Math 7502

Homework 4

Due: February 7, 2008

1. $*$ Use the two phase simplex algorithm to solve the linear program

$$\begin{array}{lrcrcrcr} \text{maximize} & x_1 & + & x_2 & + & x_3 & & \\ \text{subject to} & -x_1 & - & x_2 & + & x_3 & \leq & -2 \\ & x_1 & + & 2x_2 & + & x_3 & \leq & 5 \\ & 3x_1 & + & x_2 & + & x_3 & \leq & 8 \\ & x_1 & , & x_2 & , & x_3 & \geq & 0. \end{array}$$

We introduce slack variables x_4, x_5, x_6 to write the program in canonical form:

$$\begin{array}{lrcrcrcccr} \text{maximize} & x_1 & + & x_2 & + & x_3 & & & & \\ \text{subject to} & -x_1 & - & x_2 & + & x_3 & +x_4 & & & = -2 \\ & x_1 & + & 2x_2 & + & x_3 & & +x_5 & & = 5 \\ & 3x_1 & + & x_2 & + & x_3 & & & +x_6 & = 8 \\ & x_1 & , & x_2 & , & x_3 & x_4, & x_5, & x_6 & \geq 0. \end{array}$$

Although we identify the identity matrix in the last three columns of the system, we do not get automatically a basic feasible solution, as we have a negative coefficient -2 in the first equation. We rewrite the first equation as

$$x_1 + x_2 - x_3 - x_4 = +2.$$

Now we realize that we need to add an artificial variable x_7 and first minimize x_7. This leads to the first phase of the program to be

$$\begin{array}{lrcrcrccccr} \text{minimize} & & & & & & & & & x_7 & \\ \text{subject to} & x_1 & + & x_2 & - & x_3 & -x_4 & & & +x_7 & = +2 \\ & x_1 & + & 2x_2 & + & x_3 & & +x_5 & & & = 5 \\ & 3x_1 & + & x_2 & + & x_3 & & & +x_6 & & = 8 \\ & x_1 & , & x_2 & , & x_3 & x_4, & x_5, & x_6, & x_7 & \geq 0. \end{array}$$

In tableau format we get

1	1	-1	-1	0	0	1	2
1	2	1	0	1	0	0	5
3	1	1	0	0	1	0	8
0	0	0	0	0	0	-1	0

This is not a valid simplex tableau, because there is a nonzero number below the identity matrix. We add the first row to the last row to get

1	1	-1	-1	0	0	1	2
1	2	1	0	1	0	0	5
3	1	1	0	0	1	0	8
1	1	-1	-1	0	0	0	2

The basic feasible solution is now $(0,0,0,0,5,8,2)$ and is not optimal (for the minimum of x_7) as we have two positive entries on the last row. We include x_1 in the basic variables and check that the smallest quotient is $2/1$, which is $< 8/3 < 5/1$. So we pivot on the a_{11} entry. We subtract the first row from the second and the fourth, and we subtract three times the first row from the third row to get

1	1	-1	-1	0	0	1	2
0	1	2	1	1	0	-1	3
0	-2	4	3	0	1	-3	2
0	0	0	0	0	0	-1	0

The basic feasible solution is now $(2,0,0,0,3,2,0)$ and is optimal, since the entries on the last row are nonpositive. In fact the last row says that the minimum x_7 is 0, so there exists a feasible solution to the original system $(2,0,0,0,3,2)$. We ignore the artificial variable and the seventh column and go back to the maximization of $x_1 + x_2 + x_3$. This gives the tableau

1	1	-1	-1	0	0	2
0	1	2	1	1	0	3
0	-2	4	3	0	1	2
1	1	1	0	0	0	0

This is not a valid simplex tableau, as below the first column i.e. the vector e_1 we have a nonzero number. We subtract the first row from the last to get the simplex tableau

