Quantum Chaos

Ergodic systems → geodesic flow

If M hyperbolic → ergodic

$M = \Gamma \backslash \mathbb{H}^2$ $\quad \Gamma = PSL_2(\mathbb{Z})$

modular surface

Arithmetic quantum chaos

P. Sarnak

$\Delta u_n + \lambda_n u_n = 0 \qquad \lambda_n \to \infty$

Δ Laplace - Beltrami operator

$= y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)$

u_n e.g. Maaß cusp forms

$\|u_n\|_{L^2} = 1$

$$\mu_n = |u_n(z)|^2 \boxed{\frac{dx\,dy}{y^2}} = d\mu(z)$$

$A \qquad \mu_n(A) = \int_A |u_n(z)|^2 \, d\mu(z)$

μ_n limiting behaviour

Schnirelman

Colin de Verdière

Zelditch

for almost all n's

$\mu_n \longrightarrow \frac{1}{} \, d\mu$

$\mathrm{vol}(M)$

$$\boxed{\mu_n(A) \longrightarrow \frac{\mathrm{area}(A)}{\mathrm{area}(M)}}$$

density 1

$$N(\lambda) = \#\{\lambda_n \le \lambda\} \sim c \cdot \lambda$$

$$\#\{n : \qquad \} \sim N(\lambda)$$

$\boxed{\text{Conj}}$ Rudnick – Sarnak refined conjecture

QUE no exceptional subsequences

hyperbolic manifolds

$M = \Gamma \backslash \mathbb{H}$ $\qquad \Gamma = \Gamma SL_2(\mathbb{Z})$

it is true QUE Maaß cusp forms

Lindenstrauss | Soundararajan

Luo – Sarnak | QUE for Eisenstein series

$$d\mu_t(z) = |E(z, \tfrac{1}{2} + it)|^2 \, d\mu(z)$$

$$t \to \infty$$

$$\Delta E(z, \tfrac{1}{2} + it) + \left(\tfrac{1}{4} + t^2\right) E(z, \tfrac{1}{2} + it) = 0$$

$$\mu_t(A) = \int_A |E(z, \tfrac{1}{2} + it)|^2 \frac{dx\,dy}{y^2}$$

$A, B \subset \Gamma \backslash \mathbb{H}$

QUE $\displaystyle \lim_{t\to\infty} \frac{\mu_t(A)}{\mu_t(B)} \to \frac{\mu(A)}{\mu(B)} = \frac{\text{area}(A)}{\text{area}(B)}$

To prove

$$\mu_t(A) \sim \frac{\boxed{6}}{\pi} \text{area}(A) \cdot \log t$$

$$\sim \frac{3}{\pi} \text{area}(A) \log\left(\frac{1}{4}+t^2\right)$$

$$\underset{t\to\infty}{\sim} \frac{\text{area}(A)}{\text{area}(M)} \log\left(\frac{1}{4}+t^2\right)$$

Th 2.2 ψ compact support continuous

$$\int_{\Gamma\backslash\mathbb{H}} \psi(z)\, d\mu_t(z) = \int_{\Gamma\backslash\mathbb{H}} \psi(z) \left|E(z, \tfrac{1}{2}+it)\right|^2 \frac{dx\,dy}{y^2}$$

$$\sim \frac{1}{\text{area}(\Gamma\backslash\mathbb{H})} \int_{\Gamma\backslash\mathbb{H}} \psi(z)\, d\mu(z) \log\left(\frac{1}{4}+t^2\right)$$

2.1. Maaß cusp forms

Fix $u_j(z)$ Hecke$^{\text{Maaß}}$

Prop 2.1

$$\int_{\Gamma\backslash\mathbb{H}} u_j(z)\, d\mu_t(z) \ll_{j,\varepsilon} \underline{|t|^{-\frac{1}{6}+\varepsilon}}$$

$$\leq M(j,\varepsilon)\, |t|^{-\frac{1}{6}+\varepsilon}$$

$$\int_{\Gamma\backslash\mathbb{H}} u_j(z)\, d\mu_t(z) \ll_j \frac{\log^2 t}{\sqrt{E}} \left|L(u_j, \tfrac{1}{2}+2it)\right|$$

