Flop-flop autoequivalences and compositions of spherical twists

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I, Federico Barbacovi, confirm that the work presented in this thesis is my own. Where information has been derived from other sources, I confirm that this has been indicated in the thesis.

Abstract

The main of focus of this thesis is the study of cohomological symmetries. Namely, given an algebraic variety, we study the symmetries of its derived category, which are also known as autoequivalences.

The thesis is split into five chapters. In § 1, we give an introduction to the material presented in the thesis, as well as a motivation as to why one might be interested in studying these topics. We encourage the reader to have a look, so as to know what is coming.

In § 2, we set up the preliminary notions we will need throughout the whole thesis. The arguments touched in this chapter comprise triangulated categories, dg-categories, and spherical functors.

In § 3, we begin to present the novel mathematics developed in this thesis. The focus of this chapter is on how to compose spherical twists around spherical functors. We describe a general recipe that takes as input two spherical functors and outputs a new spherical functor whose twist is the composition of the twists around the functors we started with, and whose cotwist is a gluing of the cotwists. We conclude the chapter by specialising the theory to the case of spherical objects and $\mathbb{P}$-objects.

In § 4, we study autoequivalences arising from geometric correspondences. We prove that such autoequivalences have a natural representation as the inverse of the spherical twist around a spherical functor, and that in some examples this geometric spherical functor agrees with the construction described in § 3.

We conclude the thesis with § 5, in which we present some possible future applications of this work. In doing so, we hope to stimulate further mathematical discussion around topics that the author of this thesis finds really exciting.

Impact statement

I expect that the results of this thesis will have an impact in various areas of Mathematics. Indeed, the thesis uses techniques from algebraic geometry, triangulated categories, and dg-categories. Thus, I believe that my work will influence and generate new research in these areas. To achieve this impact, I submitted my work to peer-reviewed journals. Parts of § 2 and § 3 have published, in a different form, in *International Mathematics Research Notices* [Bar22].

As evidence of interest in my work, I have been invited to deliver research talks world-wide. On top of the various seminars to which I delivered research talks, I was invited to present my research at five international conferences: at the Workshop *Derived, Birational, and Categorical Algebraic Geometry* in 2020 and again in 2021, at the International Conference *Categories and birational geometry* in 2020, at the *Hausdorff School Hyperkähler Geometry* in 2021, and at the *Joint International Meeting of the AMS-EMS-SMF* in 2022.

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The time has finally come to write the most important chapter. This thesis marks the end of a four-year-long journey I shared with many people, and it is time they are properly acknowledged.

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I also want to thank all the people who have expressed interest in my work and with whom I have had many interesting mathematical conversations. I have been so lucky to be invited to many international conferences, and I thank all those who invited me for the great opportunity you gave me.

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Finally, I want to thank Bianca Maria. We have grown together, and while you were my girlfriend when this journey started, I am happy and honoured that you are now my wife. You have supported me even in the hardest of times. To me, that means more than anything else. Our story began seven years ago, and we have shared many beautiful moments since then, but there is still so much we can do, and we will do it together.

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Contents

Chapter 1

Introduction

In this thesis, we study non-commutative geometry in the incarnation provided by (enhanced) triangulated categories. To understand why we might want to delve into such topics, we take a step back, motivate their introduction, and showcase their features.

Broadly speaking, algebraic geometry is concerned with the study of algebraic varieties: objects X that are locally modelled as the zero loci of polynomial functions in $\mathbb{A}^n$ for some $n \in \mathbb{N}$. As such, and as their name suggests, geometric properties of X are algebraic in nature, and algebraic geometers seek to understand these properties using algebraic techniques.

When studying properties of X, algebraic geometers typically focus on some subsidiary object that is easier manage and still embodies the relevant features of X. For example, *symmetries* of X encapsulate many information about X, where by symmetry we mean an invertible map $f\colon X \to X$ that preserves the structure we have on X. Symmetries of X are also called *automorphisms*, and one of their nicest features is that they can be packaged into a group: $\mathrm{Aut}(X)$. Then, instead of studying X, we can study $\mathrm{Aut}(X)$, and obtain information about X as a byproduct.

The study of automorphisms[1] has proved to be extremely fruitful [Can01], [BC16], [HMX13], but its usefulness is sometimes hampered by the rigidity of X. Namely, X might have very few symmetries because the constraint of preserving its structure is too restrictive. For this reason, the study of *cohomological symmetries* has gained more and more attention over the years.

In their seminal work [Ver96], Grothendieck and Verdier introduced the notion of a triangulated category, and to any algebraic variety X they attached a triangulated category called the *bounded derived category of coherent sheaves*: $\mathrm{D}^{\mathrm{b}}(X)$. Since its introduction, the bounded derived category has been object of intense mathematical research, proving to be a tool to understand X and an object worth studying in itself at the same time.

[1] The papers we refer to study birational automorphisms, but they nevertheless showcase the strength of the study of symmetries.

The symmetries of $\mathrm{D^b}(X)$, which we think of as cohomological symmetries, are triangulated endofunctors $\alpha\colon \mathrm{D^b}(X) \to \mathrm{D^b}(X)$ that are equivalences of categories. They go under the name of *autoequivalences* and can once again be packaged into a group: $\mathrm{Aut}(\mathrm{D^b}(X))$.

The group of automorphisms of X injects into the group of autoequivalences of $\mathrm{D^b}(X)$, and therefore we enlarged the number of symmetries at our disposal, as we wanted. Unfortunately, sometimes the new symmetries are just a byproduct of the structure that $\mathrm{D^b}(X)$ carries, see [BO02]. However, in some cases there exist genuinely new symmetries of $\mathrm{D^b}(X)$ that we were not able to see from X and that provide us with interesting information.

This is one of the reasons why the author of this thesis studies derived categories, and more generally triangulated categories: they are flexible and have a rich theory of symmetries.

1.1 Autoequivalences and geometry

In the previous paragraphs, we said that sometimes considering cohomological symmetries does not add anything relevant to the array of symmetries at our disposal. Let us explain what we mean in more detail.

To any automorphism $f \in \mathrm{Aut}(X)$ we can associate an autoequivalence of the derived category by taking the pushforward functor $f_* \in \mathrm{Aut}(\mathrm{D^b}(X))$, and we obtain an injection $\mathrm{Aut}(X) \hookrightarrow \mathrm{Aut}(\mathrm{D^b}(X))$, $f \mapsto f_*$. Notice that in general we would have to take the right derived functor Rf_*, but as f is an automorphism this is not needed. In the following, to ease the notation, all the functors will be implicitly derived.

We can construct autoequivalences also starting with line bundles. Indeed, given any line bundle $L \in \mathrm{Pic}(X)$ the functor $L \otimes_{\mathcal{O}_X} -$ is an autoequivalence with inverse given by $L^\vee \otimes_{\mathcal{O}_X} -$, where $L^\vee = \mathcal{H}om_X(L, \mathcal{O}_X)$.

Finally, as part of the triangulated structure of $\mathrm{D^b}(X)$, we have the shift functor [1], which is an autoequivalence. Hence, inside the autoequivalence group $\mathrm{Aut}(\mathrm{D^b}(X))$ we always find the subgroup

$$\mathrm{Aut}^{\mathrm{std}}(\mathrm{D^b}(X)) = \mathbb{Z} \times (\mathrm{Aut}(X) \ltimes \mathrm{Pic}(X)) \subset \mathrm{Aut}(\mathrm{D^b}(X))$$

that is called the subgroup of *standard autoequivalences*. Above, the copy of $\mathbb{Z}$ is generated by [1], while the semidirect product comes from the fact that automorphisms of X and tensor products with line bundles do not commute, but we have $f_*(L \otimes_{\mathcal{O}_X} -) \simeq f_*(L) \otimes_{\mathcal{O}_X} f_*(-)$ for any $f \in \mathrm{Aut}(X)$ and $L \in \mathrm{Pic}(X)$.

The reason why we call the elements of $\mathrm{Aut}^{\mathrm{std}}(\mathrm{D^b}(X))$ standard autoequivalences is that they share a particular property, that is they preserve the subcategory $\mathrm{Coh}(X) \subset \mathrm{D^b}(X)$ (which is given by those complexes whose cohomology is concentrated in degree zero) up to a shift. Even more, they preserve (up to a shift) the standard t-structure on $\mathrm{D^b}(X)$, of which $\mathrm{Coh}(X)$ is the heart.

From our point of view, standard autoequivalences are not that interesting because they do not enlarge the array of available symmetries in a meaningful way. This is because standard autoequivalences are given by compositions of symmetries that were either available since the very beginning, e.g. elements of $\mathrm{Aut}(X)$, or that came to be because of the framework we placed ourselves in: the shift functor is part of the definition of a triangulated category, and line bundles appear because we consider coherent sheaves.

For this reason, we turn our attention to the following question: in which cases are there more symmetries, *i.e.*, when is true that $\mathrm{Aut^{std}}(\mathrm{D^b}(X)) \subsetneq \mathrm{Aut}(\mathrm{D^b}(X))$?

In the remarkable paper [BO01], Bondal and Orlov prove that all autoequivalences of $\mathrm{D^b}(X)$ are standard when X is a smooth projective variety with ample or antiample canonical bundle. Thus, in this case there is nothing interesting going on.

At this point, we could shoot in the dark, pick a random algebraic variety X, and try to construct a non-standard autoequivalence, *i.e.*, an autoequivalence that does not belong to $\mathrm{Aut^{std}}(\mathrm{D^b}(X))$. However, there is a more meaningful way to pinpoint which algebraic varieties should possess non-standard autoequivalences that also shows how beautifully cohomological symmetries pair up with geometric transformations.

