Math 7502

Homework 9: Solutions

1. Consider the transportation problem (T) with cost matrix (c_{ij}), supply vector (p_i) and demand vector (q_j) displayed in the following array:

	2	8	9	4	6	2
3	5	3	9	3	8	2
13	5	6	3	15	7	16
6	9	20	10	22	17	25
9	3	7	3	14	9	14

(a) Use the *north-west vertex rule* to find a basic feasible solution of (T).

2	1	0	0	0	0
0	7	6	0	0	0
0	0	3	3	0	0
0	0	0	1	6	2

Explanation: Since supplier 1 can provide 3 units while demand at port 1 is only 2, supplier 1 ships 2 to port 1 and has 1 unit left, that is sent to port 2. However, port 2 demands in all 8 units, so it is now missing 7 units. These can be provided by supplier 2, who has 13. Then supplier 2 has 6 units left, that will be shipped to port 3. This port demands 9 units. So it needs 3 units more, to be shipped by supplier 3. This supplier has 6 units. By supplying 3 units to port 3, he can ship 3 units to port 4. This port wants 4 units, so it needs one more from supplier 4. This supplier has 9 units to provide, which are distributed among the last three ports: 1 for port 4, 6 for port 5 and 2 for port 6.

(b) Write down the linear program for the dual program (T^*).

For each of the 10 equations in the transportation problem we have a variable. Dual to the equations for the supplies s_i we have the variable λ_i and dual to the equations for demand d_j we have variables μ_j. The right hand of the equations form the vector $(s, d) = (3, 13, 6, 9, 2, 8, 9, 4, 6, 2)$. These go to the objective function to maximize in the dual program:

$$f = 3\lambda_1 + 13\lambda_2 + 6\lambda_3 + 9\lambda_4 + 2\mu_1 + 8\mu_2 + 9\mu_3 + 4\mu_4 + 6\mu_5 + 2\mu_6.$$

The constraints for the dual program come from the cost function, which we try to minimize in the primal program. There are 24 constraints, as many as the variables for the primal program. Every x_{ij} shows up in two equations, once in a supply equation and once in a demand equation (with coefficients 1). So the corresponding constraint is $\lambda_i + \mu_j \leq c_{ij}$. There are no restrictions to the sign of the dual variables. The constraints are, therefore,

$$\lambda_1 + \mu_1 \leq 5, \quad \lambda_2 + \mu_1 \leq 5, \quad \lambda_3 + \mu_1 \leq 9, \quad \lambda_4 + \mu_1 \leq 3,$$

$$\begin{array}{cccc} \lambda_1 + \mu_2 \leq 3, & \lambda_2 + \mu_2 \leq 6, & \lambda_3 + \mu_2 \leq 20, & \lambda_4 + \mu_2 \leq 7, \\ \lambda_1 + \mu_3 \leq 9, & \lambda_2 + \mu_3 \leq 3, & \lambda_3 + \mu_3 \leq 10, & \lambda_4 + \mu_3 \leq 3, \\ \lambda_1 + \mu_4 \leq 3, & \lambda_2 + \mu_4 \leq 15, & \lambda_3 + \mu_4 \leq 22, & \lambda_4 + \mu_4 \leq 14, \\ \lambda_1 + \mu_5 \leq 8, & \lambda_2 + \mu_5 \leq 7, & \lambda_3 + \mu_5 \leq 17, & \lambda_4 + \mu_5 \leq 9, \\ \lambda_1 + \mu_6 \leq 2, & \lambda_2 + \mu_6 \leq 16, & \lambda_3 + \mu_6 \leq 25, & \lambda_4 + \mu_6 \leq 14. \end{array}$$

(c) Use complementary slackness conditions to show that the *shipping matrix*

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 2 \\ 0 & 7 & 0 & 0 & 6 & 0 \\ 2 & 0 & 4 & 0 & 0 & 0 \\ 0 & 1 & 5 & 3 & 0 & 0 \end{pmatrix}$$

solves (T). Explain what you do.

We find λ_i, $i = 1, 2, 3, 4$ and μ_j, $j = 1, 2, 3, 4, 5, 6$ such that the corresponding constraints to the positive variables $x_{ij} > 0$ become equalities. We have freedom in one of them, so we set $\lambda_1 = 0$. This gives the table

$\lambda_i \setminus \mu_j$	-9	-4	-8	3	-3	2
0	5	3	9	$^{1}3$	8	$^{2}2$
10	5	$^{7}6$	3	15	$^{6}7$	16
18	$^{2}9$	20	$^{4}10$	22	17	25
11	3	$^{1}7$	$^{5}3$	$^{3}14$	9	14

We check that this choice of λ_i, μ_j satisfy all the constraints $\lambda_i + \mu_j \leq c_{ij}$. This gives the table with $\surd$ when the corresponding constraint is satisfied and $\times$ if not. Luckily here we get $\surd$ everywhere:

$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$
$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$
$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$
$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$

So the solution is optimal.