$$L(u_j, s) = \sum_{n\geq 1} \frac{\lambda_j(n)}{n^s}$$

Subconvexity of $L(u_j, s)$

<u>Meurman</u>

$$L(u_j, \tfrac{1}{2}+it) \ll_j |t|^{1/3+\epsilon}$$

<u>Proof</u> $\quad I_j(t) = \int_{\Gamma\backslash\mathbb{H}} u_j \, d\mu_t(z)$

$$= \int_{\Gamma\backslash\mathbb{H}} u_j \, E(z, \tfrac{1}{2}+it) E(z, \tfrac{1}{2}-it)\, d\mu(z)$$

$$E(z, s) = \overline{E(z, \bar{s})}$$

$$I_j(s) = \int_{\Gamma\backslash\mathbb{H}} u_j(z) E(z, \tfrac{1}{2}+it)\, E(z,s)\, d\mu(z)$$

$$s = \tfrac{1}{2}+i\tau$$

Plug $\quad s = \frac{1}{2} - it$

Unfold $\quad \int_0^\infty \int_0^1 u_j(z) E(z, \tfrac{1}{2}+it)\, y^s \, \frac{dx\, dy}{y^2}$

$$= \frac{2\rho_j(1)}{\xi(1+2it)} \sum_{n=1}^\infty \lambda_j(n)\, n^{it}\, \sigma_{1-2it}(n) \int_0^\infty$$

$$K_{it_j}(2\pi n y)\, K_{it}(2\pi n y)\, y^{s-1}\,dy$$

$$2\pi n y \to y$$

$$I_j(s) = \frac{2\rho(1)}{\zeta(1+2it)} \sum_{n=1}^{\infty} \frac{\lambda_j(n)\, n^{it}\, \sigma_{-2it}(n)}{n^s} \cdot \frac{\Gamma\left(\frac{s+it_j+it}{2}\right)\Gamma\left(\frac{s-it+it_j}{2}\right)\cdots}{\pi^s \quad \Gamma(s)}$$

$$= \frac{2\rho_j(1)}{\zeta(1+2it)} \frac{1}{\zeta(2s)} L(u_j, s-it) L(u_j, s+it)$$

$$\cdot \text{ Gamma factors}$$

Plug $s = \frac{1}{2} - it$

$$I_j(t) = \frac{2\rho_j(1)}{\zeta(1+2it)} \frac{1}{\zeta(1-2it)} L\left(u_j, \frac{1}{2} - 2it\right) L\left(u_j, \frac{1}{2}\right)$$

$$\zeta(1+2it) = \pi^{-\frac{1}{2}-it}\zeta(1+2it)\Gamma\left(\tfrac{1}{2}+it\right)$$

Gamma factors with $s = \frac{1}{2} - it$

$$= \rho_j(1)\, \pi^{-2it} \frac{\left|\Gamma\left(\frac{1}{4} + \frac{it_j}{2}\right)\right|^2}{4\left|\zeta(1+2it)\right|^2} \frac{\Gamma\left(\frac{1}{4} - \frac{it_j}{2} - it\right)\Gamma\left(\frac{1}{4} + \frac{it_j}{2} - it\right)}{\left|\Gamma\left(\frac{1}{2} + it\right)\right|^2}$$

$t \to \infty$

t_j fixed

$$\cdot\, L\left(u_j, \tfrac{1}{2} - 2it\right) L\left(u_j, \tfrac{1}{2}\right)$$

$$a \le x \le b$$

$$|\Gamma(x+iy)| \sim \sqrt{2\pi}\, e^{-\frac{\pi}{2}|y|}\, |y|^{x-1/2}$$

$$\ll t^{-1/2}$$

What can I say about

$$\frac{1}{\zeta(1+it)} \ll \log t$$

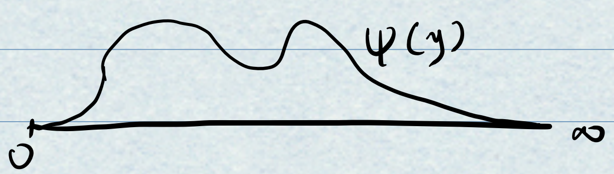
$$\frac{1}{\zeta(1+it)} \ll \log^7 t \quad \text{easier}$$

$$\mathcal{I}_j(t) \ll_j \frac{\log^2 t}{\sqrt{t}}\, |L(u_j, \tfrac{1}{2} - 2it)|$$

