Math 7502

Homework 2

Due: January 24, 2008

In this homework you will work on two problems with the simplex method, as presented in class. For both follow the instructions:

(a) Find an initial basic feasible solution for the system in canonical form.

(b) Check whether it is optimal.

(c) Search for another basic feasible solution.

(d) Move to this better solution.

(e) Go back to (b).

(f) After you find the optimal solution with the simplex method, graph the feasible region in standard form, and show the successive vertices of it visited during the simplex method.

Provide explanations for each step, as done in class.

1. (Continuation from Homework 1)

   Maximize the daily profit in manufacturing two alloys A_1 and A_2 which are different mixtures of two metals M_1 and M_2 as shown:

Metal	Proportion of metal In Alloy A_1	Proportion of metal In Alloy A_2	Daily supply in tons
M_1	0.5	0.25	10
M_2	0.5	0.75	15
Net Profit per ton	30	25	

   We wrote the system in canonical form

$$\begin{aligned} \text{maximize} \quad & 30x_1 + 25x_2 \\ \text{subject to} \quad & x_1, x_2, x_3, x_4 \geq 0 \\ & 0.5x_1 + 0.25x_2 + x_3 = 10 \\ & 0.5x_1 + 0.75x_2 + x_4 = 15 \end{aligned}$$

   where x_1 and x_2 are the production of alloy A_1 and A_2 in tons per day, and x_3 and x_4 are slack variables.

   (a) We start with basic variables x_3 and x_4 and nonbasic x_1 and x_2, as it is easy to find x_3 and x_4 given that $x_1 = x_2 = 0$. We find $x_3 = 10$ and $x_4 = 15$. So the basic feasible solution we start with is $(x_1, x_2, x_3, x_4) = (0, 0, 10, 15)$.

(b) We need to check for optimality. In this case $f(x_1, x_2) = 30x_1 + 25x_2$ gives $f(0,0) = 0$. Intuitively this cannot be the maximum daily profit. Since the coefficients of the objective function are 30 and 25, any increase of either x_1 or x_2 from the current values 0 will increase the objective function.

(c) We solve for the basic variables:

$$\begin{aligned} x_3 &= 10 - 0.5x_1 - 0.25x_2 && (1) \\ x_4 &= 15 - 0.5x_1 - 0.75x_2 && (2) \end{aligned}$$

If we keep $x_2 = 0$, we can increase x_1, while at the same time we keep $x_3 \geq 0$ and $x_4 \geq 0$.

The maximum increase Δx_1 for x_1 out of (1) can be found by setting $x_3 = 0$ in (1). The maximum increase Δx_1 for x_1 out of (2) can be found by setting $x_4 = 0$ in (2).

$$(1) \Longrightarrow \Delta x_1 = 20, \quad (2) \Longrightarrow \Delta x_1 = 30. \tag{3}$$

This means that we can increase x_1 at most 20, the minimum of the two numbers, in order to remain in the feasible region (all variables nonnegative). An increase of 20 for x_1 implies an increase Δf of $30 \cdot 20 = 600$ for the objective function.

If we keep $x_1 = 0$, we can increase x_2, while at the same time we keep $x_3 \geq 0$ and $x_4 \geq 0$.

The maximum increase Δx_2 for x_2 out of (1) can be found by setting $x_3 = 0$ in (1). The maximum increase Δx_2 for x_2 out of (2) can be found by setting $x_4 = 0$ in (2).

$$(1) \Longrightarrow \Delta x_2 = 40, \quad (2) \Longrightarrow \Delta x_2 = 15\frac{4}{3} = 20. \tag{4}$$

This means that we can increase x_2 at most 20, the minimum of the two numbers, in order to remain in the feasible region (all variables nonnegative). An increase of 20 for x_2 implies an increase Δf of $25 \cdot 20 = 500$ for the objective function.

It is clear that this way we find two basic feasible solutions: $(20, 0, 0, 5)$ and $(0, 20, 5, 0)$ giving increases of the objective function 600 and 500 respectively. Since we want to maximize the profit, we choose the first option.

(d) We move to the basic feasible solution $(20, 0, 0, 5)$. Now the basic variables are x_1 and x_4, while x_2 and x_3 are nonbasic ($= 0$).

(b') We check whether this new basic solution is optimal. We express f as a function of the nonbasic variables x_2 and x_3. Using (1), we get

$$f = 30x_1 + 25x_2 = 30 \cdot 2 \cdot (10 - 0.25x_2 - x_3) + 25x_2 = 600 + 10x_2 - 60x_3. \tag{5}$$

If we increase x_2 from its current value of 0, we can increase the objective function, as the coefficient of x_2 is $10 > 0$. So the solution we have is not optimal.

