1 Symplectic geometry

1.1 Archimedes' theorem

Theorem 1.1 (Archimedes)**.** *The area of a unit sphere* $\mathbb{S}^2$ *is equal to the area of the cylinder of radius* 1 *and a height* 2*.*

Exercise 1.2. Assume that the sphere is inscribed in the cylinder. Project the sphere radially onto the cylinder from the cylinder's symmetry axis. Prove that the map preserves the area.

Question 1.3. What about other dimensions?

1.2 Poincare's last geometric theorem

Theorem 1.4 (Poincare-Birkhoff)**.** *Every area and orientation preserving homeomorphism of an annulus that rotates the two boundaries in opposite directions has at least two fixed points.*

Exercise 1.5. Provide an example where the theorem fails when the area-preserving condition is omitted.

1.3 Symplectic forms

Example 1.6. $dx \wedge dy$ on $\mathbb{R}^2$. More generally, on $\mathbb{R}^{2n}$,

$$\omega = \sum_{i=1}^{n} dx_i \wedge dy_i. \tag{1}$$

Clearly, ω can be used to measure the *area* of any oriented surface in $\mathbb{R}^{2n}$.

Exercise 1.7. Calculate ω^n.

Example 1.8. Let M be a manifold and let T^*M be the co-tangent bundle. There is a **canonical/tautological/Liouville**, 1-form θ on the bundle.

Definition. Take a point $(x, q) \in T^*M$. Take $v \in TT^*M$. Set

$$\theta(v) = q(\pi v),$$

where $\pi : T^*M \to M$ is the projection.
Define $\omega = d\theta$. Prove that ω^n is a volume form.

Definition 1.9. A *symplectic form* ω on an even dimensional manifold M^{2n} is a 2-form such that

1. $d\omega = 0$.

2. ω^n never vanishes.

A map $f : M^{2n} \to M^{2n}$ is a symplectomorphism if $f^*\omega = \omega$.

Exercise 1.10. Let $z_1, \ldots, z_n$ be coordinates in $\mathbb{C}^n$ with $z_k = x_k + iy_k$. Take any complex-analytic submanifold $V \subset \mathbb{C}^n$. Prove that $\omega = \sum_k dx_k \wedge dy_k$ restricts to V as a symplectic form.

1.4 Motivation: classical Mechanics

Question 1.11. So, why Poincaré was interested in symplectomorphsims??

Answer: he was wondering whether the Moon will hit the Earth at some point. The Newton's equations of motion can be written

$$\ddot{\boldsymbol{x}} = -\nabla U(\boldsymbol{x}) \quad (\boldsymbol{x} \in \mathbb{R}^N, \; U - \text{a function})$$

This can be rewritten as linear equation:

$$\dot{\boldsymbol{x}} = \boldsymbol{y}, \;\; \dot{\boldsymbol{y}} = -\nabla U.$$

Observation: the flow on $\mathbb{R}^{2n}$ preserves $\omega = d\theta$, and preserves $E = \frac{1}{2}\boldsymbol{y}^2 + U$.

1.5 Hamiltonians

Definition 1.12. Let (M, ω) be a symplectic manifold and H (for Hamiltonian) be a smooth function on it. The *Hamiltonian vector field* X_H of H is the unique vector field X_H satisfying

$$dH(Y) = \omega(X_H, Y).$$

Exercise 1.13. Check that $\frac{\partial H}{\partial X_H} = 0$.

Example 1.14. Consider again the standard symplectic form $\sum_i dx_i \wedge dy_i$.

1. $X_{x_i} = -\frac{\partial}{\partial y_i}$, $X_{y_i} = \frac{\partial}{\partial x_i}$.

2. Set $H = \frac{1}{2}\sum_i (x_i^2 + y_i)^2$. Then $X_H = \sum_i(-x_i \frac{\partial}{\partial y_i} + y_i \frac{\partial}{\partial x_i})$.

Conjecture 1.15 (Arnold's conjecture)**.** *Let (M, ω) be a compact symplectic manifold. Call a symplectomorphism $F : M \to M$* Hamiltonian *if it is generated by a time-dependent Hamiltonian flow. Then*

$$\#\{\textit{fixed points of } F\} \geq$$
$$\geq \{\textit{minimal number of critical points of a smooth function on } M\}.$$

1.6 Hamiltonian reduction, $\mathbb{C}P^n$, non-squeezing.

Consider Example 1.14(2). Take the level set $H = \frac{1}{2}$ - the unit sphere $\mathbb{S}^{2n-1}$. The field X_H generates a circle action on $\mathbb{S}^{2n-1}$ and the quotient is $\mathbb{C}P^{n-1}$.

Lemma 1.16. *The restricted form $\omega|_{\mathbb{S}^{2n-1}}$ descends to a symplectic form on $\mathbb{C}P^{n-1}$.*

Proof. The orbits of the S^1-action span the kernel of $\omega|_{\mathbb{S}^{2n-1}}$ and ω is invariant under the action. □

Moral I. Complex projective manifolds are symplectic.

Moral II The symplectic structure *can see* the unit sphere - its kernel integrates to the Hopf fibration. Two smooth balls of the same volume will not be symplectomorphic.

Remark 1.17 (Symplectic cut)**.** Consider the unit ball $\{H \leq 1\} \subset \mathbb{R}^{2n}$ and quotient its boundary $\mathbb{S}^{2n-1}$ by S^1. This is $\mathbb{C}P^n$.