Prop 2.2 Incomplete Eisenstein series

$$E(\psi, z)$$

$$\int_{\Gamma\backslash\mathbb{H}} E(\psi, z)\, d\mu_t(z) \sim \frac{3}{\pi}\int_0^\infty \psi(y)\frac{dy}{y^2}\log\left(\frac{1}{4}+t^2\right)$$



$$E(\psi, z) = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \psi(\mathrm{Im}\,\gamma z)$$

$$\int E(\psi, z)\, dx\, dy = \int_0^\infty \int_0^1 \psi(y)\, dx\, dy$$

$$\int_{\Gamma\backslash\mathbb{H}} \cdots \frac{\cdots}{y^2} = \int_0^1 \int_0^\infty \cdots \frac{\cdots}{y^2}$$

$$= \int_0^\infty \psi(y)\frac{dy}{y^2}$$

$$\hat{\psi}(s) = \int_0^\infty \psi(y)\, y^{-s-1}\, dy$$

$$\psi(y) = \frac{1}{2\pi i} \int_{(\sigma)} \hat{\psi}(s)\, y^s\, ds$$

$$E(\psi, z) = \frac{1}{2\pi i} \int_{(2)} \hat{\psi}(s)\, E(z,s)\, ds$$

$$\int_{\Gamma\backslash\mathbb{H}} E(\psi, z) \left| E(z, \tfrac{1}{2}+it) \right|^2 \frac{dx\,dy}{y^2}$$

$$= \frac{1}{2\pi i} \int_{\underbrace{\Gamma\backslash\mathbb{H}}} \int_{(2)} \hat{\psi}(s)\, E(z,s) \left| E(z, \tfrac{1}{2}+it) \right|^2 \frac{dx\,dy}{y^2}$$

$$= \frac{1}{2\pi i} \int_0^\infty \int_{Re(s)=2} \int_0^1 \hat{\psi}(s)\, y^s \left| E(z, \tfrac{1}{2}+it) \right|^2 \frac{dx\,dy}{y^2}$$

Parseval

$$= \frac{1}{2\pi i} \int_0^\infty \int_{Re(s)=2} \hat{\psi}(s) y^s \sum_n |\text{Fourier of } E|^2 \frac{dy}{y^2}$$

$$= \frac{1}{2\pi i} \int_0^\infty \int_{Re(s)=2} \hat{\psi}(s)\, y^s \Big(\left| y^{\frac{1}{2}+it} + \varphi(\tfrac{1}{2}+it)\, y^{\frac{1}{2}-it} \right|^2$$

$$+ \sum_{n \geq 1} \frac{8}{|\zeta(1+2it)|^2}\, y\, |\sigma_{-2it}(n)|^2 \cdot |K_{it}(\,)|^2 \Big\} \frac{dy}{y^2}$$

$$|y^{1/2+it} + \varphi(\tfrac{1}{2}+it)y^{1/2-it}|^2 = i$$

$$y^1 + y^1|\varphi(\tfrac{1}{2}+it)|^2$$
$$+ 2\,\mathrm{Re}(y^{1+2it}\varphi(\tfrac{1}{2}+it))$$

$$\varphi(s) = \frac{\zeta(2-2s)}{\zeta(2s)} \Rightarrow \varphi(s)\varphi(1-s) = 1$$

$$= 2y + 2\,\mathrm{Re}(y^{1+2it}\varphi(\tfrac{1}{2}+it))$$

$$2\frac{1}{2\pi i}\int_{\mathrm{Re}(s)=2}\int_0^\infty \hat{\varphi}(s)\, y^{s+1}\frac{dy}{y^2} + g(t)$$

$$= 2\int_0^\infty \varphi(y)\frac{dy}{y} + g(t)$$

$$g(t) = \frac{2}{2\pi i}\int_0^\infty\int_{\mathrm{Re}(s)=2} \hat{\varphi}(s)\, y^{s+1}\,\mathrm{Re}(\varphi(\tfrac{1}{2}+it)y^{2it})\, ds\, dy$$

$$= \mathrm{Re}\int_0^\infty \varphi(y)\, y^{2it}\frac{dy}{y}\quad \varphi(\tfrac{1}{2}+it)$$

$t \to \infty$

$$\downarrow 0$$

$$\ll t^{-A} \quad A \text{ any positive}$$

$$e^{2i\ln y\, t}$$

Other Fourier coefficients

Lemma 2.1

$$\int_{\Gamma \backslash H} E(\varphi, z)\, d\mu_t(z) = 2 \int_0^\infty \varphi(y) \frac{dy}{y} + I_2(t) + O(t^{-A})$$

$$I_2(t) = \frac{8}{2\pi i\, |\zeta(1+2it)|^2} \int_{Re(s)=2} \hat{\varphi}(s) \cdot \sum_{n \geq 1} \frac{|\sigma_{-2it}(n)|^2}{n^s}$$