(c') We search for another basic feasible solution. We solve for the basic variables, which are x_1 and x_4. We get

$$x_1 = 2(10 - 0.25x_2 - x_3) \quad (6)$$

$$x_4 = 15 - 0.75x_2 - 10 + 0.25x_2 + x_3 = 5 - 0.5x_2 + x_3 \quad (7)$$

Because the coefficient of x_3 in (5) is positive, we cannot increase the objective function by taking larget values of x_3 than the current value of 0.

We search for the maximum increase of x_2 that will keep us in the feasible region.

If we keep $x_3 = 0$, we can increase x_2, while at the same time we keep $x_1 \geq 0$ and $x_4 \geq 0$.

The maximum increase Δx_2 for x_2 out of (6) can be found by setting $x_1 = 0$ in (6). The maximum increase Δx_2 for x_2 out of (7) can be found by setting $x_4 = 0$ in (7).

$$(6) \Longrightarrow \Delta x_2 = 40, \quad (7) \Longrightarrow \Delta x_2 = 10. \quad (8)$$

This means that we can increase x_2 at most 10, the minimum of the two numbers, in order to remain in the feasible region (all variables nonnegative). An increase of 10 for x_1 implies an increase Δf of $10 \cdot 20 = 100$ for the objective function, using (5).

(d') and (b") We move to the new basic feasible solution $(15, 10, 0, 0)$. We need to check whether it is optimal. Now the basic variables are x_1 and x_2 and the nonbasic variables are x_3 and x_4. We express the objective function in terms of the nonbasic variables. Using (5) and (7), we get

$$f = 600 + 10 \cdot 2 \cdot (5 - x_4 + x_3) - 60x_3 = 600 + 100 - 20x_4 + 20x_3 - 60x_3 = 700 - 20x_4 - 40x_3.$$

Since the coefficients of x_3 and x_4 are now negative, this is the optimal solution.

(f) We successively moved to the following sequence of basic feasible solutions (in canonical form): $(0, 0, 10, 15)$, $(20, 0, 0, 5)$, $(15, 10, 0, 0)$. In standard form we moved successively to the following points $(0, 0)$, $(20, 0)$, $(15, 10)$. This is seen in the figure below.

2. Maximize $3x_1 + 2x_2$ subject to the constraints: $x_1 \geq 0$, $x_2 \geq 0$,

$$3x_1 + 4x_2 \leq 60, \quad 4x_1 + 2x_2 \leq 60, \quad 10x_1 + 2x_2 \leq 120.$$

We write the system in canonical form:

$$\begin{array}{rc} \text{maximize} & 3x_1 + 2x_2 \\ \text{subject to} & 3x_1 + 4x_2 + x_3 = 60 \\ & 4x_1 + 2x_2 + x_4 = 60 \\ & 10x_1 + 2x_2 + x_5 = 120 \\ & x_i \geq 0, i = 1, 2, 3, 4, 5. \end{array}$$

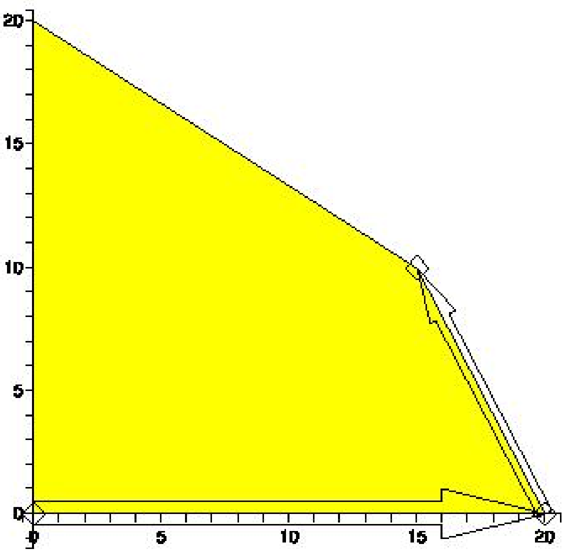

Figure 1: The order of visiting the vertices in the simplex algorithm for problem 1