1	1	-1	-1	0	0	2
0	1	2	1	1	0	3
0	-2	4	3	0	1	2
0	0	2	1	0	0	-2

The basic feasible solution $(2,0,0,0,3,2)$ is not optimal (for $x_1+x_2+x_3$), as we have positive entries on the last row. We include x_3 in the basic variables. We check that $2/4 < 3/2$, so we pivot on the entry $a_{33} = 4$. We first divide the third row by 4 to get

1	1	-1	-1	0	0	2
0	1	2	1	1	0	3
0	-1/2	1	3/4	0	1/4	1/2
0	0	2	1	0	0	-2

We add the third row to the first and subtract twice the third row from the second and fourth rows to get

1	1/2	0	-1/4	0	1/4	5/2
0	2	0	-1/2	1	-1/2	2
0	-1/2	1	3/4	0	1/4	1/2
0	1	0	-1/2	0	-1/2	-3

The basic feasible solution is now $(5/2, 0, 1/2, 0, 2, 0)$ and is not optimal, as we still have positive coefficients in the last row. We include x_2 in the basic variables. We check that $2/2 < (5/2)/(1/2)$, so we pivot on the $a_{22} = 2$ entry. We divide the second row by 2 to get

1	1/2	0	-1/4	0	1/4	5/2
0	1	0	-1/4	1/2	-1/4	1
0	-1/2	1	3/4	0	1/4	1/2
0	1	0	-1/2	0	-1/2	-3

We now subtract the second row from the fourth, subtract half the second row from the first and add half the second row to the third to get

1	0	0	-1/8	-1/4	3/8	2
0	1	0	-1/4	1/2	-1/4	1
0	0	1	5/8	1/4	1/8	1
0	0	0	-1/4	-1/2	-1/4	-4

The basic feasible solution is now $(2, 1, 1, 0, 0, 0)$ and is optimal, as the coefficients on the last row are nonpositive. The maximum of $x_1 + x_2 + x_3$ is achieved at this basic feasible solution and is 4.

Graphing in this problem is difficult, as we have 3 variables and the equations represent planes in $\mathbf{R}^3$. So we avoid a graphical approach.

2. $*$ (a) Use the two phase simplex algorithm to solve the linear program

$$\begin{array}{lrcrcl} \text{minimize} & x_1 & + & x_2 & & \\ \text{subject to} & 4x_1 & + & x_2 & \geq & 4 \\ & x_1 & + & 6x_2 & \geq & 6 \\ & 6x_1 & + & 10x_2 & \geq & 23 \\ & x_1 & , & x_2 & \geq & 0. \end{array}$$

We use slack variables to write the system in canonical form

$$\begin{array}{lrcrrrrrl}
\text{minimize} & x_1 & + & x_2 & & & & & \\
\text{subject to} & 4x_1 & + & x_2 & -x_3 & & & & = 4 \\
 & x_1 & + & 6x_2 & & -x_4 & & & = 6 \\
 & 6x_1 & + & 10x_2 & & & & -x_5 & = 23 \\
 & x_1 & , & x_2, & x_3, & x_4, & & x_5 & \geq 0.
\end{array}$$

In tableau format we have

4	1	-1	0	0	4
1	6	0	-1	0	6
6	10	0	0	-1	23
-1	-1	0	0	0	0

As we do not have at the same time the identity matrix and positive constants on the right column we introduce artificial variables x_6, x_7, x_8. The first phase of the two-phase simplex is

$$\begin{array}{lrcrrrrrrrl}
\text{minimize} & & & & & & & x_6 & +x_7 & +x_8 & \\
\text{subject to} & 4x_1 & + & x_2 & -x_3 & & & +x_6 & & & = 4 \\
 & x_1 & + & 6x_2 & & -x_4 & & & +x_7 & & = 6 \\
 & 6x_1 & + & 10x_2 & & & -x_5 & & & +x_8 & = 23 \\
 & x_1 & , & x_2, & x_3, & x_4, & x_5, & x_6, & x_7, & x_8 & \geq 0.
\end{array}$$

In tableau format we have

4	1	-1	0	0	1	0	0	4
1	6	0	-1	0	0	1	0	6
6	10	0	0	-1	0	0	1	23
0	0	0	0	0	-1	-1	-1	0

This is not a valid simplex tableau, as we have nonzero entries below the identity matrix. We add the first three rows to the last one.