- Gamma functions

$$\sum_{n \geq 1} \frac{\sigma_a(n)\sigma_b(n)}{n^s} = \frac{\zeta(s)\zeta(s-a)\zeta(s-b)\zeta(s-a-b)}{\zeta(2s-a-b)}$$

$$Z(s,t) = \sum_{n \geq 1} \frac{|\sigma_{-2it}(n)|^2}{n^s} = \frac{\zeta^2(s)\zeta(s-2it)\zeta(s+2it)}{\zeta(2s)}$$

$$\tilde{\gamma}(s,t) = \frac{\Gamma^2(s/2)\,\Gamma(s/2 - it)\,\Gamma(s/2 + it)}{8\pi^s\,\Gamma(s)}$$

$$I_2(t) = \frac{8}{2\pi i\, |\zeta(1+2it)|^2} \int_{Re(s)=2} \hat{\varphi}(s)\, Z(s,t)\, \tilde{\gamma}(s,t)\, ds$$

Shift to $Re(s) = \frac{1}{2}$

$s=1$ pole double

from $\zeta(s \pm 2it)$ poles at $1 \pm 2it$ simple

Lemma 2.2

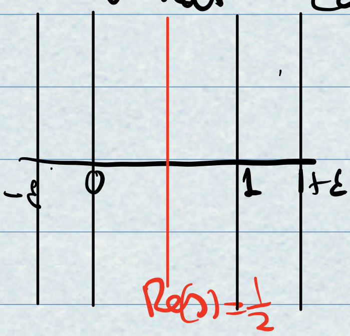
$$I_2(t) = \frac{8}{|\zeta(1+2it)|^2} \operatorname*{Res}_{s=1} \phi(s) Z(s,t) \tilde{f}(s,t)$$

$$+ O\Big(\frac{\log^2 t}{\sqrt{t}} \max_{|u-2t|<t^{\varepsilon}} |\zeta(\tfrac{1}{2}+iu)|^2\Big)$$

$$\ll t^{\frac{1}{2}+\varepsilon}$$

$$+ O(t^{-A})$$

What can we say about $\zeta(\frac{1}{2}+it)$?



$$|\zeta(1+\varepsilon+it)| \leq \sum_{n\geq 1} \left|\frac{1}{n^{1+\varepsilon+it}}\right| = \sum_{n\geq 1} \frac{1}{n^{1+\varepsilon}}$$

$\mathrm{Re}(s) \geq 1+\varepsilon$

bounded in terms of t

$$\zeta(s) = \pi^{\cdots} \frac{\Gamma(\frac{1-s}{2})}{\Gamma(s)} \zeta(1-s)$$

$\mathrm{Re}(1-s) = 1+\varepsilon$

On $\mathrm{Re}(s) = -\varepsilon$

$$\approx t^{1/2}$$

$$\zeta(-\varepsilon+it) \ll t^{1/2}$$

Re(s) exponent

$$0 \longrightarrow \frac{1}{2}$$

$$1 \longrightarrow 0$$

$$\Big\} \; \frac{1}{2} \longrightarrow \frac{1}{4}$$

Convexity estimate $\;\; \zeta\left(\frac{1}{2}+it\right) \ll_\varepsilon t^{\frac{1}{4}+\varepsilon}$

Subconvex estimate

Need

$$\zeta\left(\tfrac{1}{2}+it\right) \ll t^{\frac{1}{4}-\delta}$$

$\delta > 0$

Weyl's estimate

$$\zeta\left(\tfrac{1}{2}+it\right) \ll t^{1/6}\log t$$

$$f(s,t) = \frac{8}{|\zeta(1+2it)|^2} \quad \phi(s)\, Z(s,t)\, \tilde{\gamma}(s,t)$$

Fix

Behaviour in s

$$Z(s,t) = \frac{\zeta^2(s)\,\zeta(s-2it)\,\zeta(s+2it)}{\zeta(2s)}$$

$s = \sigma + i\tau$

$$\ll \tau^B$$

$\frac{1}{2} \le \sigma \le 2$

$$2s = \underbrace{2\sigma}_{1 \le \;\; \le 4} + 2i\tau$$

$$\frac{1}{\quad} \ll \log\tau$$

$$\overline{\zeta(2\sigma+2i\tau)}$$

$$\hat{\psi}(s) = \int_0^\infty \psi(y)\, y^{-s-1}\, dy$$

repeat int. by parts

$$\ll \frac{C_A}{(1+|\tau|)^A}$$

$$I_2(t) = \operatorname*{Res}_{s=1} f(s,t) + \operatorname*{Res}_{s=1+2it} f(s,t)$$

$$+ \operatorname*{Res}_{s=1-2it} f(s,t)$$

$$+ \frac{1}{2\pi i} \int_{(1/2)} f(s,t)\, ds$$

$\zeta(s \pm 2it)$

$$\operatorname*{Res}_{s=1\pm 2it} = \frac{8}{|\zeta(1+2it)|^2}\, \hat{\psi}(1\pm 2it)\, \tilde{\gamma}(1\pm 2it, t)$$