Exercise 1.18. Consider now the set $H \geq 1$ and contract all S^1-orbits in it's boundary $\mathbb{S}^{2n-1}$ to points. Prove that the resulting space is a simple blow up of $\mathbb{C}^n$.

Theorem 1.19 (Gromov's non-squeezing)**.** *For any $a \in (0, 1)$ one can not symplectically embed the unit ball $\{|z| \leq 1\} \subset \mathbb{C}^n$ into the subset $|z_1| < a$.*

1.7 Darboux's theorem

Theorem 1.20 (Darboux)**.** *Let (M^{2n}, ω) be a 2n-dimensional symplectic manifold, and let $p \in M^{2n}$ be a point. Then there is a coordinate chart*

$(U, x_1, \ldots, x_n, y_1, \ldots, y_n)$ *centered at* p *such that on* U

$$\omega = \sum_i dx_i \wedge dy_i.$$

Proof. We will use induction.

- Choose as x_1 any function with $dx_1(p) \neq 0$.
- Consider the Hamiltonian flow X_{x_1}. I.e. $\omega(X_{x_1}, .) = dx_1$.
- Take a hypersurface V through p, that's traversal to $X_{x_1}(p)$. This will be the hypersurface $y_1 = 0$.
- Define y_1 in a neighbourhood of p as the time needed to reach V along the flow of X_{x_1}.
- Observe that $x_1 = y_1 = 0$ is a symplectic submanifold near p. So we can use induction to find Darboux coordinates $x_2, y_2, \ldots$ on it.
- To extend $x_2, y_2 \ldots,$ to a neighbourhood of p note that X_{x_1} and X_{y_1} commute.

□

Exercise 1.21. Try to fill in some gaps in the proof, or have a look into Arnold's Mathematical methods of classical mechanics.

1.8 What about other k-forms?

Exercise 1.22. For which pairs $n \geq k$ the action of $GL(n, \mathbb{R})$ on $\Lambda^k \mathbb{R}^n$ has an open orbit?

2 Hamiltonian actions and moment maps

2.1 Back to Archimedes

Consider $\mathbb{C}^n$ and let again $\mathbb{S}^{2n-1}$ be the unit sphere.

Consider the map $\mu : \mathbb{C}^n \to \mathbb{R}^n$,

$$(z_1, \dots, z_n) \to (|z_1^2|, \dots, |z_n^2|).$$

Observation 2.1. The image $\mu(\mathbb{C}^n)$ is the positive octant $(t_i \geq 0) \subset \mathbb{R}^n$.

The image of $\mathbb{S}^{2n-1}$ is a simplex Δ^{n-1} — it's the convex hull of points $(1, 0 \dots, 0)$,..., $(0, \dots, 0, 1)$. Furthermore, the map factors through $\mathbb{C}P^{n-1}$.

Exercise 2.2 (Archimedes)**.** Prove that that for any open U in the positive octant

$$\mathrm{Vol}_{\mathbb{C}^n}(\mu^{-1}(U)) = \pi^n \mathrm{Vol}_{\mathbb{R}^n}(U).$$

Deduce a similar formula for $\mu : \mathbb{C}P^{n-1} \to \Delta^{n-1}$.

2.2 Poisson brackets and Hamiltonian actions

Definition 2.3. Let (M, ω) be a symplectic manifold. The Poisson bracket is the following operation on smooth functions:

$$\{f, g\} = \omega(X_f, X_h).$$

Exercise 2.4. Show that the Poisson bracket is a Lie bracket.

Definition 2.5. Let g be a Lie algebra and (M, ω) be a symplectic manifold. A *Hamiltonian action* of g on (M, ω) is a Lie algebra homomorphism

$$g \to (C^\infty(M), \{\ ,\ \}).$$

An action of a Lie group G on (M, ω) by symplectomorphism is called Hamiltonian if the Lie homomorphism $g \to \mathrm{Vect}(M)$ lifts to Hamiltonian action of g on (M, ω)

Example 2.6. Action of T^n on $\mathbb{R}^{2n}$.

Exercise 2.7. Prove that any smooth action of G on M induces a Hamiltonian action of G on T^*M.

2.3 Moment maps

Definition 2.8. Consider a Hamiltonian action of G on (M, ω) and let $\bar{\mu} : g \to C^\infty(M)$ be the associated Lie algebra homomorphism. Then the dual map[1] $\mu : M \to g^*$ is called the *moment map.*

Exercise 2.9. Check that the Archimedes' map $(z_1, \ldots, z_n) \to (|z_1^2|, \ldots, |z_n^2|)$ is the moment map of a Hamiltonian T^n action.

2.4 Atiyah-Bott and Delzant

Theorem 2.10 (Atiyah-Bott)**.** *Let (M, ω) be a symplectic manifold with a Hamiltonian T^k action, and let $\mu : M \to \mathbb{R}^k$ be the corresponding map. Then the image $\mu(M)$ is a convex polytope.*

Proof idea. Use equivariant Darboux to reduce this to linear actions on the standard $(\mathbb{R}^{2n}, \omega)$. □

Theorem 2.11 (Delzant)**.** *Let (M^{2n}, ω) be a symplectic manifold with an effective Hamiltonian T^{2n} action, and let $\mu : M \to \mathbb{R}^k$ be the corresponding map. Then the image $\mu(M)$ is a Dlezant polytope. I.e. simple polytope, such that for each vertex some element of $SL(n, \mathbb{Z})$ sends the adjacent edges to the coordinate axes.*

[1] Every point of M defines a linear function on g - i.e. an element of g^*