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On Differences of Eigenvalues for Flat Tori and Hyperbolic Surfaces

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Abstract. We show that the normalized differences of eigenvalues of a generic flat torus are dense in the real line. We establish a connection between density of differences of point spectrum of certain hyperbolic surfaces of finite area and continued fractions.

1. Introduction

1.1. Let $s_0 \leq s_1 \leq \dots$ be a sequence of real numbers and let $N(\lambda)$ be the counting function for them: $N(\lambda)$ is the number of s_j's which are less than or equal to λ. If $N(\lambda)$ has asymptotics of the form:

$$N(\lambda) = c\lambda^a + o(\lambda^{a-1}),$$

where c is a constant, then it is easy to see that

$$N(\lambda+\varepsilon) - N(\lambda) \sim ca\varepsilon\lambda^{a-1} \tag{1.1}$$

If $a > 1$ and we fix ε and let λ get large, the right hand side of (1.1) becomes greater than 1 and we see that for $\lambda > b(\varepsilon)$ we can find a term s_i of the sequence in the interval $(\lambda, \lambda+\varepsilon]$ and one term s_j in the interval $(\lambda+\alpha, \lambda+\alpha+\varepsilon]$, where $\alpha > 0$. Then $\alpha - \varepsilon < s_j - s_i < \alpha + \varepsilon$. So the set $\{s_j - s_i : i, j \geq 0\}$ is dense in the real line.

In this work we are interested in eigenvalues of the Laplace operator and the s_j's will be the eigenvalues or a function of them. We recall that Duistermaat and Guillemin [**6**] proved that in the generic case the counting function $N(\lambda)$ for $\sqrt{\lambda_i}$ satisfies the asymptotic formula

$$N(\lambda) = \frac{\mathrm{vol(M)}}{n(2\pi)^n}\lambda^n + o(\lambda^{n-1}), \tag{1.2}$$

where λ_i are the eigenvalues of the Laplace operator on a Riemannian manifold X^n and M is the cosphere bundle. The generic case can be described as follows: for $x \in M$ let $\rho(x)$ be the return time for the trajectory of the geodesic flow through

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x. If $\{x : \rho(x) < \infty\}$ has measure zero in M then (1.2) holds. The observation that $\{\sqrt{\lambda_i} - \sqrt{\lambda_j}\}$ is dense iff at least one geodesic is not closed is due to Helton [**7**]. The argument does not work for the differences of the actual eigenvalues.

1.2. We consider the simplest case: eigenvalues of flat tori in $\mathbb{R}^m$. The differences $\lambda_i - \lambda_j$ define an indefinite quadratic form which is of the special form: $f(\vec{x}, \vec{y}) = Q(\vec{x}) - Q(\vec{y})$ where Q is a positive definite form in m variables (so f is a form in at least 4 variables). Such an indefinite form obviously represents 0 properly, i.e., for any $\vec{x} = \vec{y} \neq \vec{0}$, $Q(\vec{x}, \vec{y}) = 0$. Then a result of Oppenheim [**13**] shows that, if f is not a multiple of a rational form, then 0 is an accumulation point of its values on $\mathbb{Z}^{2m}$. With the help of another result of Oppenheim [**12**] one sees that 0 is an accumulation point of both the positive values and the negative values of f on integer vectors. Then the values of f on integer vectors are a dense set in the real line (see Lemma 2.1). In this case the differences of the eigenvalues for the torus are dense. This case occurs exactly when at least one of the quotients $(\vec{v}_i, \vec{v}_j)/(\vec{v}_{i'}, \vec{v}_{j'})$ is irrational, where $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m$ is a basis for the lattice. One should remark here that the issue of density of the values of indefinite quadratic forms, which are not proportional to forms with rational coefficients, in n variables, $n \geq 3$, was finally settled by Margulis [**11**].

Well-known conjectures in the physics literature concern the pair correlation function and level spacing distribution of eigenvalues of quantum systems [**2**]. In the completely integrable case it is conjectured that the eigenvalues follow a Poisson process, i.e., the $\mu_j = \lambda_{j+1} - \lambda_j$ have a Poisson distribution and are independent of each other, and the level spacing distribution is

$$P(S) = \varrho e^{-\varrho S},$$

where ϱ is the mean level density. In the chaotic case the statistics are conjectured to be the same as those of ensembles of real symmetric matrices whose elements are Gaussian distributed. The level spacing distribution is approximately

$$P(S) = \frac{1}{2}\pi\varrho^2 S \exp(-\frac{1}{4}\pi\varrho^2 S^2).$$

Both the pair correlation function and the level spacing distribution are invariants giving the distribution of differences of eigenvalues, normalized so that the mean level spacing is one. In both cases (completely integrable and chaotic) the conjectures for these spectral invariants imply that the *normalized* differences are dense in $\mathbb{R}$. Our purpose is to verify this conclusion for generic flat tori. For an introduction to the mathematical problems associated with quantum chaos the reader can consult [**14**].