$|\Gamma(\tfrac{1}{2}+it)|^2$

$$\ll t^A \frac{\zeta^2(1\pm 2it)\,\zeta(1\pm 4it)}{\zeta(2\pm 4it)}$$

$$\ll t^{-A'}$$

$$\frac{1}{|\Gamma(\tfrac{1}{2}+it)\,\zeta(1+2it)|^2} \ll \log^2 t \; e^{\pi t}$$

$\tilde{\gamma}(\tfrac{1}{2}+i\tau, t)$

$$= \frac{\Gamma^2\left(\frac{1}{4}+\frac{i\tau}{2}\right)\Gamma\left(\frac{1}{4}+\frac{i\tau}{2}-it\right)\cdot\Gamma\left(\frac{1}{4}+\frac{i\tau}{2}+it\right)}{8\pi^{\frac{1}{2}+i\tau}\,\Gamma\left(\frac{1}{2}+i\tau\right)}$$

$$\ll \left|t^2-\frac{\tau^2}{4}\right|^{-\frac{1}{4}} e^{-\frac{\pi}{2}\left(\left|\frac{\tau}{2}-t\right|+\left|\frac{\tau}{2}+t\right|\right)}$$

$$Z(s,t) = \frac{\zeta^4(s)\zeta(s-2it)\zeta(s+2it)}{\zeta(2s)}$$

$\mathrm{Re}(s)=\frac{1}{2}$

$$\ll \frac{\tau^{1/2+\varepsilon}\max\left|\zeta\left(\frac{1}{2}+i(\tau-2t)\right)\right|^2}{\zeta(1+2i\tau)}$$

$\left|\frac{\tau}{2}-t\right| \ll t^{\varepsilon}$

$$\int_{\mathrm{Re}(s)=1/2} f(s,t) \ll O\left(\frac{\log^2 t}{\sqrt{t}} \max_{|u-2t|\le t^{\varepsilon}} \left|\zeta\left(\tfrac{1}{2}+iu\right)\right|^2 + O\left(t^{-\delta''}\right)\right)$$

Residue at $s=1$ $\quad f(s,t)$

$$\frac{|\zeta(1+2it)|^2}{8} f(s,t) = B(s)$$

double pole at $s=1$

$$= \zeta^2(s)\,G(s)$$

$$\zeta(s) = \frac{1}{s-1} + \gamma + \text{hol in } s \text{, close}$$

$$\zeta^2(s) = \frac{1}{(s-1)^2} + \frac{2\gamma}{s-1} + \text{hol. close to } 1$$

$$\operatorname*{Res}_{s=1} \zeta^2(s)\, G(s) = \lim_{s \to 1} \frac{d}{ds}\left((s-1)^2 \zeta^2(s)\, G(s)\right)$$

$$= 2\gamma\, G(1) + G'(1)$$

$$= G(1)\left(2\gamma + \frac{G'(1)}{G(1)}\right)$$

$$G(s) = \hat{\varphi}(s)\, Z(s,t)\, \frac{\tilde{\gamma}(s,t)}{\zeta^2(s)}$$

$$G(1) = \hat{\varphi}(1)\, \frac{|\zeta(1+2it)|^2}{\zeta(2)}\, \frac{\Gamma^2(1/2)\,|\Gamma(\tfrac{1}{2}+it)|^2}{8\pi\,\Gamma(1)}$$

$$\frac{8}{|\zeta(1+2it)|^2}\, G(1) = \int_0^\infty \varphi(y)\frac{dy}{y^2}\, \frac{8}{\pi^{-1}\cancel{|\Gamma(\tfrac{1}{2}+it)|^2}\,\cancel{|\zeta(1+2it)|^2}}$$

$$\frac{\cancel{|\zeta(1+2it)|^2}}{\zeta(2)}\, \frac{\Gamma^2(1/2)\,|\cancel{\Gamma(\tfrac{1}{2}+it)}|^2}{8\pi}$$