The Minimal Model Programme is a topic of intense mathematical research. Birational geometers' seek to classify algebraic varieties by identifying a *minimal model* in their birational class. What minimal means is beyond the scope of this introduction and of this thesis. It is enough to say that while this minimal model is unique for surfaces, already when we consider threefolds uniqueness does not hold anymore. The reason is that there are certain birational transformations (beware: they are not necessarily isomorphisms!) called *flops* that allow us to pass from a minimal model to another.

Conjecturally [BO95], the cohomological interpretation of the non-uniqueness of a minimal model is that, while we can change the geometric object, the derived category stays the same. Namely, the minimal model is not unique, but its derived category is. Hence, whenever two varieties are related by a flop, we expect their derived categories to be equivalent.

This conjecture, which has been generalised to the Bondal–Orlov–Kawamata conjecture [BO95], [Kaw02], is extremely interesting because it prescribes which geometric transformations should induce derived equivalences.[2] On top of this, it comes in handy in our search for non-standard autoequivalences.

Let us consider X_- and X_+ two smooth, projective varieties, and assume that they are

[2] Namely, flops and, more generally, K-equivalences, see [Kaw02] for the definition of the latter notion.

Calabi–Yau: $\omega_{X_-} \simeq \mathcal{O}_{X_-}$, $\omega_{X_+} \simeq \mathcal{O}_{X_+}$. One of the requirements a (birational) map must satisfy in order to be a flop is that it must "preserve" the canonical bundle. When X_- and X_+ are Calabi–Yau, this requirement is trivially fulfilled. Therefore, any birational map between two Calabi–Yau varieties is a flop,[3] and, given the above discussion, we expect birational Calabi–Yau varieties to have equivalent derived categories:

$$\Phi_- \colon \mathrm{D^b}(X_+) \xrightarrow{\simeq} \mathrm{D^b}(X_-).$$

However, the definition of a flop is symmetric, and applying the same reasoning we can infer the existence of another equivalence

$$\Phi_+ \colon \mathrm{D^b}(X_-) \xrightarrow{\simeq} \mathrm{D^b}(X_+).$$

The point is that there is no reason to expect Φ_- and Φ_+ to be inverse to each other, and therefore if we compose them we obtain a non-trivial autoequivalence of $\mathrm{D^b}(X_+)$:

$$\Phi_+\Phi_- \in \mathrm{Aut}(\mathrm{D^b}(X_+)). \tag{1.1}$$

The expectation is that (1.1), as it comes from a geometric transformation that is not an isomorphism, should be a non-standard autoequivalence.

This is often, if not almost always, true, and we will see examples of such autoequivalences in § 4. However, before turning to the mathematical advances presented in this thesis, we still have to introduce and motivate an important piece of theory that will be the main player of § 3: *spherical functors*.

1.2 How to: spherical functors

In the previous section we explained why, given two birational Calabi–Yau varieties, we expect them to have equivalent derived categories. However, we did not say how the equivalence should be constructed, or where it should come from.

This is actually one of the key problems in approaching the Bondal–Orlov–Kawamata conjecture: how does one come up with a suitable candidate for the equivalence between two birational Calabi–Yau varieties?

For the purpose of this introduction, we will focus on a question that is at the same time a generalisation and a modification of the previous one: how does one construct autoequivalences of $\mathrm{D^b}(X)$ for X an algebraic variety? To answer this question, we will make use of another beautiful example of how seemingly unrelated areas of mathematics

[3] Birational geometers might disagree, but we are using this geometric picture as a motivation, and we will skip over some technicalities.

actually have strong links between them.

Let X be a Calabi–Yau, smooth, projective variety over a field k. In his ICM address in 1994 [Kon95], Kontsevich introduced the basis of what developed into *Homological Mirror Symmetry*. Loosely speaking, drawing inspiration from physics, Kontsevich predicted that algebraic properties of X should correspond to symplectic properties of some symplectic manifold $\check{X}$, which is called the *mirror* of X. While Homological Mirror Symmetry is still more an idea than a precise conjecture, it has permitted the mathematical community to postulate an incredible amount of predictions that turned out to be true.

From our perspective, Homological Mirror Symmetry serves its purpose as follows: if symplectic properties of $\check{X}$ correspond to algebraic properties of X, then symplectic symmetries of $\check{X}$ should correspond to algebraic symmetries of X.

The flow of information between X and $\check{X}$, at least conjecturally, is realised by an equivalence of triangulated categories. On one side we have $\mathrm{D^b}(X)$, while on the other we have the *Fukaya category* of $\check{X}$: $\mathrm{Fuk}(\check{X})$. Therefore, a symplectic symmetry of $\check{X}$ should induce a symmetry of $\mathrm{Fuk}(\check{X})$, and thus a symmetry of $\mathrm{D^b}(X)$. The question now is: can we interpret this autoequivalence directly on the algebraic side without the need to know that it comes from the symplectic side of Homological Mirror Symmetry?

In their remarkable paper [ST01], Seidel and Thomas answer this question for a particular type of symplectic automorphisms: *Dehn twists*. These automorphisms arise by twisting around a Lagrangian sphere $L \subset \check{X}$, namely, a Lagrangian subvariety of $\check{X}$ that is isomorphic to a sphere.

Lagrangian spheres have the property that their complex of morphisms in the Fukaya category $\mathrm{Hom}_{\mathrm{Fuk}(\check{X})}(L, L)$ has cohomology isomorphic (as a graded algebra) to the cohomology ring of the sphere of the corresponding dimension. Therefore, there is a natural class of objects in $\mathrm{D^b}(X)$ that have the same cohomological properties as Lagrangian spheres: they are the objects $E \in \mathrm{D^b}(X)$ such that

$$\mathrm{Hom}^\bullet_{\mathrm{D^b}(X)}(E, E) = \bigoplus_{n \in \mathbb{Z}} \mathrm{Hom}_{\mathrm{D^b}(X)}(E, E[n])[-n] \simeq \mathrm{H}^\bullet(S^{\dim X}, k)$$

as graded algebras. These objects are the ones that Seidel and Thomas call *spherical objects*, and they are the key to construct autoequivalences of $\mathrm{D^b}(X)$ corresponding to Dehn twists.

We will defer for a moment how one formally constructs an autoequivalence from a spherical object,[4] and consider the final result: the *spherical twist* around the spherical

[4]Notice that the formula (1.5) does not provide a formal definition because the cone construction is not functorial for triangulated categories. For more details on this point, see the next page.

object E is an autoequivalence $T_E \in \mathrm{Aut}(\mathrm{D^b}(X))$ that acts on $F \in \mathrm{D^b}(X)$ as

$$T_E(F) = \mathrm{cone}(\mathrm{ev}\colon \mathrm{Hom}^\bullet_{\mathrm{D^b}(X)}(E, F) \otimes_k E \to F) \tag{1.2}$$

where ev is the evaluation map, see [ST01]. The formula (1.2) is our point of entrance to the theory of *spherical functors*.

Soon after Seidel and Thomas introduced spherical objects, it became clear that it should be possible to parametrise spherical objects in families, thus obtaining the notion of a spherical functor. Spherical functors developed thanks to the work of many people. Special cases of this general notion appeared in [Hor05], [Rou04], [Sze04], [Tod07], [KT07]. The first general treatment was attempted in [Ann07], but the strength of the ideas presented was hampered by the octhaedral axiom, which does not provide as much control as it is needed over the cone of the composition of two morphisms. The theory of spherical functors reached its final and most general form in [AL17], where the formalism of triangulated categories used in [Ann07] was replaced with that of dg-categories.

We will introduce spherical functors in § 2.5. For the moment, it will be enough to say that such a functor should induce for us an autoequivalence, as spherical objects did. However, a priori it is not clear how to build an autoequivalence out of a functor $\alpha\colon \mathcal{A} \to \mathcal{C}$ between two triangulated categories $\mathcal{A}$ and $\mathcal{C}$.

To understand how to do this, let us turn our attention back to (1.2) and consider the functor $\alpha\colon \mathrm{D^b}(k) \to \mathrm{D^b}(X)$ sending k to E. Then, α has a right adjoint given by $\alpha^R(F) = \mathrm{Hom}^\bullet_{\mathrm{D^b}(X)}(E, F)$ and (1.2) can be rewritten as

$$T_E(F) = \mathrm{cone}(\mathrm{ev}\colon \alpha\alpha^R(F) \to F).$$

Now notice that ev is the counit of the adjunction $\alpha \dashv \alpha^R$, and therefore we see that from the functor α we can construct an autoequivalence of $\mathrm{D^b}(X)$ that acts on $F \in \mathrm{D^b}(X)$ by taking the cone of the counit of the adjunction $\alpha \dashv \alpha^R$.

We will see in § 3 that a functor $\alpha\colon \mathcal{A} \to \mathcal{C}$ is called spherical if the endofunctors

$$T_\alpha = \mathrm{cone}(\alpha\alpha^R \to \mathrm{id}_{\mathcal{C}}) \quad \text{and} \quad C_\alpha = \mathrm{cone}(\mathrm{id}_{\mathcal{A}} \to \alpha^R\alpha)[-1],$$

which are called the *twist* and *cotwist* around α, respectively, are autoequivalences.