We now explain how to normalize the differences of the eigenvalues. If we have a good bound for the error term in Weyl's Law of the form $o(\lambda^{\frac{n}{2}-1})$, then asymptotically in an interval of length 1 we expect $c\frac{n}{2}\lambda^{\frac{n}{2}-1}$ eigenvalues, where c is a constant. The average spacing between them in the interval $(\lambda, \lambda+1]$ is $(c\frac{n}{2}\lambda^{\frac{n}{2}-1})^{-1}$. We multiply the λ_i by $c\frac{n}{2}\lambda^{\frac{n}{2}-1}$ to make the average spacing 1. Since the eigenvalues in $(\lambda, \lambda+\varepsilon]$ are approximately λ, we are left to consider $\lambda_i^{\frac{n}{2}-1}(\lambda_{i+1} - \lambda_i)$, which are the normalized gaps in the spectrum. We call the numbers $(\lambda_i^{\frac{n}{2}-1} + \lambda_j^{\frac{n}{2}-1})(\lambda_i - \lambda_j)$ the normalized differences of eigenvalues. The main theorem in this work is:

THEOREM 1.1. *For a generic flat torus $\mathcal{L} \setminus \mathbb{R}^m$ the normalized differences of eigenvalues form a dense set in the real line, i.e.:*

$$\overline{\left\{\left(\lambda_i^{\frac{m}{2}-1}+\lambda_j^{\frac{m}{2}-1}\right)(\lambda_i-\lambda_j) : \lambda_i, \lambda_j \in \operatorname{spec}(\mathcal{L} \setminus \mathbb{R}^m)\right\}} = \mathbb{R}. \tag{1.3}$$

In particular the normalized gaps $\lambda_i^{\frac{m}{2}-1}(\lambda_{i+1}-\lambda_i)$ can be made arbitrarily small (and non-zero), which implies that there are arbitrarily small gaps in the spectrum. It should be noted that for rational tori, i.e., $(\vec{v}_i, \vec{v}_j)/(\vec{v}_{i'}, \vec{v}_{j'})$ is rational for all i, i', j, j', the theorem is not true.

1.3. In Section 3 we give an approach to the same problem for certain arithmetic hyperbolic surfaces. The result (Theorem 3.1) depends on an assumption on the continued fraction expansion of a certain transcendental number. Although we use only a thin part of the spectrum, the only assumption needed is very natural and is open for numerical investigations (see Section 4). The numerical investigations can actually give concrete small gaps in the discrete spectrum of those surfaces. We note that continued fractions have been considered first for the problem of level clustering for harmonic oscillators by M. Berry and M. Tabor [**3**]. Also P. Bleher studied the level spacing of eigenvalues for two harmonic oscillators with generic ratio of frequencies using continued fractions [**4**].

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2. Proofs

THEOREM 2.1. *For a generic lattice $\mathcal{L}$ in $\mathbb{R}^m$ (in the sense of measure) and any $a < m-1$ the set*

$$\left\{(|l|^a + |l'|^a)(|l'|^2 - |l|^2) : l, l' \in \mathcal{L}\right\} \tag{2.1}$$

is dense in $\mathbb{R}$.

PROOF. Let us denote the standard Euclidean inner product of two vectors $\vec{x}$ and $\vec{y}$ by $(\vec{x}, \vec{y})$.

2.1. Let $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m$ be a basis for the lattice so that all elements in the lattice are of the form

$$l = \sum_{i=1}^{m} x_i \vec{v}_i, \tag{2.2}$$

where $x_i \in \mathbb{Z}$. We also set

$$l' = \sum_{i=1}^{m} (x_i + 1)\vec{v}_i. \tag{2.3}$$

We consider the difference

$$
\begin{aligned}
|l'|^2 - |l|^2 &= \left| \sum_{i=1}^{m} (x_i + 1)\vec{v}_i \right|^2 - \left| \sum_{i=1}^{m} x_i \vec{v}_i \right|^2 \\
&= 2 \left(\sum_{i=1}^{m} x_i \vec{v}_i, \sum_{i=1}^{m} \vec{v}_i \right) + \left(\sum_{i=1}^{m} \vec{v}_i, \sum_{i=1}^{m} \vec{v}_i \right) \\
&= \sum_{i=1}^{m} 2\,(\vec{v}_i, \vec{w})\, x_i + |\vec{w}|^2 ,
\end{aligned}
\tag{2.4}
$$

where $\vec{w} = \sum_{i=1}^{m} \vec{v}_i$. Obviously $\vec{w} \neq \vec{0}$ and we have $(\vec{v}_t, \vec{w}) \neq 0$ for some t, which we assume is actually m. Each of the sets $(\vec{v}_t, \vec{w}) \neq 0$, $t = 1, 2, \ldots, m$ is a non-empty open set in $\mathbb{R}^{m^2}$, since $(\vec{v}_t, \vec{w}) = 0$ is a quadratic equation and, by choosing $\vec{v}_i = \vec{0}$ for $i \neq t$, we see that it is a hypersurface. So the restriction $(\vec{v}_m, \vec{w}) \neq 0$ is no loss of generality. Set