$$= \frac{6}{\pi}\int_0^\infty \varphi(y)\, \frac{dy}{y^2}$$

$$\frac{G'}{G}(1) = \frac{\hat{\varphi}'}{\hat{\varphi}}(1) + \frac{\zeta'(1+2it)}{\zeta(1+2it)} + \frac{\zeta'}{\zeta}(1-2it)$$

$$-\frac{\zeta'}{\zeta}(2)$$

$$+\frac{\Gamma'}{\Gamma}(1/2)-\frac{\Gamma'}{\Gamma}(1)-\log\pi$$

$$+\frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{1}{2}-it\right)+\frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{1}{2}+it\right)$$

$$\frac{\Gamma'}{\Gamma}\left(\frac{1}{2}+it\right)+\frac{\Gamma'}{\Gamma}\left(\frac{1}{2}-it\right)=\log\left(\frac{1}{4}+t^2\right)+O(t^{-1})$$

$$2\log t$$

What about $\frac{\zeta'}{\zeta}(1+it) \ll \log^9 t$

Hadamard de la Vallée Poussin $\frac{\zeta'}{\zeta}(1+it) \ll \log t$

Weyl $\frac{\zeta'}{\zeta}(1+it) \ll \frac{\log t}{\log\log t}$

$$\int_{\mathbb{H}} F(z)\,d\mu_t(z) \to 0$$

Γ^{∞}

- $F(z) = u_j(z) \qquad t \to \infty$
- $F(z) = E(\psi, z)$

$$\sim \frac{\int_{\Gamma \backslash \mathbb{H}} F(z) \frac{dx\,dy}{y^2} \cdot 2\log t}{\operatorname{area}(\Gamma \backslash \mathbb{H})}$$

$L^2(\Gamma \backslash \mathbb{H}) =$ Cusp forms + incomplete Eiseins

finite sum of cusp form

$$F(z) = G_1 + G_2 + H(z)$$

$$\| H(z) \|_\infty < \varepsilon$$

$$\int_{\Gamma \backslash \mathbb{H}} F(z)\, d\mu_t(z) = \int G_1\, d\mu_t + \int G_2\, d\mu_t$$

$$\downarrow$$

$$0$$

$$+ \int_{\Gamma \backslash \mathbb{H}} H\, d\mu_t$$

$h_1(z) = \psi_1(y) \geq 0$ small in $\| \ \|_\infty$

$$E(h_1, z) \geq |H(z)|$$

$$\int_{\Gamma \backslash \mathbb{H}} E(h_1, z)\, d\mu(z) < 10\varepsilon$$

$$\left| \int_{\Gamma\backslash\mathbb{H}} H \, d\mu_t \right| \le \int_{\Gamma\backslash\mathbb{H}} E(h_1, z)\, d\mu_t(z)$$

$$\sim c \log t \int_{\Gamma\backslash\mathbb{H}} E(h_1, z)\, dx\,dy/y^2$$

$$\frac{1}{\zeta(1+it)} \ll \log t \qquad \zeta(1+it) \ll \log t$$

Meurmann $\quad L(u_j, \frac{1}{2}+it) \ll t^{1/3+\varepsilon}$ Subcon

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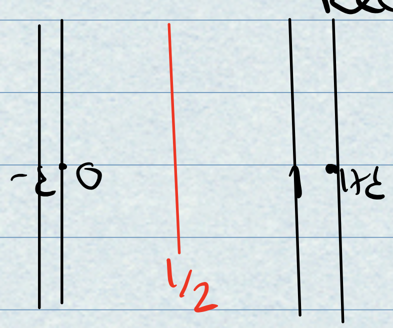
Weyl $\quad \zeta(\frac{1}{2}+it) \ll t^{1/6} \log t$

Convexity for $L(u_j, \frac{1}{2}+it)$

$$L(u_j, s) = \pi^{\cdots} \frac{\Gamma(\frac{1-s+it_j}{2})\Gamma(\frac{1-s-it_j}{2})}{\Gamma(\frac{s+it_j}{2})\Gamma(\frac{s-it_j}{2})} L(u_j, 1-s)$$

t_j fixed

$\mathrm{Re}(s)=0 \qquad \ll t^1$



$$L(u_j, \tfrac{1}{2}+it) \ll t^{1/2+\varepsilon}$$

$$\frac{\zeta'}{\zeta}(1+it) \ll \frac{\log t}{\log\log t}$$