4	1	-1	0	0	1	0	0	4
1	6	0	-1	0	0	1	0	6
6	10	0	0	-1	0	0	1	23
11	17	-1	-1	-1	0	0	0	33

The basic feasible solution is $(0, 0, 0, 0, 0, 4, 6, 23)$ and is not optimal, due to the positive entries on the last row. We decide to include x_2 in the basic variables, since $17 > 11$ (Dantzig's rule). Since $6/6 < 23/10 < 4/1$ we pivot on the $a_{22} = 6$ entry.

4	1	-1	0	0	1	0	0	4
1/6	1	0	-1/6	0	0	1/6	0	1
6	10	0	0	-1	0	0	1	23
11	17	-1	-1	-1	0	0	0	33

23/6	0	-1	1/6	0	1	-1/6	0	3
1/6	1	0	-1/6	0	0	1/6	0	1
13/3	0	0	5/3	-1	0	-5/3	1	13
49/6	0	-1	11/6	-1	0	-17/6	0	16

The basic feasible solution is $(0, 1, 0, 0, 0, 4, 0, 23)$ and is not optimal (for the minimum of $x_6 + x_7 + x_8$), since we have positive entries on the last row. Because $49/6 > 11/6$ we choose to include x_1 in the basic variables (Dantzig's rule). Since

$$\frac{3}{23/6} < \frac{13}{13/3} < \frac{1}{1/6}$$

we pivot on the $a_{11} = 23/6$ entry.

1	0	-6/23	1/23	0	6/23	-1/23	0	18/23
1/6	1	0	-1/6	0	0	1/6	0	1
13/3	0	0	5/3	-1	0	-5/3	1	13
49/6	0	-1	11/6	-1	0	-17/6	0	16

1	0	-6/23	1/23	0	6/23	-1/23	0	18/23
0	1	1/23	-4/23	0	-1/23	4/23	0	20/23
0	0	26/23	34/23	-1	-26/23	-34/23	1	221/23
0	0	26/23	34/23	-1	-49/23	-57/23	0	221/23

The basic feasible solution is $(18/23, 20/23, 0, 0, 0, 0, 0, 221/23)$ and is not optimal (for the minimum of $x_6 + x_7 + x_8$), since we have positive entries on the last row. Since $34/23 > 26/23$, we include x_4 in the basic variables (Dantzig's rule). Since

$$\frac{221/23}{34/23} < \frac{20/23}{1/23}$$

we pivot on the $a_{34} = 26/23$ entry. We get

1	0	-6/23	1/23	0	6/23	-1/23	0	18/23
0	1	1/23	-4/23	0	-1/23	4/23	0	20/23
0	0	13/17	1	-23/34	-13/17	-1	23/34	13/2
0	0	26/23	34/23	-1	-49/23	-57/23	0	221/23

1	0	-5/17	0	1/34	5/17	0	-1/34	1/2
0	1	3/17	0	-2/17	-3/17	0	2/17	2
0	0	13/17	1	-23/34	-13/17	-1	23/34	13/2
0	0	0	0	0	-1	-1	-1	0

The basic feasible solution is $(1/2, 2, 0, 13/2, 0, 0, 0, 0)$ and is optimal, as the entries on the last row are nonpositive. The maximum of $-x_6 - x_7 - x_8$ is 0, i.e. the minimum of $x_6 + x_7 + x_8$ is 0. Consequently the original program has a feasible point and we can proceed to the second phase of the two-phase simplex algorithm by erasing the columns of the artificial variables. We get

1	0	-5/17	0	1/34	1/2
0	1	3/17	0	-2/17	2
0	0	13/17	1	-23/34	13/2
-1	-1	0	0	0	0

This is not a valid simplex tableau, because we have nonzero entries below the identity matrix. We add the first two rows to the last one to get

1	0	-5/17	0	1/34	1/2
0	1	3/17	0	-2/17	2
0	0	13/17	1	-23/34	13/2
0	0	-2/17	0	-3/34	5/2

The basic feasible solution is now $(1/2, 2, 0, 13/2, 0)$ and is optimal, since the entries in the last row are nonpositive. The maximum of $-x_1 - x_2$ is $-5/2$, i.e. the minimum of $x_1 + x_2$ is $5/2 = 1/2 + 2$.