$$
L(\vec{x}) = \sum_{i=1}^{m-1} \frac{(\vec{v}_i, \vec{w})}{(\vec{v}_m, \vec{w})} x_i \tag{2.5}
$$

which is a linear form in the variables $x_1, x_2, \ldots, x_{m-1}$. We recall some terminology and results from the theory of diophantine approximation. We denote by $||x||$ the distance from x to the nearest integer. A linear form L in k variables is called singular [**5**, p. 92] iff for each $\varepsilon > 0$ the inequalities:

$$
||L(\vec{x})|| < \varepsilon X^{-k}, \quad |x_i| \leq X
$$

for $i = 1, 2, \ldots, k$ have an integer solution $\vec{x} \neq \vec{0}$ for all $X \geq X_0(\varepsilon)$. Otherwise it is called regular. It is easy to see [**5**, p. 92] that the set of $(\vartheta_1, \vartheta_2, \ldots, \vartheta_k) \in \mathbb{R}^k$ such that the form

$$
L(\vec{x}) = \sum_{i=1}^{k} \vartheta_i x_i
$$

is singular is a set of measure zero. Morover (see [**5**, p. 93]) the following theorem holds:

A necessary and sufficient condition for the form $L(\vec{x})$ to be regular is the existence of a $\delta > 0$ such that

$$
||L(\vec{x}) - \alpha|| \cdot (\max_{i=1,2,\ldots,k} |x_i|^k) < \delta \tag{2.6}
$$

has infinitely many integral solutions $\vec{x}$ for each real number α.

If the form is regular, we get for fixed α the existence of infinitely many $\vec{x} = (x_1, x_2, \ldots, x_k) \neq \vec{0}$ such that for each $\vec{x}$ we can find an integer x_{k+1} with

$$
|L(\vec{x}) - \alpha + x_{k+1}| < \delta / (\max_{i=1,2,\ldots,k} |x_i|)^k. \tag{2.7}
$$

We get the following inequality for x_{k+1}:

$$\begin{aligned} |x_{k+1}| &\leq |x_{k+1} + L(\vec{x}) - \alpha| + |L(\vec{x})| + |\alpha| \\ &< \delta / \max_{i\leq k} |x_i|^k + |\alpha| + \sum_{i=1}^{k} |\vartheta_i||x_i| \\ &< \delta + |\alpha| + \left(\sum_{i=1}^{k} |\vartheta_i|\right) \max_{i\leq k} |x_i| \end{aligned} \tag{2.8}$$

From (2.8) it follows that we can find an M such that for infinitely many of the integral solutions $\vec{x}$ we have

$$|x_{k+1}| \leq M \max_{i=1,2,\ldots,k} |x_i|. \tag{2.9}$$

If the form (2.5) is regular then

$$(|l|^a + |l'|^a)(|l'|^2 - |l|^2) = 2(|l|^a + |l'|^a)(\vec{v}_m, \vec{w}) \left(L(\vec{x}) + x_m + \frac{|\vec{w}|^2}{2(\vec{v}_m, \vec{w})}\right), \tag{2.10}$$

where $\vec{x} = (x_1, x_2, \ldots, x_{m-1})$. Using (2.9) we see that the absolute value of the right side of (2.10) is less than

$$C \left(\max_{j=1,2,\ldots,m-1} |x_j|^2\right)^{a/2} \frac{\delta}{\max |x_j|^{m-1}} = \frac{C\delta}{\max |x_j|^{m-1-a}} \tag{2.11}$$

for infinitely many $\vec{x} \in \mathbb{Z}^{m-1}$, where C is a constant depending on the lattice (e.g. the inner products $(\vec{v}_i, \vec{v}_j)$) and M, which again depends on the lattice. As long as $a < m - 1$ we see that, since we have infinitely many solutions, we can make the right-hand-side of (2.11) as small as wanted.

2.2. We need to know that for a generic lattice the difference $|l'|^2 - |l|^2$ is zero for at most one integral vector $(x_1, x_2, \ldots, x_m)$. Considering the basis as an element of $GL(m, \mathbb{R})$:

$$G = \begin{pmatrix} \vec{v}_1 \\ \vec{v}_2 \\ \vdots \\ \vec{v}_m \end{pmatrix} \tag{2.12}$$

we will show that the set of matrices with $|l'|^2 = |l|^2$ for more than one vector $(x_1, x_2, \ldots, x_m)$ has Lebesgue measure zero in $\mathbb{R}^{m^2}$. If $|l'|^2 = |l|^2$, equation (2.4) gives