Spherical functors provide an extremely fruitful way to construct autoequivalences, and we will study their behaviour extensively in § 3. For the moment, let just point out one thing: in the previous paragraph we took cones of natural transformations between functors, but this is not possible when working with plain triangulated categories. We need a stronger framework, and thus we need to consider some type of *enhancement*. We will deal with the question of enhancing triangulated categories in § 2.1, and we refer the interested reader to that section.

1.3 What is in this thesis?

In the previous sections we introduced the two main players of this thesis: autoequivalences and spherical functors. We now move on to explain which questions about autoequivalences and spherical functors are addressed in the thesis, and what are the mathematical advances presented in this work.

In § 1.2, we said that starting from a spherical functor we can construct two autoequivalences: the twist and the cotwist. Actually, much more is true. Namely, it is a result due to Segal that *any* autoequivalence of a triangulated category can be realised as a spherical twist around a spherical functor [Seg18]. A word of caution: once again this result does not hold for plain triangulated categories, and we need to work with enhanced triangulated categories and functors. However, to make the language more fluent, we will drop the adjective enhanced. Hence, all triangulated categories and functors from now on are implicitly assumed to be enhanced.

Knowing that any autoequivalence can be realised as a spherical twist prompts us with an array of questions, we propose three of them. Fix $\mathcal{C}$ a triangulated category and $\Phi_{\mathcal{A}}, \Phi_{\mathcal{B}} \in \mathrm{Aut}(\mathcal{C})$. Then,

(i) is there a preferred[5] way to write $\Phi_{\mathcal{A}} = T_{\alpha_{\mathcal{A}}}$ for some spherical functor $\alpha_{\mathcal{A}}$?

(ii) what is the information about $\alpha_{\mathcal{A}}$ contained in $\Phi_{\mathcal{A}}$, and viceversa?

(iii) if $\Phi_{\mathcal{A}} = T_{\alpha_{\mathcal{A}}}$ and $\Phi_{\mathcal{B}} = T_{\alpha_{\mathcal{B}}}$ for two spherical functors $\alpha_{\mathcal{A}}$ and $\alpha_{\mathcal{B}}$, what can we say about $\Phi_{\mathcal{B}}\Phi_{\mathcal{A}}$?

Arguably, question (i) is not well posed because it is not clear what "preferred" should mean. Actually, we will see in § 5 that the results of § 3 hint to the fact that having different representations as a spherical twist can be an advantage, rather than a disadvantage.

Question (ii) can be approached in different ways. For example, rather than looking at the functor $\alpha_{\mathcal{A}} \colon \mathcal{A} \to \mathcal{C}$, we could consider the source category $\mathcal{A}$. Then, it was shown in [HLS16] that if $\mathcal{A}$ has a semiorthogonal decomposition $\mathcal{A} = \langle \mathcal{S}_2, \mathcal{S}_1 \rangle$ (see Definition 2.3.1) that satisfies some properties, then $\Phi = T_{\alpha_{\mathcal{A}}} = T_{\alpha_{\mathcal{A}}|_{\mathcal{S}_2}} T_{\alpha_{\mathcal{A}}|_{\mathcal{S}_1}}$, and we obtain a factorisation of Φ.[6]

[5] We do not wonder about the uniqueness of $\alpha_{\mathcal{A}}$ because it can be easily shown that a representation as a spherical twist is not unique (not even if we require the functor to be conservative, *i.e.*, without kernel, see [Chr20, § 4.1].)

[6] The converse is not true: given a 2 spherical object E, in [Seg18] Segal proves that T_E^2 can be realised at the twist around a spherical functor whose source category is $\mathrm{D}(k[\varepsilon]/\varepsilon^2)^c$, $\deg(\varepsilon) = -1$, see Definition 2.3.8 for the definition of compact objects. While T_E^2 has a factorisation, the category

Question (iii) is the one in which we are most interested, and it is the first problem we solve in this thesis. The reason why we are interested in answering this question is that it helps us to shed light on the structure of the autoequivalence group of $\mathcal{C}$: understanding how spherical twists around spherical functors compose might allow us to see relations that would have otherwise remain hidden, see § 5.

Let us fix $\alpha_{\mathcal{A}} \colon \mathcal{A} \to \mathcal{C}$ and $\alpha_{\mathcal{B}} \colon \mathcal{B} \to \mathcal{C}$ two spherical functors. Our aim is to represent the autoequivalence $T_{\alpha_{\mathcal{B}}} T_{\alpha_{\mathcal{A}}}$ as the spherical twist around a spherical functor. The following theorem, which is Theorem 3.1.4, tells us how to do this. A precise statement would require us to introduce too much notation and would derail this introduction. Thus, we refer the interested reader to § 3 for the formal statement, further details, and pointers to the relevant parts in the thesis.

Theorem 1.3.1. *Let $\alpha_{\mathcal{A}} \colon \mathcal{A} \to \mathcal{C}$ and $\alpha_{\mathcal{B}} \colon \mathcal{B} \to \mathcal{C}$ be two spherical functors. Then, there exists a category $\mathcal{B} \sqcup_{\varphi} \mathcal{A}$, $\varphi = \alpha_{\mathcal{B}}^R \alpha_{\mathcal{A}}$, and a spherical functor $\beta \colon \mathcal{B} \sqcup_{\varphi} \mathcal{A} \to \mathcal{C}$ such that*

(i) $T_\beta = T_{\alpha_{\mathcal{B}}} T_{\alpha_{\mathcal{A}}}$

(ii) the cotwist around β can be described in terms of the cotwists around $\alpha_{\mathcal{B}}$ and $\alpha_{\mathcal{A}}$.

The author of this thesis would like to remark that he published the above theorem, in a different form, in [Bar22].

Let us now explain why we think of the above theorem as a *gluing* procedure, and why we call functors such as β above *glued spherical functors*.

The category $\mathcal{B} \sqcup_{\varphi} \mathcal{A}$ was introduced by Tabuada [Tab07] and is a way to glue two categories along a functor. Indeed, $\mathcal{B} \sqcup_{\varphi} \mathcal{A}$ is called the gluing of $\mathcal{B}$ and $\mathcal{A}$ along φ. Thanks to [KL15, Proposition 4.6], we know that $\mathcal{B} \sqcup_{\varphi} \mathcal{A}$ has an SOD $\mathcal{B} \sqcup_{\varphi} \mathcal{A} = \langle \mathcal{B}, \mathcal{A} \rangle$ with right gluing functor[7] given by φ, and it is a simple computation to show that the restriction of β to $\mathcal{B}$ and $\mathcal{A}$ is given by $\alpha_{\mathcal{B}}$ and $\alpha_{\mathcal{A}}$, respectively.[8]

Therefore, both the category $\mathcal{B} \sqcup_{\varphi} \mathcal{A}$ and the functor β are constructed via a gluing procedure, and it is for this reason that we think of Theorem 1.3.1 as a recipe to glue two spherical functors into a single one.

After proving theorem Theorem 1.3.1, we provide examples of its application to spherical objects § 3.4 and $\mathbb{P}$-objects § 3.5. In § 5, we explain some possible future uses of Theorem 1.3.1.

Although the above theorem is remarkably interesting in itself, it has an aura of formality about it. It answers an interesting question, but we might wonder whether such construction appears naturally in geometric situations.

$\mathrm{D}(k[\varepsilon]/\varepsilon^2)^c$ cannot have a semiorthogonal decomposition because its Serre duality functor is the shift by 1.

[7]See Definition 2.3.5.

[8]More abstractly, the fact that $\beta|_{\mathcal{B}} \simeq \alpha_{\mathcal{B}}$ and $\beta|_{\mathcal{A}} \simeq \alpha_{\mathcal{A}}$ follows from the adjunction of [Efi20, Proposition 4.5].

The search for geometric examples of glued spherical functors as those constructed in Theorem 1.3.1 kicked off our second research project, whose results are presented in this thesis.

In § 1.1, we explained that flops are a source of non-standard autoequivalences. It turns out that spherical twists around spherical objects almost always give examples of non-standard autoequivalences. For this reason, if one wants to find examples of glued spherical functors as in Theorem 1.3.1, they might as well start from autoequivalences induced by flops.

This random guess is not really random, as there is evidence that backs it. Namely, it has been shown in various papers, e.g. [ADM19], [HLS16], [DS14], that autoequivalences coming from flops factorise as compositions of inverses of spherical twists around spherical functors, and these factorisations have a geometric meaning.[9]

Given this evidence, we started to investigate derived equivalences coming from flops in the hope to find geometric examples of Theorem 1.3.1. In order to be able to explain the results of this research project, we have to take a step back. First of all, we have to clarify what it means that two Calabi–Yau varieties X_- and X_+ are related by a flop. This means that there exists a (possibly singular) variety Y and proper, birational maps[10]

$$\begin{array}{ccccc} X_- & & & & X_+ \\ & \searrow^{f_-} & & \swarrow^{f_+} & \\ & & Y & & \end{array}$$

such that for any divisor on $D_- \subset X_-$ with the property that $-D_-$ is f_--nef, the proper transform of D_- is an f_+-nef divisor on X_+.

From this picture emerges a natural candidate for the equivalence $\mathrm{D^b}(X_-) \simeq \mathrm{D^b}(X_+)$ predicted by the Bondal–Orlov–Kawamata conjecture. Namely, we take the fibre product

$$\begin{array}{ccccc} & & X_- \times_Y X_+ & & \\ & \swarrow^{p_-} & & \searrow^{p_+} & \\ X_- & & & & X_+ \\ & \searrow^{f_-} & & \swarrow^{f_+} & \\ & & Y & & \end{array} \tag{1.3}$$

[9]We hope the reader will forgive us for our lack of formality. In the previous phrase, and in the following, when we say "geometric meaning" we mean that the factorisation is not abstractly constructed, but that it appears from an in-depth study of the geometric picture.