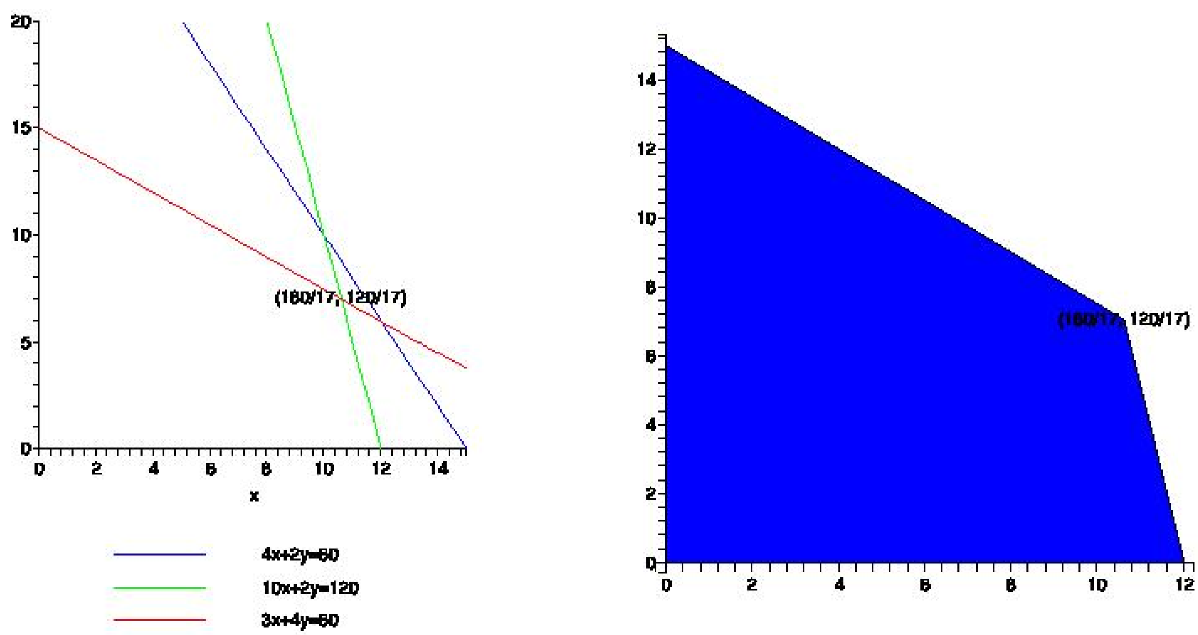

Figure 2: The feasible region in problem 2

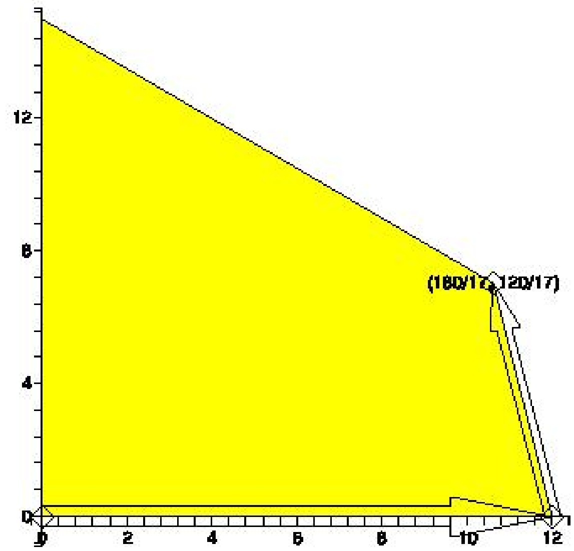

Figure 3: The order of visiting the vertices in the simplex algorithm in problem 2

(a) We start with a basic feasible solution with $x_1 = x_2 = 0$. This way x_1 and x_2 are nonbasic variables and x_3, x_4, x_5 are basic variables. We easily find from the system above the values

$$x_3 = 60, \quad x_4 = 60, \quad x_5 = 120,$$

so that the basic feasible solution is $(x_1, x_2, x_3, x_4, x_5) = (0, 0, 60, 60, 120)$.

(b) The corresponding value of the objective function $f = 3x_1 + 2x_2$ is 0. This function has coefficients of x_1 and x_2 positive, so any increase of the values of x_1 or x_2 from the current values of 0, will increase the objective function. So the basic feasible solution $(0, 0, 60, 60, 120)$ is not optimal.

(c) We solve the system for the basic variables:

$$x_3 = 60 - 3x_1 - 4x_2 \quad (9)$$

$$x_4 = 60 - 4x_1 - 2x_2 \quad (10)$$

$$x_5 = 120 - 10x_1 - 2x_2 \quad (11)$$

If we keep $x_2 = 0$, we can increase x_1, while at the same time we keep $x_3 \geq 0$, $x_4 \geq 0$ and $x_5 \geq 0$.

The maximum increase Δx_1 for x_1 out of (9) can be found by setting $x_3 = 0$ in (9). The maximum increase Δx_1 for x_1 out of (10) can be found by setting $x_4 = 0$ in (10).

The maximum increase Δx_1 for x_1 out of (11) can be found by setting $x_5 = 0$ in (11).

$$(9) \Longrightarrow \Delta x_1 = 20, \quad (10) \Longrightarrow \Delta x_1 = 15, \quad (11) \Longrightarrow \Delta x_1 = 12. \tag{12}$$

This means that we can increase x_1 at most 12, the minimum of the three numbers, in order to remain in the feasible region (all variables nonnegative). An increase of 12 for x_1 implies an increase Δf of $3 \cdot 12 = 36$ for the objective function.