$*$ (b) Solve the same problem graphically and explain what the two phase simplex algorithm does geometrically (on the graph).

The arrows show that we started at $(0, 0)$ which was NOT feasible for the original problem. Then we moved to two more non feasible points for the original problem $(0, 1)$ and $(18/23, 20/23)$. Finally we reached a basic feasible solution for the original problem at $(1/2, 2)$ at the end of the first phase of the method. This point is now in the feasible region and is optimal, so we do not need to continue to find a better basic feasible solution to the original program.

3. Let $x_1, x_2, \ldots, x_k$ be points in $\mathbf{R}^n$. We say that $y \in \mathbf{R}^n$ is a convex combination of $x_1, \ldots, x_k$ if we can find scalars $\lambda_1, \lambda_2, \ldots, \lambda_k$ such that

$$y = \sum_{j=1}^{k} \lambda_j x_j, \quad \lambda_j \geq 0, \quad j = 1, \ldots, k, \quad \sum_{j=1}^{k} \lambda_j = 1.$$

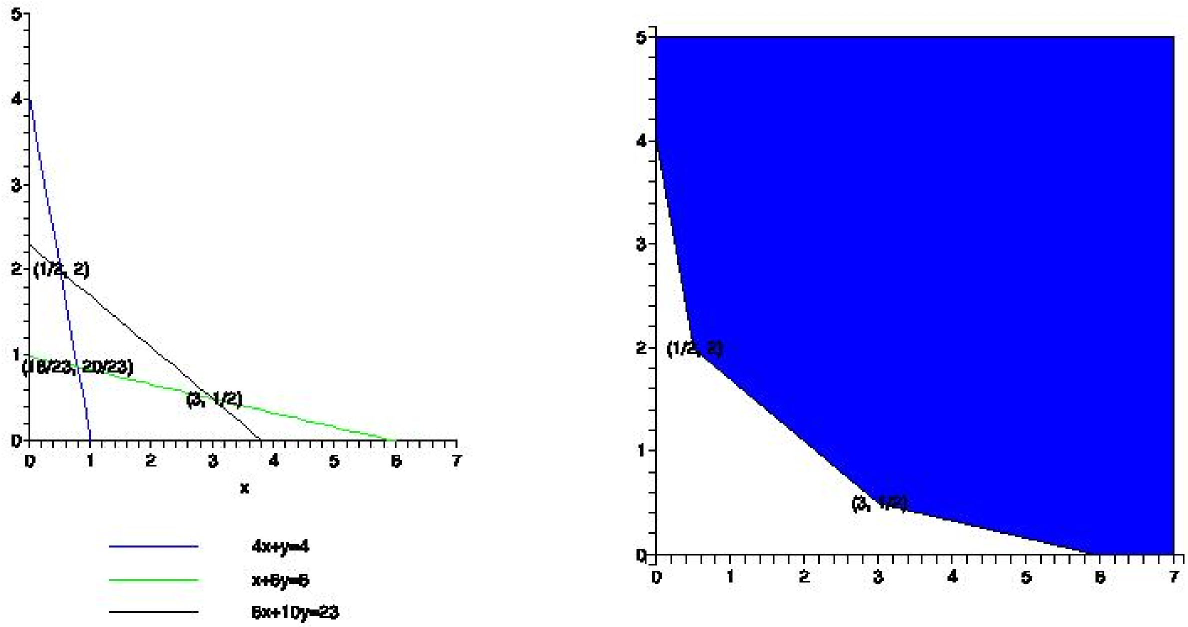

Figure 1: The unbounded feasible region for problem 2

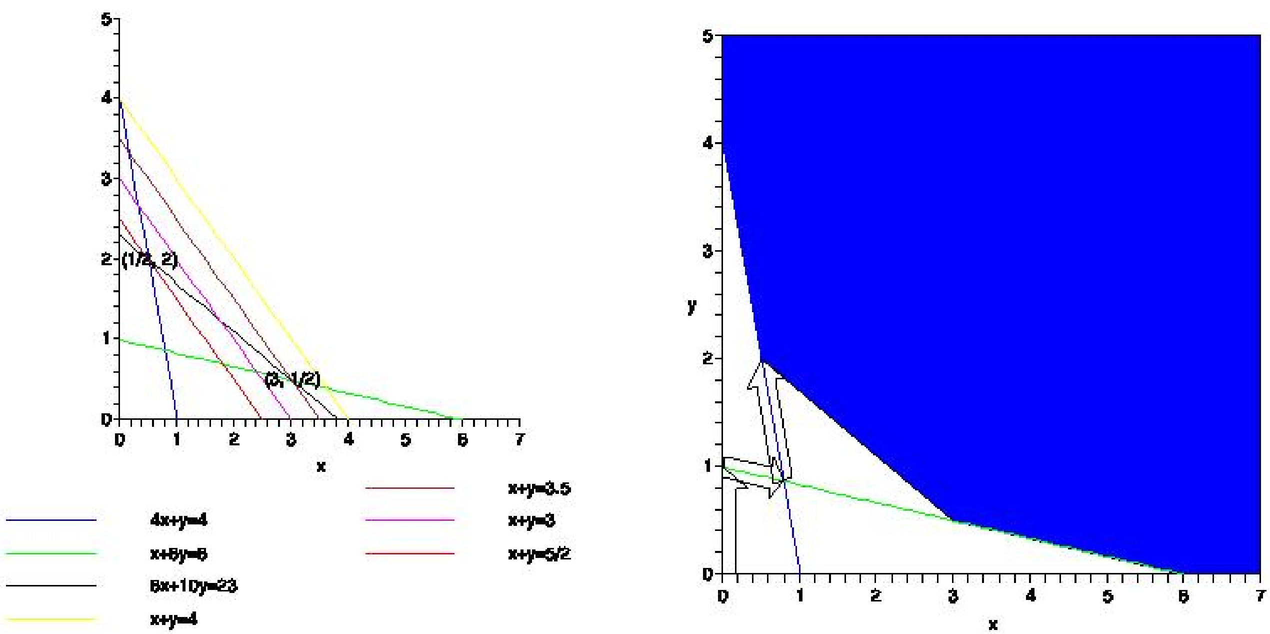