[10]More precisely, the maps $f_\pm$ have to be *small contractions*.

and we consider the functors[11]

$$\Phi_- = (p_-)_* p_+^* \colon \mathrm{D}_{\mathrm{qc}}(X_+) \to \mathrm{D}_{\mathrm{qc}}(X_-) \quad \text{and} \quad \Phi_+ = (p_+)_* p_-^* \colon \mathrm{D}_{\mathrm{qc}}(X_-) \to \mathrm{D}_{\mathrm{qc}}(X_+). \tag{1.4}$$

Then, if Φ_- and Φ_+ are equivalences, they induce equivalences[12] between the respective bounded derived categories of coherent sheaves, as we wanted.

This approach works in many, but not all, cases (there are only two known families of counterexamples [Nam04], [Kaw06], [Cau12a]) and it provides us with two equivalences that one can easily see are not inverse to each other. Therefore, we obtain a non-trivial autoequivalence $\Phi_+\Phi_- \in \mathrm{Aut}(\mathrm{D}^{\mathrm{b}}(X_+))$.

It is this autoequivalence that, in some examples, was shown to have a factorisation in terms of inverses of spherical twists around spherical functors, and it is on this autoequivalence that we focus our attention.

At this point, we have an autoequivalence $\Phi_+\Phi_-$ that is known to have a factorisation as a composition of inverses of spherical twists around spherical functors, and we would like to show that this autoequivalence gives an example of a glued spherical functor as constructed in Theorem 1.3.1. However, to match these two pictures, we lack a key player: a spherical functor whose spherical twist has inverse isomorphic to the autoequivalence $\Phi_+\Phi_-$.

If we managed to find such a functor in a "geometric" way, then it would be natural to expect that this functor is an example of the construction of Theorem 1.3.1.

For this reason, the first question we turn our attention to is: given a diagram as (1.3) such that the functors (1.4) are equivalences, can we find a spherical functor whose twist has inverse isomorphic to the autoequivalence $\Phi_+\Phi_-$?

To answer this question, recall that the autoequivalence $\Phi_+\Phi_-$ is given by $\Phi_+\Phi_- = (p_+)_* p_-^* (p_-)_* p_+^*$. Even though we have not specified this, we will always assume to be in the situation where $(p_-)_*\mathcal{O}_{X_- \times_Y X_+} \simeq \mathcal{O}_{X_-}$, and similarly for p_+.[13] Therefore, the pull-up functors p_-^* and p_+^* are fully faithful and never land in the common kernel $\mathcal{K} := \ker(p_-)_* \cap \ker(p_+)_*$, and this subcategory of $\mathrm{D}_{\mathrm{qc}}(X_- \times_Y X_+)$ does not play any role in the action of $\Phi_+\Phi_-$. Thus, it seems reasonable to investigate the Verdier quotient

$$\mathrm{D}_{\mathrm{qc}}(X_- \times_Y X_+)/\mathcal{K}$$

[11] Here $\mathrm{D}_{\mathrm{qc}}(-)$ is the unbounded derived category of quasi-coherent sheaves; strictly speaking, when X_- and X_+ are smooth we could work directly with $\mathrm{D}^{\mathrm{b}}(-)$, however, we will need to pass to $\mathrm{D}_{\mathrm{qc}}(-)$ to be able to harness all the strengths of cocomplete triangulated categories, see § 4.

[12] X_- and X_+ being smooth, the claim follows by restricting to compact objects, see Definition 2.3.8 and Remark 4.2.5

[13] Notice that this is a requirement about the higher cohomology sheaves of the structure sheaves of the fibres of p_- and p_+, respectively. Indeed, by Zariski's Main Theorem the underived pushfoward of the structure sheaf of $X_- \times_Y X_+$ via p_- and p_+ is the structure sheaf of X_- and X_+, respectively.

rather than the whole category $\mathrm{D}_{\mathrm{qc}}(X_- \times_Y X_+)$. It turns out that this is the only passage required to realise $\Phi_+\Phi_-$ as the inverse of a spherical twist. Indeed, we can prove

Theorem 1.3.2. *Consider a diagram as (1.3) such that*

1. *the functors (1.4) are equivalences*
2. *we have isomorphisms* $(p_-)_*\mathcal{O}_{X_- \times_Y X_+} \simeq \mathcal{O}_{X_-}$ *and* $(p_+)_*\mathcal{O}_{X_- \times_Y X_+} \simeq \mathcal{O}_{X_+}$

and write

$$(\overline{p}_-)_* \colon \mathrm{D}_{\mathrm{qc}}(X_- \times_Y X_+)/\mathcal{K} \to \mathrm{D}_{\mathrm{qc}}(X_-) \quad \text{and} \quad (\overline{p}_+)_* \colon \mathrm{D}_{\mathrm{qc}}(X_- \times_Y X_+)/\mathcal{K} \to \mathrm{D}_{\mathrm{qc}}(X_+)$$

for the functors induced by $(p_-)_*$ *and* $(p_+)_*$ *on the Verdier quotient of* $\mathrm{D}_{\mathrm{qc}}(X_- \times_Y X_+)$ *by* $\mathcal{K} = \ker(p_-)_* \cap \ker(p_+)_*$.

Then, the functors

$$\begin{gathered} \Psi_- \colon \ker(\overline{p}_+)_* \hookrightarrow \mathrm{D}_{\mathrm{qc}}(X_- \times_Y X_+)/\mathcal{K} \xrightarrow{(\overline{p}_-)_*} \mathrm{D}_{\mathrm{qc}}(X_-) \\ \text{and} \\ \Psi_+ \colon \ker(\overline{p}_-)_* \hookrightarrow \mathrm{D}_{\mathrm{qc}}(X_- \times_Y X_+)/\mathcal{K} \xrightarrow{(\overline{p}_+)_*} \mathrm{D}_{\mathrm{qc}}(X_+) \end{gathered} \tag{1.5}$$

are spherical and the inverse of their spherical twists are given $\Phi_-\Phi_+$ *and* $\Phi_+\Phi_-$*, respectively.*

The above theorem is Corollary 4.1.7 in the main body of the thesis. In § 4, we are actually able to prove a stronger and more general statement about cocomplete triangulated categories, namely Theorem 4.1.3. However, presenting the complete statement of Theorem 4.1.3 here would divert our discussion into technical digressions we do not want to address at the moment.

For the time being, let us just notice two things. First, that Theorem 1.3.2 is categorical in nature, and thus it is not surprising that it can be proved for more general categories than derived categories of algebraic varieties.[14] Second, that Theorem 1.3.2 as stated has a big drawback: it involves taking a Verdier quotient, which is well known to be a bad-behaved operation. The great advantage of Theorem 4.1.3 is that it replaces the Verdier quotient $\mathrm{D}_{\mathrm{qc}}(X_- \times_Y X_+)/\mathcal{K}$ with a subcategory of $\mathrm{D}_{\mathrm{qc}}(X_- \times_Y X_+)$, which is much easier to work with. This exchange is possible because we are working with cocomplete triangulated categories, and therefore (almost) any Verdier quotient can also be realised as a subcategory of the parent category.

[14]The possibility of such a generalisation was pointed out to the author by an anonymous referee whom the author would like to thank.

Having proved Theorem 1.3.2, we now have a candidate to be our glued spherical functor. Namely, our guess is that the functor Ψ_+ in (1.5) gives an example of a glued spherical functor every time $\Phi_+\Phi_-$ factorises as a composition of inverses of spherical twists around spherical functors in a geometric way, such as in the cases treated in [ADM19], [HLS16], and [DS14].

However, before attempting to match Theorem 1.3.1 and Theorem 1.3.2, we briefly turn our attention to another question. Namely, to prove Theorem 1.3.2 we passed from $\mathrm{D}^{\mathrm{b}}(-)$ to $\mathrm{D}_{\mathrm{qc}}(-)$, and we would like to understand to what extent we are able to obtain information about the bounded derived category from Theorem 1.3.2.

As we explained before, if X_- and X_+ are smooth, then the passage to $\mathrm{D}_{\mathrm{qc}}(-)$ is not restrictive. However, we could wonder what happens if the smoothness assumption is dropped, and this is the next question we provide an answer to. The answer is that, regardless of the smoothness properties of X_- and X_+, if we assume that Φ_- and Φ_+ are equivalences and that they preserve boundedness, then we can prove an analogue of Theorem 1.3.2 by replacing $\mathrm{D}_{\mathrm{qc}}(X_- \times_Y X_+)$ with $\mathrm{D}^{\mathrm{b}}(X_- \times_Y X_+)$. We do so in Theorem 4.2.2.

The drawback of Theorem 4.2.2 is that it does not have an analogue to Theorem 4.1.3. Namely, we are forced to work with a Verdier quotient, and this is often problematic. See also Remark 4.2.3 and Remark 4.2.5.

Having completed this small digression, we can open another short one. Namely, Bodzenta and Bondal were the first authors to consider the quotient category $\mathrm{D}_{\mathrm{qc}}(X_- \times_Y X_+)/\mathcal{K}$, and they did so in their paper [BB15], where they assumed p_- and p_+ to have fibres of dimension at most one. In *ibidem*, they proved, among other things, results similar to Theorem 4.2.2, and we would like to understand what is the relationship between their results and ours. We do so in § 4.3.

