag{8.4}$$

We start by examining powers of primes p^a. An elementary argument shows that $p = 5$, $a = 2$ give one nullvector and $p = 3$ and $a = 4$ give two. Also $p = 2$ and $a = 8$ give two nullvectors and no other prime powers give one or two nullvectors (for details see [P]). So $\Gamma^0(25)$ has one nullvector and $\Gamma^0(49)$, $\Gamma^0(256)$, and $\Gamma^0(81)$ have two nullvectors each. Now we examine the case of composite levels. We assume that m, n are both different from 1 and are relatively prime. Since f_0 and f_2 are multiplicative functions the number of nullvectors for $\Gamma^0(mn)$ is

$$\frac{f_0(mn)-f_2(mn)}{2}=\frac{f_0(m)-f_2(m)}{2}f_0(n)+\frac{f_0(n)-f_2(n)}{2}f_2(m). \quad (8.5)$$

We investigate when this number can be 1 or 2. First we note that $f_0(k)=1$ only if $k=1$, since f_0 is multiplicative and is not 1 on prime powers by formula (8.2). Similarly a case by case study of (8.3), (8.4) shows that $f_2(k)=1$ only if $k=1$.

In order that the left-hand side of (8.5) is 1, one of the summands on the right-hand side has to be 1 and the other 0. If $(f_0(m)-f_2(m))/2 \neq 0$, then $f_0(n)$ has to be 1, i.e., $n=1$. If $(f_0(m)-f_2(m))/2=0$, then $f_2(m)=1$, i.e., $m=1$. So we cannot have only one nullvector if N has distinct prime factors.

At last we examine how composite levels can give two nullvectors. If the left-hand side of (8.5) is 2, the right-hand side of (8.5) is $0+2$ or $1+1$ or $2+0$. The case $1+1$ is impossible, since it implies that $f_0(m)=f_2(n)=1 \Rightarrow m=n=1$.

In the case $0+2$, $\Gamma^0(m)$ has no nullvectors and $f_2(m)((f_0(n)-f_2(n))/2)=2$. If $m\neq 1$, as assumed, $f_2(m)=2$ and $\Gamma^0(n)$ has 1 nullvector, i.e., $n=5^2$. In this case m has to be prime, as follows from (8.3), (8.4) and is different from 5.

The last case is the case $2+0$. Then $\Gamma^0(n)$ has no nullvectors and $((f_0(m)-f_2(m))/2)\, f_0(n)=2$. Since $n\neq 1$, $f_0(n)\neq 1$, so it has to be 2 and there is only one nullvector in $\Gamma^0(m)$, so $m=5^2$. All $\Gamma^0(p)$ with p prime have $f_0(p)=2$, i.e., have two cusps. But this is not true for higher powers of primes and, consequently, no composite levels have two cusps, as follows from formula (8.2). We find again that $n=p$, a prime different from 5.

This completes the proof the lemma.

It follows from [HU1] that, if $N \mid m^2$, with $m\leqslant 18$, then $\Gamma^0(N)$ carries no exceptional eigenvalues.

The examples we want to consider are the following: The surface $\Gamma^0(25)\backslash\mathbb{H}$ is a surface of genus 0 with 6 cusps and two ramified points of order 2. The surface $\Gamma^0(25)\backslash\mathbb{H}$ has one nullvector. There is no cusp form with eigenvalue $\frac{1}{4}$. The deformation space is five dimensional. The destruction of the nullvector is associated with the non-vanishing of a certain Rankin–Selberg convolution. This is discussed in detail later in this section.

The surface $\Gamma^0(49)\backslash\mathbb{H}$ is an elliptic curve with eight punctures and two ramified points of order 3. Its Teichmüller space is ten dimensional. It has two nullvectors and $\frac{1}{4}$ is not an eigenvalue. We come back to this surface in Section 9.