Figure 2: Graphical solution for problem 2 and the path of the two phase simplex

(a) Let S be the set of convex combinations of $x_1, \dots, x_k$. Prove that S is a convex set. The set S is called the convex hull of $x_1, \dots, x_k$.

Let y and z be in S, i.e. are convex combinations of $x_1, \dots, x_k$. Then we can find scalars $\lambda_1, \lambda_2, \dots, \lambda_k$ and $\mu_1, \mu_2, \dots, \mu_k$ such that

$$y = \sum_{j=1}^{k} \lambda_j x_j, \quad z = \sum_{j=1}^{k} \mu_j x_j, \quad \lambda_j, \mu_j \geq 0, \quad j = 1, \dots, k, \quad \sum_{j=1}^{k} \lambda_j = 1, \quad \sum_{j=1}^{k} \mu_j = 1.$$

Let $t \in [0, 1]$. We need to show that $(1-t)y + tz \in S$. We have

$$(1-t)y + tz = \sum_{j=1}^{k} ((1-t)\lambda_j + t\mu_j)x_j.$$

The coefficients $(1-t)\lambda_j + t\mu_j$ are nonnegative, as linear combinations of nonnegative numbers. Moreover,

$$\sum_{j=1}^{k} (1-t)\lambda_j + t\mu_j = (1-t)\sum_{j=1}^{k} \lambda_j + t\sum_{j=1}^{k} \mu_j = (1-t) \cdot 1 + t \cdot 1 = 1 - t + t = 1.$$

This proves that $(1-t)y + tz$ is a convex combination of x and y.