$$\sum_{i=1}^{m} x_i 2(\vec{v}_i, \vec{w}) + |\vec{w}|^2 = 0. \tag{2.13}$$

If we have two distinct such vectors $\vec{x}, \vec{y}$, there are integers $c_1, c_2, \ldots, c_m$ not all zero with

$$\sum_{i=1}^{m} c_i \left(\vec{v}_i, \sum_{j-1}^{m} \vec{v}_j\right) = 0, \tag{2.14}$$

which is another quadratic equation in m^2 variables. If $c_j \neq 0$ we set $\vec{v}_i = \vec{0}$ for $i \neq j$, so (2.14) implies that $c_j(\vec{v}_j, \vec{v}_j) = 0$ which cannot hold. So (2.14) represents a hypersurface. Therefore the set of such matrices G has measure zero and the claim is proven.

2.3. Now we need to show that generically in $GL(m,\mathbb{R})$ the form (2.5) is regular. We consider the map $f:\mathbb{R}^{m^2}\to\mathbb{R}^{m-1}$ given by

$$f(\vec{v}_1,\vec{v}_2,\dots,\vec{v}_m)=\left(\frac{(\vec{v}_1,\sum_{i=1}^m\vec{v}_i)}{(\vec{v}_m,\sum_{i=1}^m\vec{v}_i)},\frac{(\vec{v}_2,\sum_{i=1}^m\vec{v}_i)}{(\vec{v}_m,\sum_{i=1}^m\vec{v}_i)},\dots,\frac{(\vec{v}_{m-1},\sum_{i=1}^m\vec{v}_i)}{(\vec{v}_m,\sum_{i=1}^m\vec{v}_i)}\right) \tag{2.15}$$

and we let B be the subset of $\mathbb{R}^{m-1}$ such that the form

$$L(\vec{x})=\sum_{j=1}^{m-1}\vartheta_j x_j \tag{2.16}$$

is singular iff $(\vartheta_1,\vartheta_2,\dots,\vartheta_{m-1})\in B$. The set B has Lebesgue measure zero. We want to prove that $f^{-1}(B)$ is again a set of measure zero. We also consider the map $T:\mathbb{R}^{m-1}\to\mathbb{R}$ given by

$$T(s_1,s_2,\dots,s_{m-1})=\frac{1}{s_1+s_2+\cdots+s_{m-1}+1}. \tag{2.17}$$

We have that $T(B)$ has measure zero and it is enough to prove that $(T\circ f)^{-1}(T(B))$ has measure zero. Using (2.15), (2.17), we easily calculate

$$T\circ f(\vec{v}_1,\vec{v}_2,\dots,\vec{v}_m)=\frac{(\vec{v}_m,\sum_{i=1}^m\vec{v}_i)}{|\sum_{i=1}^m\vec{v}_i|^2}.$$

It is enough to prove that $T\circ f$ is a submersion, because in this case the inverse image of a set of measure zero has measure zero. It suffices to prove that there are no critical points of $T\circ f$ on the open set $GL(m,\mathbb{R})\subset\mathbb{R}^{m^2}$. We set $\vec{v}_j=(a_{j1},a_{j2},\dots,a_{jm})$. Then a trivial calculation gives:

$$\frac{\partial}{\partial a_{mi}}\frac{(\vec{v}_m,\sum_{j=1}^m\vec{v}_j)}{|\sum_{j=1}^m\vec{v}_j|^2}=0\Leftrightarrow\left(\sum_{k=1}^{m-1}a_{ki}+2a_{mi}\right)|\vec{w}|^2=2\left(\sum_{k=1}^{m}a_{ki}\right)(\vec{v}_m,\vec{w}) \tag{2.18}$$

and

$$\frac{\partial}{\partial a_{1i}}\frac{(\vec{v}_m,\sum_{j=1}^m\vec{v}_j)}{|\sum_{j=1}^m\vec{v}_j|^2}=0\Leftrightarrow a_{mi}|\vec{w}|^2=2\left(\sum_{k=1}^{m}a_{ki}\right)(\vec{v}_m,\vec{w}) \tag{2.19}$$

Equations (2.18), (2.19) together imply:

$$a_{1i}+a_{2i}+\cdots+a_{mi}=0$$

for each $i=1,2,\dots,m$, which gives that $\vec{w}=\vec{0}$, i.e., the vectors $\vec{v}_1,\vec{v}_2,\dots,\vec{v}_m$ are linearly dependent. Consequently there are no critical points and the map $T\circ f$ is a submersion.

2.4. We saw that for a generic lattice $\mathcal{L}$ the expression $|l'|^2-|l|^2|$ is zero at most once, so, if the lattice is also generic in the sense that the linear form (2.5) corresponding to it is regular, then there are infinitely many $\vec{x}$'s such that the expression $(|l'|^a+|l|^a)||l'|^2-|l|^2|$ is non-zero and is less than (2.11). So 0 is an accumulation point of the positive and the negative values of $(|l'|^a+|l|^a)(|l|^2-|l'|^2)$.