If we keep $x_1 = 0$, we can increase x_2, while at the same time we keep $x_3 \geq 0$, $x_4 \geq 0$ and $x_5 \geq 0$.

The maximum increase Δx_2 for x_2 out of (9) can be found by setting $x_3 = 0$ in (9). The maximum increase Δx_2 for x_2 out of (10) can be found by setting $x_4 = 0$ in (10). The maximum increase Δx_2 for x_2 out of (11) can be found by setting $x_5 = 0$ in (11).

$$(9) \Longrightarrow \Delta x_2 = 15, \quad (10) \Longrightarrow \Delta x_2 = 30, \quad (11) \Longrightarrow \Delta x_2 = 60. \tag{13}$$

This means that we can increase x_2 at most 15, the minimum of the three numbers, in order to remain in the feasible region (all variables nonnegative). An increase of 15 for x_2 implies an increase Δf of $2 \cdot 15 = 30$ for the objective function.

Since the largest increase of the objective function occurs when we set $x_2 = 0$ and $\Delta x_1 = 12$, we choose as basic variables x_1 and x_3 and x_4 and nonbasic x_2 and x_5, i.e we remove x_5 and include x_1. The new basic feasible solution is $(12, 0, 24, 12, 0)$, using (9), (10), (11).

We need to check whether it is optimal or not. For this we rewrite the objective function in terms of the nonbasic variables x_2 and x_5. Eq. (11) gives $x_1 = (1/10)(120 - 2x_2 - x_5)$ so that

$$f = 3 \cdot \frac{1}{10}(120 - 2x_2 - x_5) + 2x_2 = 36 - \frac{6}{10}x_2 - \frac{3}{10}x_5 + 2x_5 = 36 + \frac{14}{10}x_2 - \frac{3}{10}x_5.$$

Since the coefficient of x_5 is negative, we cannot increase x_5 and increase the objective function. On the other hand, if we increase x_2, we can increase f. So we decide to include x_2 in the basic variables. We solve the equations (9), (10), (11) to express the basic variables x_1, x_3 and x_4, in terms of the nonbasic x_2 and x_5. We get

$$x_1 = 12 - 0.2x_2 - 0.1x_5 \tag{14}$$

$$x_3 = 60 - 3 \cdot (12 - 0.2x_2 - 0.1x_5) - 4x_2 = 24 - 3.4x_2 + 0.3x_5 \tag{15}$$

$$x_4 = 60 - 4(12 - 0.2x_2 - 0.1x_5) - 2x_2 = 12 - 1.2x_2 + 0.4x_5 \tag{16}$$

If we keep $x_5 = 0$, we can increase x_2, while at the same time we keep $x_3 \geq 0$, $x_4 \geq 0$ and $x_1 \geq 0$.

The maximum increase Δx_2 for x_2 out of (14) can be found by setting $x_1 = 0$ in (14). The maximum increase Δx_2 for x_2 out of (15) can be found by setting $x_3 = 0$ in (10). The maximum increase Δx_2 for x_2 out of (16) can be found by setting $x_4 = 0$ in (16).

$$(14) \Longrightarrow \Delta x_2 = 60, \quad (15) \Longrightarrow \Delta x_2 = \frac{120}{17}, \quad (16) \Longrightarrow \Delta x_2 = 100. \tag{17}$$

This means that we can increase x_2 at most 120/17, the minimum of the three numbers, in order to remain in the feasible region (all variables nonnegative). An increase of 120/17 for x_2 implies an increase Δf of $\frac{14}{10} \cdot \frac{120}{17} = \frac{168}{17}$ for the objective function. The new basic feasible solution is $\left(\frac{180}{17}, \frac{120}{17}, 0, 12 - \frac{144}{17}, 0\right)$. We get these numbers by plugging $x_2 = 120/17$ and $x_5 = 0$ into (14), (15), (16).

We need to check whether this basic feasible solution is optimal or not. We express the objective function in terms on the nonbasic variables x_3 and x_5. Out of (15) we get $x_2 = (10/34)(24 - x_3 + 0.3x_5)$ so that

$$f = 36 + 1.4 \cdot \frac{10}{34}(24 - x_3 + 0.3x_5) - 0.3x_5 = 36 + \frac{168}{17} - \frac{7}{17}x_3 - \frac{3}{17}x_5.$$

Since the coefficients of x_3 and x_5 are negative, we can no longer increase f and the basic solution found in optimal. The maximum of f is $780/17 = 45.8823...$.