(b) Let y be a convex combination of a and $b \in \mathbf{R}^n$. Assume also that a, b are convex combinations of $x_1, \dots, x_k$. Prove that y is a convex combination of $x_1, \dots, x_k$.

We are given that we can find a $t \in [0, 1]$ such that

$$y = (1-t)a + tb.$$

Moreover, since a and b are convex combinations of $x_1, \dots, x_k$, we can find scalars $\lambda_1, \lambda_2, \dots, \lambda_k$ and $\mu_1, \mu_2, \dots, \mu_k$ such that

$$a = \sum_{j=1}^{k} \lambda_j x_j, \quad b = \sum_{j=1}^{k} \mu_j x_j, \quad \lambda_j, \mu_j \geq 0, \quad j = 1, \dots, k, \quad \sum_{j=1}^{k} \lambda_j = 1, \quad \sum_{j=1}^{k} \mu_j = 1.$$

Then

$$y = (1-t)a + tb = \sum_{j=1}^{k} ((1-t)\lambda_j + t\mu_j)x_j.$$

The coefficients $(1-t)\lambda_j + t\mu_j$ are nonnegative, as linear combinations of nonnegative numbers. Moreover,

$$\sum_{j=1}^{k} (1-t)\lambda_j + t\mu_j = (1-t)\sum_{j=1}^{k} \lambda_j + t\sum_{j=1}^{k} \mu_j = (1-t) \cdot 1 + t \cdot 1 = 1 - t + t = 1.$$

(c) Show that the convex hull of $(0,0)$, $(1,0)$, $(0,1)$ and $(1,1)$ is the square $[0,1]\times[0,1]$. The difficult part is to show that every point in the square is a convex combination of the four extreme points of the square. Write (x,y) as a convex combination of $(0,y)$ and $(1,y)$ first and use (b).

Let y be a convex combination of the four points, i.e. for some scalars $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ we have

$$y = \lambda_1(0,0) + \lambda_2(1,0) + \lambda_3(0,1) + \lambda_4(1,1) = (\lambda_2 + \lambda_4, \lambda_3 + \lambda_4),$$

$$\lambda_j \geq 0, \quad j = 1, \ldots, 4, \quad \sum_{j=1}^{4} \lambda_j = 1.$$

To show that y belongs to the square, we need to show that its coordinates are in the interval $[0,1]$. As $\lambda_j \geq 0$, we have

$$\lambda_2 + \lambda_4 \geq 0, \quad \lambda_3 + \lambda_4 \geq 0.$$

On the other hand, the sum of the four coefficients is 1, while they are all nonnegative. This implies that

$$\lambda_2 + \lambda_4 \leq \sum_{j=1}^{4} \lambda_j = 1, \quad \lambda_3 + \lambda_4 \leq \sum_{j=1}^{4} \lambda_j = 1.$$

The converse: Let $z \in [0,1] \times [0,1]$. We need to write $z = (x,y)$ as a convex combination of the four points. We first notice that, given that $x \in [0,1]$, that

$$(x,y) = (1-x)(0,y) + x(1,y),$$

i.e. (x,y) is a convex combination of $(0,y)$ and $(1,y)$. Now these two points are convex combinations of the vertices:

$$(0,y) = (1-y)(0,0) + y(0,1), \quad (1,y) = (1-y)(1,0) + y(1,1),$$

as $y \in [0,1]$. Using (b) we conclude that (x,y) is a convex combination of the four points.

Remark: If one insists, one can write the convex combination more explicitly

$$(x,y) = (1-x)(0,y)+x(1,y) = (1-x)(1-y)(0,0)+(1-x)y(0,1)+x(1-y)(1,0)+xy(1,1).$$