We now consider the expression $(|l'|^a + |l|^a)(|l'|^2 - |l|^2)$ as a homogeneous function of degree $2 + a$ on integral vectors of dimension $2m$ by dropping (2.3) and setting

$$l' = \sum_{i=1}^{m} y_i \vec{v}_i,$$

where $y_i \in \mathbb{Z}$. The following well-known lemma proves that the expression $(|l'|^a + |l|^a)(|l'|^2 - |l|^2)$ can be made as close to any real number as wanted and completes the proof of theorem 2.1. □

LEMMA 2.1. *If $Q(\vec{y})$ is a function on integral vectors, homogeneous of degree n with 0 being an accumulation point of its positive values and of its negative values, then*

$$\overline{\{Q(\vec{y}) : \vec{y} \in \mathbb{Z}^m\}} = \mathbb{R}. \tag{2.20}$$

PROOF. Let $z > 0$ be fixed and let $\varepsilon > 0$, $z > \varepsilon$. Since

$$\frac{z^{1/n} - (z-\varepsilon)^{1/n}}{Q(\vec{y})^{1/n}}$$

can be made arbitrarily large by choosing $Q(\vec{y}) > 0$ and small, we can find an integer d and an integral vector $\vec{y}$ such that

$$\frac{z^{1/n}}{Q(\vec{y})^{1/n}} > d > \frac{(z-\varepsilon)^{1/n}}{Q(\vec{y})^{1/n}},$$

which implies

$$z > Q(d\vec{y}) > z - \varepsilon.$$

We work similarly for $z < 0$. This completes the proof of the lemma. □

PROOF OF THEOREM 1.1. The eigenvalues of $\mathcal{L} \setminus \mathbb{R}^m$ are $4\pi^2 |l^*|^2$ where $l^* \in \mathcal{L}^*$, $\mathcal{L}^*$ the dual lattice of $\mathcal{L}$. Genericity for $\mathcal{L}$ is equivalent to genericity for $\mathcal{L}^*$ and $\frac{m}{2} - 1 < m - 1$. □

3. Hyperbolic Surfaces

3.1. In this section we approach the problem of density of the set

$$\{\lambda_i - \lambda_j, i, j \geq 0\},$$

where the λ_i's are in the discrete spectrum of certain finite area non-compact hyperbolic surfaces of the form $\Gamma \setminus \mathbb{H}$. We can take Γ to be a principal congruence subgroup of level N, where N is the least common multiple of the discriminants of two real quadratic fields F_1 and F_2 with $F_1 \cap F_2 = \mathbb{Q}$. We assume both F_1 and F_2 have narrow class number one and D_1 and D_2 are their discriminants and η_1, η_2 the fundamental units in the rings of integers o_{F_1} and o_{F_2}, i.e., generators for $o_{F_1}^*/\{\pm 1\}$, $o_{F_2}^*/\{\pm 1\}$. We can assume that η_1, η_2 are both greater than 1. For l a non-zero integer we have Grossencharacters ξ_1^l and ξ_2^l defined on integral ideals of F_1 and F_2 with conductors o_{F_1} and o_{F_2}, respectively, as follows

$$\xi_j^l((\alpha)) = \left|\frac{\alpha}{\alpha'}\right|^{\frac{i\pi l}{\log \eta_j}},$$

where $j = 1, 2$ and α' is the conjugate of α in F_j. Setting

$$\beta_j = \frac{\pi}{\log \eta_j}$$

for $j = 1, 2$, we get Maaß cusp forms for $\Gamma(D_j)$ of the form

$$g_j^l(z) = \sum_{\mu \in o_{F_j}/E_j^+} \xi_j^l(\mu) y^{\frac{1}{2}} K_{il\beta_j}(2\pi|N_j(\mu)|y) \exp(2\pi i N_j(\mu)x),$$

where the summation is over all integers of F_j modulo the totally positive units of o_{F_j}, denoted by E_j^+, and N_j is the norm in the field F_j and $K_s(x)$ is the MacDonald Bessel function. The corresponding eigenvalues for the Laplace operator on $\Gamma(D_j) \setminus \mathbb{H}$ are $\frac{1}{4} + (l\beta_j)^2$. Let now Γ be any group contained in $\Gamma(N)$ where N is the least common multiple of D_1 and D_2. Then all the numbers

$$\frac{1}{4} + \left(\frac{l\pi}{\log \eta_j}\right)^2,$$

where l is a positive integer and $j = 1, 2$, are in the discrete spectrum (even cuspidal spectrum) of $\Gamma \setminus \mathbb{H}$. When we subtract the eigenvalues coming from F_2 from those of F_1 we get

$$\left(\frac{l\pi}{\log \eta_1}\right)^2 - \left(\frac{m\pi}{\log \eta_2}\right)^2 = \frac{\pi^2}{(\log \eta_1)^2}\left(l^2 - \left(\frac{\log \eta_1}{\log \eta_2}\right)^2 m^2\right). \tag{3.1}$$