Also $\Gamma^0(81)\backslash\mathbb{H}$ is a surface of genus 4 with 12 cusps and no ramification points. It has 2 nullvectors and $\frac{1}{4}$ is not an eigenvalue. Its deformation space is 21 dimensional. The surface $\Gamma^0(54)\backslash\mathbb{H}$ has also genus 4 and 12 cusps and no ramification points. Consequently, the two groups $\Gamma^0(81)$ and $\Gamma^0(54)$ have the same deformation space, since they have the same signature. However, on $\Gamma^0(54)$ there are no nullvectors and $\frac{1}{4}$ is not an eigenvalue, since $54 \nmid 18^2$. This is the first example of a Teichmüller space where the multiplicity at $\frac{1}{2}$ is not only generically 0, but also there exists a lower dimensional subset where it is not 0. This proves the corollary:

COROLLARY 7. *The scattering matrix at $\frac{1}{2}$ is not a continuous function on the Teichmüller space of the surface $\Gamma^0(81)\backslash\mathbb{H}$.*

In the rest of this section we investigate closer the example of $\Gamma^0(25)$. In particular we want to determine the nullvector, i.e., the Eisenstein series that does not vanish at $s=\frac{1}{2}$. We follow [HU1] again. If $B_\chi^\chi(z/m, s)$ are the Eisenstein series with characters, where χ is a proper character modulo q, $q \mid m$, $mq \mid 25$, and

$$B_\chi^\chi(z, s) = \sum_{\substack{c=-\infty \\ (c,d)\neq(0,0)}}^{\infty} \sum_{d=-\infty}^{\infty} \frac{\chi(c)\,\chi(d)\,y^s}{|cz+d|^{2s}} \tag{8.6}$$

then we have the functional equation

$$B_\chi^\chi(z, s) = \pi^{2s-1}\frac{(-s)!}{(s-1)!}\frac{\tau(\chi)}{\tau(\bar\chi)}q^{1-2s}B_{\bar\chi}^{\bar\chi}(z, 1-s). \tag{8.7}$$

We also denote

$$E_\chi^\chi(z, s) = \sum_{(c,d)=1} \frac{\chi(c)\,\chi(d)\,y^s}{|cz+d|^{2s}} \tag{8.8}$$

and then we have

$$E_\chi^\chi = B_\chi^\chi(z, s)/L(2s, \chi^2). \tag{8.9}$$

For $\Gamma^0(25)$, χ can be the trivial character, in which case $q=1$ and m can be 1 or 5 or 25, or q is 5, $m=5$, and χ is a proper character mod 5. The multiplicative group mod 5 is generated by 2 and there are three proper characters mod 5, denoted here χ_1, $\bar\chi_1$, and χ which is the quadratic character mod 5. We have $\chi_1(2)=i$, $\bar\chi_1(2)=-i$, and $\chi(2)=-1$ and $\chi_1^2=\chi$. For ψ the trivial character or the quadratic character mod 5, $L(2s, \psi^2)=\zeta(2s)$. The functions $\zeta(2s)$ and $\zeta(2-2s)$ both have a pole at $s=\frac{1}{2}$, while, for χ quadratic mod 5, the functions $L(2s, \chi)$ and $L(2-2s, \chi)$ do not. It is easy to see that

$$\lim_{s\to 1/2}\frac{\zeta(2-2s)}{\zeta(2s)} = -1. \tag{8.10}$$

Then the functional equation for $E_{\psi}^{\psi}(z, s)$ for ψ trivial or quadratic mod 5 implies that those Eisenstein series vanish at $s=\frac{1}{2}$. However, for χ quadratic mod 5:

$$\lim_{s \to 1/2} \frac{L(2-2s, \chi)}{L(2s, \chi)} = \frac{L(1, \chi)}{L(1, \chi)} = 1. \tag{8.11}$$

Then the functional equation implies that:

$$E_{\chi_1}^{\chi_1}\left(\frac{z}{5}, \frac{1}{2}\right) = \frac{\tau(\bar{\chi}_1)}{\tau(\chi_1)} E_{\chi_1}^{\bar{\chi}_1}\left(\frac{z}{5}, \frac{1}{2}\right). \tag{8.12}$$