Once these loose ends have been tied up, we can move on to pursue our initial goal: to prove that Ψ_+ is, in some cases, an example of a glued spherical functor as constructed in Theorem 1.3.1. We consider two families of examples: standard flops § 4.4.1 and Mukai flops § 4.4.2. In both cases we prove that Ψ_+ does indeed provide an example of a glued spherical functor. The relevant statements are Theorem 4.4.1 and Theorem 4.4.13, respectively. We do not present them here because it would require the introduction of too much notation, and we refer the reader to the relevant sections in the thesis.

We conclude § 4 by considering some examples where we were not able to match Theorem 1.3.2 with Theorem 1.3.1 even though we know that $\Phi_+\Phi_-$ does factorise. We provide some heuristic explanation as to why the situation in these examples is more complicated, and a guess as to what could be a possible solution to the hurdles we encountered.

Finally, in § 5 we provide some possible future applications of the mathematics presented in this thesis, with the hope that they will foster further mathematical research.

The author hopes that this introduction, albeit long, provided a comprehensive glimpse of the material presented in this thesis, and that it will encourage the reader to continue reading after the current page.

Chapter 2

Preliminaries

In this first chapter, we set up the scene for presenting the main mathematical advances developed in this thesis. Given the amount of material we need to introduce, we favour narrative flow over definition boxes, and we give a number only to the definitions that will play an important role in the following chapters.

On top of recalling well known mathematical notions, in this chapter we also prove some technical lemmas and propositions that will be useful in § 3 and § 4, see for example § 2.3.2.

Some of the arguments presented in this chapter have been presented in a similar but different form in the author's published work [Bar22].

2.1 The art of enhancing categories and functors

It is now common knowledge that triangulated categories lack the functorial properties needed to allow their study in families. Namely, given two triangulated categories $\mathcal{A}$ and $\mathcal{B}$, the category of triangulated functors from $\mathcal{A}$ to $\mathcal{B}$ is not a triangulated category.

This is a gigantic drawback because as soon as one wants to study autoequivalences of triangulated categories, it is obvious that we need to be able to take cones of natural transformations.

Presently, there are three ways to overcome this issue. They are dg-categories, A_∞-categories, and $(\infty, 1)$-categories.

When working over a commutative ring k, these formalisms are equivalent, see [Coh13]. However, this is not true anymore when we consider more general setups, e.g. when we take categories linear over the sphere spectrum.

The reasons why one might want to choose any of the above formalisms over the others are mostly of personal taste. However, it is fair to say that these different formalisms incarnate different beliefs and allow the use of different techniques. Namely, dg-categories and A_∞-categories provide stricter models that allow to carry out explicit computations, but sometimes they hide the universal reason why some statements are true. On the other hand, $(\infty, 1)$-categories render explicit computations more complicated, but highlight universal properties.

In the course of this thesis, and in particular in § 2 and § 3, we will use the formalism of dg-categories. However, when possible, we will also point the reader to the relevant references in the formalism of $(\infty, 1)$-categories.

2.2 Conventions

Throughout the thesis, k will be a fixed field. In § 4.4, it will be an algebraically closed field of characteristic zero.

Given a triangulated category $\mathcal{A}$ and two objects $A_1, A_2 \in \mathcal{A}$, we will write

$$\operatorname{Hom}^\bullet_{\mathcal{A}}(A_1, A_2) = \bigoplus_{n\in\mathbb{Z}} \operatorname{Hom}_{\mathcal{A}}(A_1, A_2[n])[-n]$$

for the standard graded enhancement of any triangulated category, see e.g. [RVdB20].

Furthermore, we will employ the following conventions:

- Triangulated categories will be denoted by the letters $\mathcal{A}, \mathcal{B}, \mathcal{C}, \ldots$
- Dg-categories will be denoted by the letters $\boldsymbol{\mathcal{A}}, \boldsymbol{\mathcal{B}}, \boldsymbol{\mathcal{C}}, \ldots$
- Objects of triangulated categories will be denoted by capital latin letters $A, B, C, \ldots$
- Objects of dg-categories will be denoted by lowercase latin letters $a, b, c, \ldots$
- Functors between triangulated categories will be denoted by lowercase Greek letters $\alpha, \beta, \ldots$, with the exception of the inclusion of a subcategory $\mathcal{S} \subset \mathcal{A}$ for which we write $i_{\mathcal{S}}$
- Functors between dg-categories will be denoted by capital latin letters. This clash of notation with objects of triangulated categories is justified by the fact that dg-functors are essentially quasi-isomorphism classes of dg-bimodules, which in turn are elements of a triangulated category
- Spherical functors between triangulated categories will be denoted by the letter Ψ. The only exception is when the spherical functor is induced by a dg-bimodule M, in which case we write α_M for the spherical functor
- We write T_Ψ and C_Ψ, respectively, for the twist and cotwist around a spherical functor Ψ

- Autoequivalences of triangulated categories will be denoted by the letter Φ
- Morphisms between objects in a triangulated category will be denoted by lowercase English letters $f, g, h, \ldots$, with the exception of the connecting morphism in the distinguished triangle defining the cotwist around a spherical functor for which we write σ
- All the functors appearing are implicitly derived, with the exception of tensor products over dg-algebras and dg-categories, which we derive explicitly.
- All the subcategories appearing are assumed to be full unless otherwise stated.

We want to reassure the reader that the possible clash of notation between objects of dg-categories and morphisms between objects in a triangulated category will not happen because we will never have more than three dg-categories at once at any point in the thesis, and we always start to name morphisms from the letter f.

Finally, throughout the whole thesis we will adopt the following convention when defining a dg-algebra: if we do not mention the differential, it means that it is identically zero.

2.3 Triangulated categories

2.3.1 Semiorthogonal decompositions

In this first section we introduce *semiorthogonal decompositions* of triangulated categories. This notion is now quite classical and it was introduced by Bondal and Kapranov in [BK89].

Definition 2.3.1 ([BK89]). Let $\mathcal{A}$ be a triangulated category. Let $\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_r \subset \mathcal{A}$ be triangulated subcategories of $\mathcal{A}$, and write $i_{\mathcal{S}_k} : \mathcal{S}_k \hookrightarrow \mathcal{A}$ for their inclusions. We say that $\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_r$ give a semiorthogonal decomposition (SOD) of $\mathcal{A}$ if

1. $\mathrm{Hom}_{\mathcal{A}}(i_{\mathcal{S}_k}(S_k), i_{\mathcal{S}_j}(S_j)) = 0$ for any $S_k \in \mathcal{S}_k$, $S_j \in \mathcal{S}_j$ and $k > j$
2. for any $A \in \mathcal{A}$ there exist objects $E_i \in \mathcal{A}$, $i = 1, \ldots, r$, and maps
$$0 = E_r \to E_{r-1} \to \cdots \to E_1 \to E_0 = A$$
such that $\mathrm{cone}(E_i \to E_{i-1}) \in \mathcal{S}_i$ for any $i = 1, \ldots, r$

In this case, we write $\mathcal{A} = \langle \mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_r \rangle$.

Definition 2.3.2. Given an SOD $\mathcal{A} = \langle \mathcal{S}_1, \mathcal{S}_2, \dots, \mathcal{S}_r \rangle$ we define its *projection functors* as

$$\pi_{\mathcal{S}_i} \colon \mathcal{A} \to \mathcal{S}_i \qquad A \mapsto \mathrm{cone}(E_i \to E_{i-1})$$

where $A \in \mathcal{A}$ and the E_i's are defined in Definition 2.3.1 (2).

Remark 2.3.3. The semiorthogonality property Definition 2.3.1 (1) implies that the filtration of Definition 2.3.1 (2) is functorial, by which we mean that the E_i's, the maps $E_i \to E_{i-1}$, and the A_i's are uniquely defined up to unique isomorphism. Thus, the projection functors are well defined. Moreover, it is easy to see that π_1 is the left adjoint to the inclusion $i_{\mathcal{S}_1} \colon \mathcal{S}_1 \hookrightarrow \mathcal{A}$, and that π_r is the right adjoint to the inclusion $i_{\mathcal{S}_r} \colon \mathcal{S}_r \hookrightarrow \mathcal{A}$, see [Bon89, Lemma 3.1].

For the convenience of the reader, and because it will be the main case of interest in most of the thesis, we spell out the definition in the case $r = 2$.

Let us consider two triangulated subcategories $\mathcal{S}_1, \mathcal{S}_2 \subset \mathcal{A}$. Then, according to the above definition, they give an SOD of $\mathcal{A}$ if $\mathrm{Hom}_{\mathcal{A}}(i_{\mathcal{S}_2}(S_2), i_{\mathcal{S}_1}(S_1)) = 0$ for any $S_i \in \mathcal{S}_i$, $i = 1, 2$, and for any $A \in \mathcal{A}$ there exists a distinguished triangle (which is unique up to unique isomorphism, see Remark 2.3.3)

$$i_{\mathcal{S}_2}(A_{\mathcal{S}_2}) \to A \to i_{\mathcal{S}_1}(A_{\mathcal{S}_1}) \tag{2.1}$$

where $A_{\mathcal{S}_1} \in \mathcal{S}_1$, $A_{\mathcal{S}_2} \in \mathcal{S}_2$.