Up to a multiplicative constant, we get an indefinite quadratic form in two variables $l^2 - d^2m^2$, where $d = \log \eta_1 / \log \eta_2$. In this case d is transcendental, since, by the Gelfond-Schneider theorem [**1**], if α_1 and α_2 are non-zero algebraic numbers with $\log \alpha_1$ and $\log \alpha_2$ linearly independent over the rationals, then $\log \alpha_1$ and $\log \alpha_2$ are linearly independent over the algebraic numbers. Here $p_1 \log \eta_1 + p_2 \log \eta_2 = 0$ implies $\eta_1^{p_1}\eta_2^{p_2} = 1$ and this cannot hold for p_1 and p_2 integers, not both zero, since $F_1 \cap F_2 = \mathbb{Q}$.

3.2. For a form $l^2 - d^2m^2$ with d irrational it is known that it represents an arbitrary small number iff the continued fraction expansion of d has unbounded partial quotients. More precisely [**8**, Th. 23]: If $d > 0$ has bounded partial quotients and ε is sufficiently small, the inequality:

$$\left|d - \frac{p}{q}\right| < \frac{\varepsilon}{q^2} \tag{3.2}$$

has no solution in the integers $p, q > 0$. Then

$$\left|q^2d^2 - p^2\right| = |qd - p|\,|qd + p| \geq \frac{\varepsilon}{q}\,|qd + p| \geq \varepsilon d$$

for all $p, q \neq 0$. On the other hand, if d has unbounded partial quotients then the inequality (3.2) has an infinite number of solutions and we can take p and q to be the p_k and q_k, where p_k/q_k are the convergents of d. We have $\lim_k (p_k/q_k) = d$. Then

$$\left|q_k^2d^2 - p_k^2\right| = q_k^2\left|d - \frac{p_k}{q_k}\right|\left|d + \frac{p_k}{q_k}\right| < \varepsilon\left|d + \frac{p_k}{q_k}\right|$$

and the result follows. Although there has been extensive work on lower bounds for linear forms in logarithms of algebraic numbers (see [**1**]), there are no results in the other direction. In particular it is not known whether numbers like $\log \eta_1 / \log \eta_2$ have bounded or unbounded partial quotients, although it is highly expected that they have unbounded partial quotients. No simple example of numbers with bounded partial quotients other than quadratic irrationals is known and the best guess is that there are no other natural examples (see [**9**]). It is to be remarked that the set of numbers having bounded partial quotients has measure 0 (see [**8**, Th. 29]). From the discussion above we conclude:

THEOREM 3.1. *If the partial quotients of* $\log \eta_1 / \log \eta_2$ *are unbounded then the differences of the eigenvalues in the discrete spectrum of* $\Gamma \setminus \mathbb{H}$*,* $\Gamma \subset \Gamma(N)$*,* $N =$ l.c.m.(D_1, D_2) *are dense in the real line.*

EXAMPLE 3.1. Let $F_1 = \mathbb{Q}(\sqrt{5})$ and $F_2 = \mathbb{Q}(\sqrt{2})$. The fundamental units are: $\eta_1 = \frac{1}{2}(1+\sqrt{5})$ and $\eta_2 = 1 + \sqrt{2}$. Both fields have class number 1 and, since the norms of the fundamental units are -1, they even have narrow class number 1. The discriminants are 5 and 8 respectively and $N = 40$. We examine the partial quotients of $\log \eta_1 / \log \eta_2$ numerically in the next section.

REMARK 3.1. The lengths of closed geodesics for the surface $SL(2,\mathbb{Z}) \setminus \mathbb{H}$ which is covered by $\Gamma(N) \setminus \mathbb{H}$ are multiples of $2 \log \varepsilon_d$, where $\varepsilon_d = \frac{1}{2}(n + m\sqrt{d})$ is the smallest solution greater than 1 of Pell's equation $n^2 - dm^2 = 4$. Here d runs over positive integers congruent to 0 or 1 (mod 4). Thus ε_d is a unit in the field $\mathbb{Q}(\sqrt{d})$. If we choose the F_j's to be $\mathbb{Q}(\sqrt{d_j})$ with $d_j \equiv 1 \pmod 4$, we see that the hypothesis of theorem 3.1 can be expressed in terms of the continued fraction of the quotient of the lengths of two closed geodesics in $SL(2,\mathbb{Z}) \setminus \mathbb{H}$ associated with the Pell's equations $n^2 - d_1 m^2 = 4$ and $n^2 - d_2 m^2 = 4$.