The last statement shows that $E_{\chi_1}^{\chi_1}(z/5, 1/2)$ and $E_{\chi_1}^{\bar{\chi}_1}(z/5, 1/2)$ are linearly dependent and the space of Eisenstein series at $s=\frac{1}{2}$ is spanned by

$$E_{\chi_1}^{\chi_1}\left(\frac{z}{5}, \frac{1}{2}\right) = B_{\chi_1}^{\chi_1}\left(\frac{z}{5}, \frac{1}{2}\right) \Big/ L(1, \chi). \tag{8.13}$$

The nullvector has Fourier expansion at ∞

$$\frac{4\pi}{5\sqrt{5}} \tau(\chi_1) \sum_{n=1}^{\infty} \sum_{ck=n} \chi_1(c)\, \overline{\chi_1(k)} \cos\left(\frac{2\pi n x}{25}\right) \sqrt{y}\, K_0\left(\frac{2\pi n y}{25}\right) \tag{8.14}$$

as follows from [HU1].

Remark. In a similar fashion one can determine the two nullvectors for $\Gamma^0(49)$ and $\Gamma^0(81)$. We omit the proof but we describe the two nullvectors for $\Gamma^0(49)$,

$$E_{\chi}^{\chi}\left(\frac{z}{7}, \frac{1}{2}\right) = B_{\chi}^{\chi}\left(\frac{z}{7}, \frac{1}{2}\right) \Big/ L(1, \chi^2)$$

and

$$E_{\chi^2}^{\chi^2}\left(\frac{z}{7}, \frac{1}{2}\right) = B_{\chi^2}^{\chi^2}\left(\frac{z}{7}, \frac{1}{2}\right) \Big/ L(1, \bar{\chi}^2),$$

where χ is primitive mod 7 with $\chi(3) = e^{\pi i/3}$.

We now consider the L-function associated with the first variation of the nullvector $E_{\chi_1}^{\chi_1}(z/5, 1/2)$. The first variation of the nullvector $E_{\chi_1}^{\chi_1}(z/5, 1/2)$ is $(\dot{\Delta} E_{\chi_1}^{\chi_1}(z/5, 1/2), E_{\chi_1}^{\chi_1}(z/5, 1/2))_{L^2}$ where $\dot{\Delta} = -4 \operatorname{Re}(Q(z)\, y^2(\partial/\partial \bar{z})(y^2(\partial/\partial \bar{z})))$ (see [PS1]). Among the Eisenstein series $E_i(z, s)$, indexed by the cusps, one of them at least is non-vanishing at $s = 1/2$. This one, say $E_k(z, s)$, has to be a non-zero multiple of the resonance $E_{\chi_1}^{\chi_1}(z/5, 1/2)$ when evaluated at $s = 1/2$. Then the non-vanishing of the first variation of the resonance is equivalent to the non-vanishing of $(\dot{\Delta} E_{\chi_1}^{\chi_1}(z/5, 1/2), E_k(z, s))_{L^2}$ at $s = 1/2$. This inner product can be unfolded to give the Rankin–Selberg convolution of $Q(z)$ and $E_{\chi_1}^{\chi_1}(z/5, 1/2)$, both expanded at the k cusp. So we need to find which of the six Eisenstein series (indexed by the cusps) does not vanish at $s=1/2$. Following Huxley we number the cusps as -1, -5, $-5/2$, $-5/3$, $-5/4$, -25 and we set $E_{-1}(z,s)$, $E_{-5}(z,s)$, $E_{-5/2}(z,s)$, $E_{-5/3}(z,s)$, $E_{-5/4}(z,s)$, $E_{-25}(z,s)$ as the corresponding Eisenstein series. Then the formula (valid for all admissible characters)

$$E_{\chi}^{\chi}\left(\frac{z}{m}, s\right) = \sum_{m=gh} \chi(g)\left(\frac{h}{g}\right)^{s} \sum_{k \mid N/h} \chi(k) \sum \chi(e)\left(\frac{N}{(h^2k^2, N)}\right)^{s} E_{-hk/e}(z,s) \tag{8.15}$$