Notice that, when $\mathcal{A} = \langle \mathcal{S}_1, \mathcal{S}_2 \rangle$, by Remark 2.3.3 the inclusions $i_{\mathcal{S}_1} \colon \mathcal{S}_1 \hookrightarrow \mathcal{A}$ and $i_{\mathcal{S}_2} \colon \mathcal{S}_2 \hookrightarrow \mathcal{A}$ have a left and right adjoint given by the projection functors $\pi_{\mathcal{S}_1}$ and $\pi_{\mathcal{S}_2}$, respectively. In this case, we write

$$i^L_{\mathcal{S}_1} = \pi_{\mathcal{S}_1} \quad \text{and} \quad i^R_{\mathcal{S}_2} = \pi_{\mathcal{S}_2}.$$

Remark 2.3.4. By what we explained above, in (2.1) we have $A_{\mathcal{S}_2} = \pi_{\mathcal{S}_2}(A) = i^R_{\mathcal{S}_2}(A)$ and $A_{\mathcal{S}_1} = \pi_{\mathcal{S}_1}(A) = i^L_{\mathcal{S}_1}(A)$. Thus, when $\mathcal{A} = \langle \mathcal{S}_1, \mathcal{S}_2 \rangle$ we have the distinguished triangle

$$i_{\mathcal{S}_2} i^R_{\mathcal{S}_2} \to \mathrm{id}_{\mathcal{A}} \to i_{\mathcal{S}_1} i^L_{\mathcal{S}_1}. \tag{2.2}$$

Let us now turn back our attention to the general case. By definition, when we have an SOD $\mathcal{A} = \langle \mathcal{S}_1, \dots, \mathcal{S}_r \rangle$ morphisms from objects of $\mathcal{S}_k$ to objects of $\mathcal{S}_j$ are zero for $k > j$. On the other hand, morphisms from objects of $\mathcal{S}_j$ to objects of $\mathcal{S}_k$ can be rather mysterious. Gluing functors help us to shed some light on these morphisms.

To the knowledge of the author, the first appearance of gluing functors in the literature was [KL15, Definition 2.4].

Definition 2.3.5. Let $\mathcal{A}$ be a triangulated category with an SOD $\mathcal{A} = \langle \mathcal{S}_1, \mathcal{S}_2, \dots, \mathcal{S}_r \rangle$. We say that the pair $(\mathcal{S}_j, \mathcal{S}_k)$, $k > j$, has a *left gluing functor* if there exists a functor $\phi_{jk} \colon \mathcal{S}_j \to \mathcal{S}_k$ and bifunctorial isomorphisms

$$\mathrm{Hom}_{\mathcal{A}}(i_{\mathcal{S}_j}(S_j)[-1], i_{\mathcal{S}_k}(S_k)) \simeq \mathrm{Hom}_{\mathcal{S}_k}(\phi_{jk}(S_j), S_k)$$

for any $S_j \in \mathcal{S}_j$, $S_k \in \mathcal{S}_k$.

Similarly, we say that the pair $(\mathcal{S}_j, \mathcal{S}_k)$, $k > j$, has a *right gluing functor* if there exists a functor $\phi_{kj} \colon \mathcal{S}_k \to \mathcal{S}_j$ and bifunctorial isomorphisms

$$\mathrm{Hom}_{\mathcal{A}}(i_{\mathcal{S}_j}(S_j)[-1], i_{\mathcal{S}_k}(S_k)) \simeq \mathrm{Hom}_{\mathcal{S}_j}(S_j, \phi_{kj}(S_k))$$

for any $S_j \in \mathcal{S}_j$, $S_k \in \mathcal{S}_k$.

Remark 2.3.6. If we have an SOD $\mathcal{A} = \langle \mathcal{S}_1, \mathcal{S}_2, \dots, \mathcal{S}_r \rangle$ and for some $k \in \{1, \dots, r\}$ the functor $i_{\mathcal{S}_k}$ has a left adjoint $i^L_{\mathcal{S}_k}$, then $\phi_{jk} = i^L_{\mathcal{S}_k} i_{\mathcal{S}_j}[-1]$ is a left gluing functor for the pair $(\mathcal{S}_j, \mathcal{S}_k)$ for any $k > j$. Similarly, if $i_{\mathcal{S}_j}$ has a right adjoint $i^R_{\mathcal{S}_j}$, then $\phi_{kj} = i^R_{\mathcal{S}_j} i_{\mathcal{S}_k}[1]$ is a right gluing functor for the pair $(\mathcal{S}_j, \mathcal{S}_k)$ for any $k > j$.

Remark 2.3.7. Notice that, by the Yoneda lemma, if a left gluing functor for the couple $(\mathcal{S}_j, \mathcal{S}_k)$ exists, then it is unique up to isomorphism. Similarly, if a right gluing functor exists, it is unique up to isomorphism.

Gluing functors will play a prominent role in § 3. For some examples of gluing functors, the reader can have a look at § 2.4.7.

2.3.2 Inducing SODs and compactness

Among all triangulated categories we will be interested in those containing arbitrary small direct sums. This is because having all small direct sums allows us to use theorems such as Brown representability and the adjoint functor theorem [Nee96], which tell us when a functor has a right adjoint, see Remark 2.3.29.

Definition 2.3.8. Let $\mathcal{A}$ be a triangulated category.

i) $\mathcal{A}$ is called *cocomplete* if it is closed under arbitrary small direct sums.

ii) An object $A \in \mathcal{A}$ is called *compact* if for every family of objects $B_i \in \mathcal{A}$ the canonical morphism

$$\oplus \mathrm{Hom}_{\mathcal{A}}(A, B_i) \to \mathrm{Hom}_{\mathcal{A}}(A, \oplus B_i)$$

is an isomorphism. The subcategory of compact objects is denoted by $\mathcal{A}^c$.

iii) A subcategory $\mathcal{S} \subset \mathcal{A}$ is called *localising* if it is closed under small direct sums in $\mathcal{A}$.

iv) If $\mathcal{B}$ is another cocomplete triangulated category, we say that a functor $\alpha\colon \mathcal{A} \to \mathcal{B}$

a) is *cocontinuous* if for every family of objects $A_i \in \mathcal{A}$ the canonical morphism

$$\oplus\alpha(A_i) \to \alpha(\oplus A_i),$$

obtained by applying α to the morphisms $A_j \to \oplus A_i$, is an isomorphism

b) *preserves compactness* if $\alpha(\mathcal{A}^c) \subset \mathcal{B}^c$

Remark 2.3.9. Requiring $\mathcal{A}$ to be cocomplete is almost always not restrictive because one can formally add direct sums by passing to the *ind-completion* $\mathrm{Ind}(\mathcal{A})$ and then recover $\mathcal{A}$ by taking the compact objects of $\mathrm{Ind}(\mathcal{A})$, see e.g. [KS06, Exercise 6.1 (iii)] or [Lur09, Proposition 5.3.5.11] for a more recent account in the framework of $(\infty, 1)$-categories. Notice that this argument works only if $\mathcal{A}$ is idempotent complete[1] because otherwise $\mathcal{A} \subsetneq \mathrm{Ind}(\mathcal{A})^c$.

Remark 2.3.10. Notice that if $\mathcal{S} \subset \mathcal{A}$ is localising, then $i_{\mathcal{S}}\colon \mathcal{S} \hookrightarrow \mathcal{A}$ is cocontinuous.

Remark 2.3.11. If $\alpha : \mathcal{A} \to \mathcal{B}$ is a cocontinuous functor, then its essential image $\operatorname{im}\alpha \subset \mathcal{B}$ is a localising subcategory of $\mathcal{B}$.

Remark 2.3.12. Notice that, by definition, if we have a distinguished triangle of functors $\alpha_1 \to \alpha_2 \to \alpha_3$ and two of the three functors are cocontinuous, so is the third.

Remark 2.3.13. Not all abstractly cocomplete subcategories of a cocomplete category are localising. For example, consider $f\colon X \to Y$ a map of Noetherian schemes over a field k of characteristic zero such that f_* does not preserve compactness and $f_*\mathcal{O}_X \simeq \mathcal{O}_Y$, e.g. a resolution of rational singularities. Then, $f^\times = (f_*)^R$ is a fully faithful functor by [Nee18a, Remark 6.1.1]. In particular, $f^\times \mathrm{D}_{\mathrm{qc}}(Y) \subset \mathrm{D}_{\mathrm{qc}}(X)$ is abstractly cocomplete, but the inclusion functor is not cocontinuous because its left adjoint is f_*, which does not preserve compactness.

Example 2.3.14. If X is a separated scheme of finite type over a field k, and $\mathrm{D}_{\mathrm{qc}}(X)$ is the derived category of quasi-coherent sheaves on X, then $\mathrm{D}_{\mathrm{qc}}(X)$ is cocomplete and $\mathrm{D}_{\mathrm{qc}}(X)^c = \mathrm{D}_{\mathrm{perf}}(X)$, *i.e.*, the compact objects in $\mathrm{D}_{\mathrm{qc}}(X)$ are the perfect complexes: complexes which are locally quasi-isomorphic to a complex of locally free sheaves of finite rank, see [Nee96].

The next notion we concern ourselves with answers the following question: when is it true that subcategories of categories possessing an SOD inherit an SOD?

[1] A category $\mathcal{A}$ is called *idempotent complete* if given any $A \in \mathcal{A}$ and any $p\colon A \to A$ such that $p^2 = p$ there exists $E \in \mathcal{A}$ and morphisms $i\colon E \to A$, $r\colon A \to E$, such that $ri = \mathrm{id}_A$ and $ir = p$.

Definition 2.3.15 ([Kuz11, § 3])**.** Let $\mathcal{S} \subset \mathcal{A}$ be a triangulated subcategory and assume that we have an SOD $\mathcal{A} = \langle \mathcal{S}_1, \mathcal{S}_2, \dots, \mathcal{S}_r \rangle$ with projection functors $\pi_{\mathcal{S}_i}$, $i = 1, \dots, r$. We say that the SOD of $\mathcal{A}$ *induces* an SOD of $\mathcal{S}$ if for any $i = 1, \dots, r$ we have $\pi_{\mathcal{S}_i}(\mathcal{S}) \subset \mathcal{S}$.