4. Numerical Investigations

4.1. In this section we present numerical investigations for the continued fractions $[a_1, a_2, \dots]$ of the number

$$d = \frac{\log \eta_1}{\log \eta_2} = \frac{\log(\frac{1}{2}(1+\sqrt{5}))}{\log(1+\sqrt{2})}. \tag{4.1}$$

We follow the approach taken in [**10**] for algebraic numbers. Tables 1, 2, 3 are to be read horizontally and the numbers $a_1, a_2, \dots$ are laid out in rows of ten integers. Table 4 gives the frequency count N for the number k appearing in the continued fractions and p, which is the probability that (for a generic number a) the n-th integer a_n in the continued fraction for a is equal to k. This probability is, according to a theorem of Kuzmin ([**8**, p. 92]), $\log_2((k+1)^2/k(k+2))$. We see that the number d behaves very closely to what is expected for almost all numbers.

The following are the large partial quotients. If they appear more than once the number of times they appear is given in parentheses:

41	42(2)	44	50	53	55	58	71	75
83(3)	89	102	121	147	161	166	170	171
182	352	358	459	469	621	957	2170	31804

The probability $q_{N,k}$ that the first N partial quotients of a random number are all less than k is γ_k^N, if we assume that numbers in $(0,1)$ have the distribution:

$$\Pr\{X \leq c\} = \log_2(1+c) \quad \text{for} \quad c \in (0,1)$$

and the randon variables $a_n(X)$ are independent. Then

$$\gamma_k = \Pr\{a_n(X) < k\} = 1 - \log_2\left(1 + \frac{1}{k}\right).$$

The maximum value of the first 1000 partial quotients observed for d is 31804 and $a_{870} = 31804$. The probability that a random number would have a value as large as 31804 among its first 1000 partial quotients is

$$p_{1000,31804} = 1 - q_{1000,31804} \approx 0.0443.$$

Intuitively we see that the smaller $p_{N,k} = 1 - q_{N,k}$ is, the more unusual our number is. This may suggest that d is unusual but, if we take into account the second largest partial quotient $a_{67} = 2170$, we get

$$p_{1000,2170} \approx 0.4857.$$

We expect that a large partial quotient like 31804 does not show for quite a while after the first thousand partial quotients.

4.2. The numerical investigations undertaken can provide a specific gap in the discrete spectrum of $\Gamma(40) \setminus \mathbb{H}$ as follows: since

$$\left| d - \frac{p_n}{q_n} \right| \leq \frac{1}{q_n q_{n+1}},$$

we easily see that

$$\left| q_n^2 d^2 - p_n^2 \right| < 2 \frac{q_n}{q_{n+1}}.$$

In order to make the gap in the discrete spectrum small, we try to make q_n/q_{n+1} as small as possible and the inequality

$$\frac{1}{a_{n+1} + 1} < \frac{q_n}{q_{n+1}} < \frac{1}{a_{n+1}}$$

(which follows from $q_{n+1} = a_{n+1} q_n + q_{n-1}$) suggests to look for large a_{n+1}. In our case $a_{n+1} = a_{870} = 31804$ and $q_{869}^2 d^2 - p_{869}^2 \approx 0.0000343329$. With the normalization for the eigenvalues (so that the curvature is -1) the gap in the discrete spectrum is:

$$\frac{\pi^2}{(\log \eta_1)^2} \left(q_{869}^2 d^2 - p_{869}^2 \right) \approx 0.001463276$$

All the numerical calculations were done on *Mathematica*.

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Table 1. a_n, $n \leq 340$

	1	2	3	4	5	6	7	8	9	10
0	1	1	4	1	14	1	12	1	6	1
10	7	4	4	1	2	7	11	6	5	1
20	21	1	4	1	1	5	1	2	1	2
30	8	1	5	2	1	1	1	1	2	3
40	1	166	2	2	2	1	1	8	1	8
50	4	1	1	1	2	2	1	1	6	4
60	1	3	2	1	1	1	2170	1	4	8
70	5	1	1	8	1	8	3	3	10	1
80	1	2	1	8	8	1	4	1	1	1
90	1	1	1	1	1	1	8	2	10	25
100	1	1	11	1	1	1	27	1	1	13
110	1	3	1	7	1	3	2	3	1	23
120	2	11	1	1	2	2	10	1	83	3
130	14	16	31	19	1	1	7	4	2	3
140	1	2	4	19	1	4	5	1	1	4
150	89	1	4	1	1	41	1	2	1	1
160	1	1	2	1	1	3	1	1	3	1
170	2	2	1	2	1	2	1	3	1	12
180	1	2	2	2	2	2	1	1	8	1
190	3	9	3	12	3	4	2	35	25	1
200	2	2	2	7	6	14	1	1	13	1
210	1	3	1	2	3	3	2	1	4	161
220	2	1	7	1	1	2	5	16	2	2
230	4	171	19	1	1	1	1	1	2	2
240	1	1	1	2	2	3	12	1	1	15
250	83	1	7	1	1	22	7	5	5	1
260	21	2	3	50	2	1	1	40	1	1
270	15	1	19	6	1	11	1	6	1	6
280	3	12	2	4	1	2	3	1	5	1
290	1	3	1	2	2	3	1	4	1	7
300	2	16	2	6	1	3	1	1	1	1
310	957	3	1	12	1	2	16	1	4	14
320	1	18	2	2	1	1	21	17	2	1
330	11	1	2	1	15	1	1	1	1	2