(see [HU1]) where the third sum extends over e mod$(hk, N/hk)$, for $s=1/2$ gives a system of six equations with six unknowns. We recall that

$$E_1^1\left(z, \frac{1}{2}\right) = E_1^1\left(\frac{z}{5}, \frac{1}{2}\right) = E_1^1\left(\frac{z}{25}, \frac{1}{2}\right) = E_{\chi_1^2}^{\chi_1^2}\left(\frac{z}{5}, \frac{1}{2}\right) = 0. \tag{8.16}$$

To solve the system one needs also (8.12) and the fact that $\tau(\chi_1)/\tau(\bar{\chi}_1) \neq 1$, which follows from the fact that $\tau(\chi_1) = -2\sin(6\pi/5) - 2i\sin(2\pi/5)$ is not purely imaginary and as a result $\tau(\bar{\chi}_1) = \chi_1(-1)\overline{\tau(\chi_1)} = \overline{-\tau(\chi_1)} \neq \tau(\chi_1)$. Then we get that $E_{-5/2}(z, 1/2)$ and $E_{-5/3}(z, 1/2)$ do not vanish and are multiples of $E_{\chi_1}^{\chi_1}(z/5, 1/5)$. So we have to study

$$\left(\dot{\Delta} E_{\chi_1}^{\chi_1}\left(\frac{z}{5}, \frac{1}{2}\right), E_{-5/3}(z,s)\right)_{L^2} \tag{8.17}$$

at $s=1/2$. As remarked earlier, it is the Rankin-Selberg convolution of the Dirichlet series of a holomorphic cusp form Q of weight 4 and the L-series of the resonance $E_{\chi_1}^{\chi_1}(z/5, 1/2)$. We aim at writing this convolution as an explicit Euler product, if we assume that Q is an eigenfunction of the Hecke operators $T_4(p)$, $(p, 5)=1$. Under this assumption, if

$$Q(z) = \sum_{n=1}^{\infty} a(n) e^{2\pi i n z} \tag{8.18}$$

at $-5/2$, then

$$\begin{aligned} L(s, Q) &= \prod_p (1 - a(p)\, p^{-s} + p^{3-2s})^{-1} \\ &= \prod_p (1 - \alpha_1(p)\, p^{-s})^{-1} (1 - \alpha_2(p)\, p^{-s})^{-1}. \end{aligned} \tag{8.19}$$

The second expression we get by factoring the Hecke polynomials. By adjusting the coefficient in front of $E_{\chi_1}^{\chi_1}(z/5, 1/2)$, we can assume the null-vector has a series at ∞ of the form

$$\sum_{n \neq 0}^{\infty} b(n)\, y^{1/2} e^{2\pi i n x/25} K_0\left(\frac{2\pi n y}{25}\right) \tag{8.20}$$

with $b(1)=1$ and $b(n)=\sum_{ck=n}\chi_1(c)\,\bar{\chi}_1(k)$. Then the coefficients of the nullvector are multiplicative: $b(mn)=b(m)\,b(n)$ for $(m,n)=1$. Moreover they satisfy

$$b(p^{r+2})=b(p^{r+1})\,b(p)-b(p^r) \tag{8.21}$$

which can be proved easily using the fact that χ_1 is a character. Essentially the above argument shows that the normalized nullvector is an eigenfunction for the Hecke operators $T(p)$, $(p,5)=1$. The space of functions of moderate growth splits into subspaces which are eigenspaces simultaneously of Δ and the Hecke operators, since Δ commutes with them. In this case the eigenspace for 1/4 is one dimensional, so a nullvector is also eigenvector for the Hecke algebra. We need the Fourier coefficients of the nullvector at the cusp $-5/2$. Since $E^{\chi_1}_{\bar{\chi}_1}(z/5, 1/2)$ is a Hecke eigenform, it has the same coefficients $b(n)$ for $(5,n)=1$ at all cusps.