Remark 2.3.16. If $\mathcal{S} \subset \mathcal{A} = \langle \mathcal{S}_1, \mathcal{S}_2, \dots, \mathcal{S}_r \rangle$ and the SOD of $\mathcal{A}$ induces an SOD of $\mathcal{S}$, then we have $\mathcal{S} = \langle \mathcal{S}_1 \cap \mathcal{S}, \mathcal{S}_2 \cap \mathcal{S}, \dots, \mathcal{S}_r \cap \mathcal{S} \rangle$.

The following lemma explains when an SOD of a cocomplete triangulated category $\mathcal{A}$ induces an SOD of $\mathcal{A}^c$.

Lemma 2.3.17. *Let $\mathcal{A} = \langle \mathcal{S}_1, \dots, \mathcal{S}_r \rangle$ be an SOD of a cocomplete triangulated category. Assume that the $\mathcal{S}_i$'s are localising subcategories, and that there exist left gluing functors $\phi_{ij} : \mathcal{S}_i \to \mathcal{S}_j$ for any $i < j$. If ϕ_{ij} preserves compactness for any $i < j$, then the SOD of $\mathcal{A}$ induces an SOD of $\mathcal{A}^c$. Furthermore, the inclusion functors $\mathcal{S}_i \hookrightarrow \mathcal{A}$, $i = 1, \dots, r$, preserve compactness and we have $\mathcal{A}^c = \langle \mathcal{S}_1^c, \dots, \mathcal{S}_r^c \rangle$.*

Remark 2.3.18. If $\mathcal{A} = \langle \mathcal{S}_1, \dots, \mathcal{S}_r \rangle$ is an SOD and the $\mathcal{S}_i$ are localising subcategories, then the projection functors $\pi_{\mathcal{S}_i}$ are cocontinuous. Even more is true, namely if E^j, $j \in J$, is a family of objects in $\mathcal{A}$, then the filtration in Definition 2.3.1 (2) for $\oplus_j E^j$ is the direct sum of the filtrations for the objects E^j. This property follows from the uniqueness of the filtration, see Remark 2.3.3, and by the fact that the categories $\mathcal{S}_i$ are localising.

Proof of Lemma 2.3.17. We prove the case $r = 2$ for simplicity, the general case can be proved with similar arguments.

Notice that an object $A \in \mathcal{A}$ is compact if and only if the functor $\operatorname{Hom}^\bullet_{\mathcal{A}}(A, -)$ is cocontinuous. We will use this equivalent characterisation in the course of the proof because it will allow us to speak of distinguished triangles rather than of long exact sequences. Moreover, to ease the notation, we write $\phi = \phi_{12}$.

First, we prove that

$$\mathcal{S}_1 \cap \mathcal{A}^c = \mathcal{S}_1^c \quad \text{and} \quad \mathcal{S}_2 \cap \mathcal{A}^c = \mathcal{S}_2^c. \tag{2.3}$$

The inclusions $\mathcal{S}_1 \cap \mathcal{A}^c \subset \mathcal{S}_1^c$ and $\mathcal{S}_2 \cap \mathcal{A}^c \subset \mathcal{S}_2^c$ are obvious, therefore we only have to prove the reverse inclusions. The functor $i_{\mathcal{S}_2}$ has a cocontinuous right adjoint by Remark 2.3.18, thus it preserves compactness, and we get $\mathcal{S}_2 \cap \mathcal{A}^c \supset \mathcal{S}_2^c$. Now take $S_1 \in \mathcal{S}_1^c$ and $A \in \mathcal{A}$, and apply $\operatorname{Hom}^\bullet_{\mathcal{A}}(i_{\mathcal{S}_1}(S_1), -)$ to the distinguished triangle (2.1) for A. We get the distinguished triangle

$$\operatorname{Hom}^\bullet_{\mathcal{A}}(i_{\mathcal{S}_1}(S_1), i_{\mathcal{S}_1} i^L_{\mathcal{S}_1}(A)) \to \operatorname{Hom}^\bullet_{\mathcal{A}}(i_{\mathcal{S}_1}(S_1), A) \to \operatorname{Hom}^\bullet_{\mathcal{A}}(i_{\mathcal{S}_1}(S_1), i_{\mathcal{S}_2} i^R_{\mathcal{S}_2}(A)).$$

Let us consider the previous triangle as a triangle of functors in the variable $A \in \mathcal{A}$. Then, the functor $\operatorname{Hom}^\bullet_{\mathcal{A}}(i_{\mathcal{S}_1}(S_1), i_{\mathcal{S}_1} i^L_{\mathcal{S}_1}(A)) \simeq \operatorname{Hom}^\bullet_{\mathcal{S}_1}(S_1, i^L_{\mathcal{S}_1}(A))$ is cocontinuous in A because S_1

is compact and $i^L_{\mathcal{S}_1}$ is cocontinuous by Remark 2.3.18. Moreover, the functor

$$\mathrm{Hom}^\bullet_{\mathcal{A}}(i_{\mathcal{S}_1}(S_1), i_{\mathcal{S}_2} i^R_{\mathcal{S}_2}(A)) \simeq \mathrm{Hom}^\bullet_{\mathcal{S}_2}(\phi(S_1)[1], i^R_{\mathcal{S}_2}(A))$$

is cocontinuous in A because S_1 is compact, ϕ preserves compactness, and $i^R_{\mathcal{S}_2}$ is cocontinuous by Remark 2.3.18. Hence, the functor $\mathrm{Hom}^\bullet_{\mathcal{A}}(i_{\mathcal{S}_1}(S_1), A)$ is cocontinuous in A, *i.e.*, $i_{\mathcal{S}_1}(S_1)$ is compact and $\mathcal{S}_1^{\mathrm{c}} \subset \mathcal{S}_1 \cap \mathcal{A}^{\mathrm{c}}$. Thus, (2.3) is proved.

Now that we have established (2.3), to conclude we only have to prove that the SOD of $\mathcal{A}$ induces an SOD of $\mathcal{A}^{\mathrm{c}}$. Indeed, then it will follow that $\mathcal{A}^{\mathrm{c}} = \langle \mathcal{S}_1 \cap \mathcal{A}^{\mathrm{c}}, \mathcal{S}_2 \cap \mathcal{A}^{\mathrm{c}} \rangle = \langle \mathcal{S}_1^{\mathrm{c}}, \mathcal{S}_2^{\mathrm{c}} \rangle$.

By Definition 2.3.15, to prove that the SOD of $\mathcal{A}$ induces an SOD of $\mathcal{A}^{\mathrm{c}}$ we have to prove that $i^L_{\mathcal{S}_1}$ and $i^R_{\mathcal{S}_2}$ send $\mathcal{A}^{\mathrm{c}}$ to $\mathcal{S}_1 \cap \mathcal{A}^{\mathrm{c}}$ and $\mathcal{S}_2 \cap \mathcal{A}^{\mathrm{c}}$, respectively. However, by (2.3) this is equivalent to say that $i^L_{\mathcal{S}_1}$ and $i^R_{\mathcal{S}_2}$ preserve compactness. We prove the latter claim. The functor $i^L_{\mathcal{S}_1}$ preserves compactness because its right adjoint is cocontinuous by Remark 2.3.10. To prove that $i^R_{\mathcal{S}_2}$ preserves compactness we take $S_2 \in \mathcal{S}_2$ and $A \in \mathcal{A}^{\mathrm{c}}$, we apply $\mathrm{Hom}^\bullet_{\mathcal{A}}(-, i_{\mathcal{S}_2}(S_2))$ to the distinguished triangle (2.1) for A, and we proceed as we did above to show that $\mathrm{Hom}^\bullet_{\mathcal{S}_2}(i^R_{\mathcal{S}_2}(A), S_2)$ is cocontinuous in the variable $S_2 \in \mathcal{S}_2$. □

Similarly, one can prove

Lemma 2.3.19. *Let $\mathcal{A} = \langle \mathcal{S}_1, \dots, \mathcal{S}_r \rangle$ be an SOD of a cocomplete triangulated category. Assume that the $\mathcal{S}_i$'s are localising subcategories, and that there exist right gluing functors $\phi_{ji} : \mathcal{S}_j \to \mathcal{S}_i$ for any $i < j$. If ϕ_{ji} preserves compactness for any $i < j$, then the SOD of $\mathcal{A}$ induces an SOD of $\mathcal{A}^{\mathrm{c}}$. Furthermore, the inclusion functors $\mathcal{S}_i \hookrightarrow \mathcal{A}$, $i = 1, \dots, r$, preserve compactness and we have $\mathcal{A}^{\mathrm{c}} = \langle \mathcal{S}_1^{\mathrm{c}}, \dots, \mathcal{S}_r^{\mathrm{c}} \rangle$.*

Remark 2.3.20. In general, it is not true that SODs of $\mathcal{A}$ induce SODs of $\mathcal{A}^{\mathrm{c}}$. For a counterexample, see Example 2.4.31.

Before concluding this subsection, we focus on a particular type of SOD that will play a fundamental role in § 4.