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Table 2. a_n, $341 \leq n \leq 670$

	1	2	3	4	5	6	7	8	9	10
340	1	1	11	12	1	1	1	1	1	1
350	6	9	1	5	2	4	42	2	3	11
360	2	12	3	1	3	1	3	2	1	1
370	1	1	2	1	5	9	1	1	1	1
380	2	2	1	5	1	170	1	1	10	2
390	17	14	34	2	2	5	1	4	2	1
400	3	1	1	3	14	2	53	1	1	2
410	1	6	1	1	1	5	4	1	5	1
420	1	2	1	2	2	1	6	459	1	1
430	1	3	2	1	9	2	4	5	1	2
440	2	25	1	4	19	8	3	1	2	3
450	2	1	1	1	1	6	2	1	9	2
460	13	3	6	10	2	2	75	1	2	9
470	4	1	3	1	16	1	8	10	1	7
480	1	7	121	1	5	1	7	9	1	5
490	2	17	1	1	3	1	3	8	1	1
500	2	2	1	1	1	2	3	4	2	182
510	1	1	5	1	4	3	1	20	1	6
520	6	8	2	3	2	20	8	1	9	1
530	21	14	2	1	9	2	3	2	3	1
540	22	1	1	1	5	1	34	1	1	1
550	71	7	3	1	2	2	2	3	4	1
560	5	2	10	2	1	2	1	2	3	3
570	1	4	1	2	3	3	1	1	1	1
580	1	4	2	3	1	1	4	2	1	1
590	1	6	1	7	1	24	1	5	5	1
600	5	1	5	1	2	2	2	1	1	1
610	1	1	13	1	1	2	10	1	2	8
620	1	28	13	2	7	9	7	147	1	12
630	1	1	2	3	1	1	17	12	1	14
640	1	1	1	2	1	1	5	4	2	3
650	1	2	621	1	1	1	8	1	23	1
660	2	4	1	11	1	1	6	6	1	4

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TABLE 3. a_n, $671 \leq n \leq 1000$

	1	2	3	4	5	6	7	8	9	10
670	1	6	1	7	1	16	2	1	2	4
680	2	15	11	35	4	1	5	2	3	1
690	2	5	2	3	3	2	1	2	20	2
700	10	3	2	7	1	6	1	6	1	1
710	4	1	5	1	9	4	2	1	3	3
720	4	5	1	1	1	1	3	1	1	5
730	1	1	2	1	1	9	1	1	12	1
740	1	1	2	3	1	5	1	27	12	1
750	1	3	3	14	1	1	1	3	3	2
760	1	1	469	2	2	3	2	1	2	1
770	4	1	1	358	15	9	1	2	1	18
780	1	1	1	31	1	9	27	3	1	1
790	1	1	102	1	3	1	2	36	83	1
800	58	6	42	4	2	1	2	1	1	1
810	20	1	8	1	20	1	5	1	2	10
820	1	1	7	3	4	6	1	20	2	1
830	1	15	1	1	1	2	1	6	2	6
840	12	2	1	7	6	1	1	1	1	3
850	5	3	2	5	1	5	10	1	3	4
860	1	9	3	6	2	5	36	6	1	31804
870	7	10	1	3	3	1	3	1	6	7
880	1	2	3	13	1	3	5	2	5	17
890	1	55	1	2	1	44	22	1	2	1
900	7	1	2	4	5	2	3	1	1	8
910	14	1	7	7	1	1	1	1	2	2
920	2	1	1	4	352	2	2	40	3	17
930	2	6	3	1	9	2	2	2	2	1
940	1	1	5	1	2	2	3	4	12	1
950	1	2	2	6	3	2	3	1	1	3
960	2	1	2	3	3	2	1	6	2	1
970	1	1	1	8	28	12	1	1	1	1
980	4	1	1	1	8	1	1	1	1	2
990	2	1	3	1	6	4	1	2	1	3

Table 4. Frequencies

k	N	p	k	N	p
1	413	.4150	17	6	.0045
2	173	.1699	18	2	.0040
3	89	.0931	19	5	.0036
4	48	.0589	20	6	.0033
5	41	.0406	21	4	.0030
6	33	.0297	22	3	.0027
7	25	.0227	23	2	.0025
8	21	.0179	24	1	.0023
9	16	.0145	25	3	.0021
10	12	.0120	27	3	.0018
11	9	.0101	28	2	.0017
12	15	.0086	31	2	.0014
13	6	.0074	34	2	.0012
14	10	.0064	35	2	.0011
15	6	.0056	36	2	.0011
16	6	.0050	40	2	.0009