The Dirichlet series for the nullvector has an Euler product of the form (excluding the prime 5):

$$\prod_{p\neq 5}(1-b(p)\,p^{-s}+p^{-2s})^{-1}. \tag{8.22}$$

It is easy to verify that

$$b(p)=\begin{cases}0 & \text{if } \ p\equiv 2,3 \bmod 5\\ -2 & \text{if } \ p\equiv 4 \bmod 5\\ 2 & \text{if } \ p\equiv 1 \bmod 5.\end{cases} \tag{8.23}$$

The Hecke polynomials can be factored as follows: For $p\equiv 4 \bmod 5$ we have $1+2p^{-s}+p^{-2s}=(1+p^{-s})^2$. For $p\equiv 1 \bmod 5$ we have $1-2p^{-s}+p^{-2s}=(1-p^{-s})^2$. For $p\equiv 2,3 \bmod 5$ we have $1+p^{-2s}=(1-ip^{-s})(1+ip^{-s})$. So the Dirichlet series for the nullvector has the Euler product (excluding the prime 5):

$$\prod_{p\neq 5}(1-\chi_1(p)\,p^{-s})^{-1}\,(1-\bar{\chi}_1(p)\,p^{-s})^{-1}. \tag{8.24}$$

Now it is well known how to get the Euler product for the Rankin–Selberg convolution (see [SH]). Ignoring the prime 5 and the term $\prod_p(1-p^{3-2s})$ we get

$$\prod_{p\neq 5}[(1-\chi_1(p)\,\alpha_1(p)\,p^{-s})(1-\bar{\chi}_1(p)\,\alpha_1(p)\,p^{-s})$$
$$\times(1-\chi_1(p)\alpha_2(p)\,p^{-s})(1-\bar{\chi}_1(p)\,\alpha_2(p)\,p^{-s})]^{-1} \tag{8.25}$$

which gives

$$\prod_{p \neq 5} [(1 - \chi_1(p)\, a(p) p^{-s} + \chi_1(p)^2\, p^{3-2s}) \times (1 - \bar{\chi}_1(p)\, a(p)\, p^{-s} + \bar{\chi}_1(p)^2\, p^{3-2s})]^{-1}. \tag{8.26}$$

We define, as usual, twisted L-series by

$$L(s, Q, \chi_1) = \sum_{n=1}^{\infty} \frac{\chi_1(n)\, a(n)}{n^s} = \prod_p \sum_{r=0}^{\infty} \frac{\chi_1(p^r)\, a(p^r)}{p^{rs}} \tag{8.27}$$

and similarly we define $L(s, Q, \bar{\chi}_1)$. We see that

$$\sum_{r=0}^{\infty} \chi_1(p^r)\, a(p^r)\, X^r = (1 - \chi_1(p)\, a(p)\, X + \chi_1(p)^2\, p^3 X^2)^{-1} \tag{8.28}$$

since

$$\sum_{r=0}^{\infty} a(p^r)\, X^r = (1 - a(p) X + p^3 X^2)^{-1} \tag{8.29}$$

which is the formal expression for the coefficients of a Hecke eigenform. So, if we exclude the prime 5, the Rankin–Selberg convolution we are interested in is essentially the product:

$$\prod_{p \neq 5} (1 - p^{3-2s})\, L(s, Q, \chi_1)\, L(s, Q, \bar{\chi}_1). \tag{8.30}$$

We have seen that the nullvector is unstable and the generic multiplicity of the point $\frac{1}{2}$ is 0 on the Teichmüller space of $\Gamma^0(25)$ if the Rankin–Selberg convolution discussed above is non-zero at the middle of its critical line, i.e., the point 2. After the calculation above we see that we have to consider whether the two twisted L-series above vanish at 2, which is again the middle of their critical line.

