Math 7502

Homework 5

Due: February 21, 2008

1. Consider the linear program (P)

$$\begin{array}{lrcrcrrrl} \text{minimize} & 2x_1 & + & 5x_2 & + & 3x_3 & +5x_4 & +3x_5 & \\ \text{subject to} & x_1 & + & 3x_2 & + & x_3 & +2x_4 & +3x_5 & \geq 4 \\ & 2x_1 & + & 2x_2 & - & 2x_3 & +3x_4 & +x_5 & \geq 3 \\ & x_1 & , & x_2 & , & x_3, & x_4, & x_5 & \geq 0. \end{array}$$

∗ (i) Write down the dual program (D).

∗ (ii) Solve the dual program (D) using the simplex method.

∗ (iii) Solve the dual program graphically.

∗ (iv) Use complementary slackness to find an optimal solution to the primal program (P).

(v) Use the two-phase simplex method to solve the program (P). You should appreciate how much faster complementary slackness is.

∗ (vi) Assume that the primal program corresponds to a diet problem: We have 5 types of food A_1, A_2, A_3, A_4, A_5 providing two types of nutrients C and M. The following table summarizes the nutritional content of the types of food, their cost, and the daily requirements for a healthy diet. We ignore units.

	A_1	A_2	A_3	A_4	A_5	Daily need
C	1	3	1	2	3	4
M	2	2	-2	3	1	3
Cost	2	5	3	5	3	

Notice that the negative number can be interpreted as follows: not only A_3 does not provide nutrient M but it removes from the body 2 units of nutrient M.

Give an economic interpretation of the dual program. In particular interpret the weak duality theorem, the strong duality theorem and the complementary slackness theorem.

2. Recall the definition of a convex function: Given $f : \mathbf{R} \to \mathbf{R}$ and $x,\ y \in \mathbf{R}$, $t \in [0,1]$, we have

$$f((1-t)x + ty) \leq (1-t)f(x) + tf(y).$$

(i) Prove Jensen's inequality: Given nonnegative scalars λ_i, $i = 1, 2, \ldots, k$ with $\sum_{i=1}^{k} \lambda_i = 1$, and points x_i, $i = 1, \ldots, k$, we have for a convex function $f(x)$ the inequality

$$f\left(\sum_{i=1}^{k} \lambda_i x_i\right) \leq \sum_{i=1}^{k} \lambda_i f(x_i).$$

Hint: Use induction with a clever choice in the inductive step to produce $k-1$ nonnegative numbers with sum equal to 1.

(ii) Prove that if f is differentiable and $f'(x)$ is an increasing function, then f is convex.

Hint: Use the Mean Value Theorem.

(iii) Prove that $-\ln(x)$ is a convex function for $x > 0$.