9. Deformations by Characters

We review the setting (see [S]). The first homology group of $\Gamma \backslash \mathbb{H}$ is isomorphic to $\Gamma/[\Gamma, \Gamma]$. Its dual group consists of the unitary characters χ of Γ and $A_{\text{cusp}} = \{\chi \mid \chi(p) = 1, p \in \Gamma, p \text{ parabolic}\}$. The cohomology classes in the first de Rham cohomology which can be represented by forms of compact support have a square integrable harmonic representative (which can be taken to be cuspidal; i.e., if $w = w_0\, dy + w_1\, dx$, $\int_C w_0 = \int_C w_1 = 0$ for C a path corresponding to a parabolic). Fix $z_0 \in \Gamma \backslash \mathbb{H}$ and

$\pi: \Gamma \to \Gamma/[\Gamma, \Gamma]$ the natural projection from $\pi_1(\Gamma\backslash\mathbb{H}) \to H_1(\Gamma\backslash\mathbb{H}, \mathbb{R})$. For any cuspidal harmonic square integrable form w we set

$$\chi_w(\gamma) = e^{2\pi i \int_{\pi(\gamma)} w} \tag{9.1}$$

which is a cuspidal character in the connected component of the trivial character in A_{cusp}.

The deformation depends on a real parameter θ and the corresponding spectral problem concerns L^2-functions satisfying:

$$f(\gamma z) = \chi_{\theta w}(\gamma) f(z). \tag{9.2}$$

The resolvent kernel for this problem has been continued analytically by Venkov [V]. The Lax–Phillips scattering theory has been extended to include this problem in [PS2]. So the theory developed here can be extended too. The first variation of a nullvector e is given by

$$\int_{\Gamma\backslash\mathbb{H}} \dot{L}e \cdot e \frac{dx\, dy}{y^2} \tag{9.3}$$

and here $\dot{L} = \langle de, w \rangle$. This condition can be expressed in terms of L-functions as follows: We unfold the integral and perform an integration by parts to get (apart from constants and Gamma factors) $L(\frac{1}{2})$ where

$$L(s) = \sum_{n \neq 0} \frac{\varrho(n)\, b(n)}{n^{s+1}} \tag{9.4}$$

is the Rankin–Selberg convolution of

$$e(z) = \sum_n \varrho(n)\, y^{1/2} K_0(2\pi\, |n|\, y) e^{2\pi i n x} \tag{9.5}$$

and

$$\omega_0(z) = \sum_{n \neq 0} b(n)\, y^{1/2} K_0(2\pi\, |n|\, y) e^{2\pi i n x}. \tag{9.6}$$

If $\Gamma\backslash\mathbb{H}$ has genus $g \geq 1$, so there exists cohomology, the L-series above becomes the L-series of a Rankin–Selberg convolution of a holomorphic cusp form of weight 2 for Γ and the nullvector.

The surface $\Gamma^0(49)\backslash\mathbb{H}$ has genus 1, so there exists a one-dimensional space of holomorphic differentials of the first kind, generated, say, by w. In fact w is a newform, since $\Gamma^0(7)\backslash\mathbb{H}$ and $\Gamma(1)\backslash\mathbb{H}$ are of genus 0 and have no cusp forms of weight 2. Since we have two nullvectors, one deals with the degenerate case. All the conditions for removing the degeneracy and killing the nullvectors can be expressed in terms of the special values of various twisted L-series of w at the middle of their critical line (up to trivial factors). The twists depend on the Fourier coefficients of the nullvectors, which are given by (non-quadratic) characters mod 7 (see the example of $\Gamma^0(25)$ in Section 8). These L-series will be factors of the Hasse–Weil zeta function of the elliptic curve over the field $F = \mathbb{Q}(e^{2\pi i/7})$, since $\mathrm{Gal}(\mathbb{Q}(e^{2\pi i/7}) = (\mathbb{Z}/7\mathbb{Z})^*$. The Birch Swinnerton–Dyer conjecture [BI] relates the non-vanishing of the Hasse–Weil zeta function at $s=1$ to g_F, which is the rank of the F-points of the elliptic curve. This rank is finite by the Mordell–Weil theorem and the conjecture is that the Hasse–Weil zeta function vanishes to order g_F at $s=1$.

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