Definition 2.3.21. Let $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3, \mathcal{S}_4 \subset \mathcal{A}$ be four triangulated subcategories. We say that they give a *four periodic SOD* of $\mathcal{A}$ if we have the following SODs

$$\mathcal{A} = \langle \mathcal{S}_1, \mathcal{S}_2 \rangle = \langle \mathcal{S}_2, \mathcal{S}_3 \rangle = \langle \mathcal{S}_3, \mathcal{S}_4 \rangle = \langle \mathcal{S}_4, \mathcal{S}_1 \rangle.$$

The following lemma shows that four periodic SODs always induce SODs of compact objects.

Lemma 2.3.22. *Assume we have a four periodic SOD*

$$\mathcal{A} = \langle \mathcal{S}_1, \mathcal{S}_2 \rangle = \langle \mathcal{S}_2, \mathcal{S}_3 \rangle = \langle \mathcal{S}_3, \mathcal{S}_4 \rangle = \langle \mathcal{S}_4, \mathcal{S}_1 \rangle \tag{2.4}$$

where $\mathcal{S}_j$ is a localising subcategory for every j. Then, the SODs in (2.4) induce SODs of $\mathcal{A}^c$. Furthermore, $\mathcal{S}_j^c = \mathcal{S}_j \cap \mathcal{A}^c$ for every j, and we have a four periodic SOD

$$\mathcal{A}^c = \langle \mathcal{S}_1^c, \mathcal{S}_2^c \rangle = \langle \mathcal{S}_2^c, \mathcal{S}_3^c \rangle = \langle \mathcal{S}_3^c, \mathcal{S}_4^c \rangle = \langle \mathcal{S}_4^c, \mathcal{S}_1^c \rangle.$$

Proof. By the symmetry of the situation, it is enough to prove that $\mathcal{A} = \langle \mathcal{S}_1, \mathcal{S}_2 \rangle$ induces an SOD of $\mathcal{A}^c$, that $\mathcal{S}_j^c = \mathcal{S}_j \cap \mathcal{A}^c$ for $j = 1, 2$, and that we have $\mathcal{A}^c = \langle \mathcal{S}_1^c, \mathcal{S}_2^c \rangle$. We prove this statement.

From the fourth SOD in (2.4) and Remark 2.3.3, we see that $i_{\mathcal{S}_1}$ has a right adjoint $i_{\mathcal{S}_1}^R$. Moreover, as $\mathcal{S}_1$ and $\mathcal{S}_4$ are localising, by Remark 2.3.18 the functor $i_{\mathcal{S}_1}^R$ is cocontinuous. Hence, by Remark 2.3.6 the SOD $\mathcal{A} = \langle \mathcal{S}_1, \mathcal{S}_2 \rangle$ has a right gluing functor $\phi_{21} = i_{\mathcal{S}_1}^R i_{\mathcal{S}_2}[1]$ that is cocontinuous, and therefore the statement follows from Lemma 2.3.19. □

2.3.3 Admissibility

A notion that is strictly related to that of an SOD is that of *admissibility*.

Definition 2.3.23. Let $\mathcal{A}$ be a triangulated category and $\mathcal{S} \subset \mathcal{A}$ be a triangulated subcategory. We say that $\mathcal{S}$ is *left admissible* if the inclusion functor $i_{\mathcal{S}} \colon \mathcal{S} \hookrightarrow \mathcal{A}$ has a left adjoint. Similarly, $\mathcal{S}$ is *right admissible* if $i_{\mathcal{S}}$ has a right adjoint. We say that $\mathcal{S}$ is admissible if it is both left and right admissible.

By Remark 2.3.3, we know that if we have an SOD $\mathcal{A} = \langle \mathcal{S}_1, \ldots, \mathcal{S}_r \rangle$, then $\mathcal{S}_1$ is left admissible and $\mathcal{S}_r$ is right admissible. There is a converse statement due to Bondal that we now recall.

Given $\mathcal{S} \subset \mathcal{A}$, we define its right orthogonal as

$$\mathcal{S}^\perp = \{A \in \mathcal{A} : \operatorname{Hom}^\bullet_{\mathcal{A}}(S, A) = 0 \quad \forall S \in \mathcal{S}\}.$$

Similarly we define its left orthogonal ${}^\perp\mathcal{S}$.

Lemma 2.3.24 ([Bon89])**.** *Let $\mathcal{A}$ be a triangulated category and $\mathcal{S} \subset \mathcal{A}$ be a triangulated subcategory. Then, $\mathcal{S}$ is left admissible if and only if we have an SOD $\mathcal{A} = \langle \mathcal{S}, {}^\perp\mathcal{S} \rangle$. Similarly, $\mathcal{S}$ is right admissible if and only if we have an SOD $\mathcal{A} = \langle \mathcal{S}^\perp, \mathcal{S} \rangle$.*

2.3.4 Generation

We conclude this section by dealing with the notion of *generation* in the world of triangulated categories. Furthermore, we prove Lemma 2.3.31, which helps us in constructing SODs.

Definition 2.3.25. Let $\mathcal{A}$ be a triangulated category and $\mathcal{S} \subset \mathcal{A}$ be a triangulated subcategory. The subcategory $\mathcal{S}$ is called *thick* if it closed under taking direct summands, *i.e.*, if for any $A_1, A_2 \in \mathcal{A}$ the condition $A_1 \oplus A_2 \in \mathcal{S}$ implies $A_1, A_2 \in \mathcal{S}$.

Example 2.3.26. For a cocomplete triangulated category $\mathcal{A}$, the subcategory of compact objects $\mathcal{A}^{\mathrm{c}}$ is thick.

Definition 2.3.27. Let $\mathcal{A}$ be a cocomplete triangulated category and $\{A_i\}$ be a family of objects in $\mathcal{A}$. We define $\langle\{A_i\}\rangle^{\mathrm{thick}}$ as the smallest thick triangulated subcategory of $\mathcal{A}$ containing the A_i's. Similarly, we define $\langle\{A_i\}\rangle^{\oplus}$ as the smallest cocomplete triangulated subcategory of $\mathcal{A}$ containing the A_i's.

We say that $\mathcal{A}$ is *compactly generated* if there exists a family of objects $\{A_i\} \subset \mathcal{A}^{\mathrm{c}}$ such that $\mathcal{A} = \langle\{A_i\}\rangle^{\oplus}$.

Remark 2.3.28. If $\mathcal{A}$ is a cocomplete triangulated category such that $\mathcal{A} = \langle\{A_i\}\rangle^{\oplus}$ for some $\{A_i\} \subset \mathcal{A}^{\mathrm{c}}$, then $\mathcal{A}^{\mathrm{c}} = \langle\{A_i\}\rangle^{\mathrm{thick}}$. See e.g. [Sta18, Tag 09SR].

Remark 2.3.29. Now that we have defined what it means for a cocomplete triangulated category $\mathcal{A}$ to be compactly generated, we can explain our interest in cocomplete triangulated categories.

If $\alpha\colon \mathcal{A} \to \mathcal{B}$ is a functor between cocomplete triangulated categories with a right adjoint $\alpha^R\colon \mathcal{A} \to \mathcal{B}$, then α is a cocontinuos functor. The converse is true if we assume that $\mathcal{A}$ is compactly generated. Namely, [Nee96, Theorem 4.1], also known as *the adjoint functor theorem*, proves that if $\mathcal{A}$ is compactly generated and α is cocontinuous, then α has a right adjoint.

Example 2.3.30. If X is a separated scheme of finite type over a field k, then $\mathrm{D}_{\mathrm{qc}}(X)$ is compactly generated by $\mathrm{D}_{\mathrm{qc}}(X)^{\mathrm{c}} = \mathrm{D}_{\mathrm{perf}}(X)$, see [Nee96].

Lemma 2.3.31. *Let $\mathcal{A}$ be a triangulated category and $\mathcal{S} \subset \mathcal{A}$ be a localising subcategory. Assume that $\mathcal{S}$ is generated by a family of objects which are compact in $\mathcal{A}$, namely $\mathcal{S} = \langle\{S_i\}\rangle^{\oplus}$ for some $\{S_i\} \subset \mathcal{A}^{\mathrm{c}}$. Then, we have $\mathcal{A} = \langle \mathcal{S}^{\perp}, \mathcal{S}\rangle$.*

Proof. By [Nee96, Theorem 4.1] $i_{\mathcal{S}}$ has a right adjoint, *i.e.*, $\mathcal{S}$ is right admissible. Then, the result follows from Lemma 2.3.24. □

2.4 Differential graded categories

As we explained in § 2.1, we will need to work in a richer framework than the one of triangulated categories. For this reason, in this section we introduce dg-categories, their derived categories, and functors between them. There is almost no novel mathematics in this chapter, and the reader well acquainted with dg-categories can safely move on to § 3. Our main reference for the material on dg-categories is [AL17].

Let k be a fixed field. The category **Mod**-k is the category of differential graded modules (dg-modules) over k, *i.e.*, graded k-modules $V = \bigoplus_{n\in\mathbb{Z}} V^n$ equipped with a k-linear endomorphism d_V, called the *differential*, such that $d_V(V^i) \subset V^{i+1}$ and $d_V^2 = 0$. For any two dg k-modules (V, d_V), (W, d_W), we define the hom space between them as

$$\mathrm{Hom}_{\mathbf{Mod}\text{-}k}((V, d_V), (W, d_W)) = \bigoplus_{n\in\mathbb{Z}} \mathrm{Hom}^n((V, d_V), (W, d_W)),$$

where $f \in \mathrm{Hom}^n((V, d_V), (W, d_W))$ is a homomorphism of k-vector spaces $f\colon V \to W$ such that $f(V^p) \subset W^{p+n}$. We endow this graded k-module with the differential given by

$$d(\{f_n\}) = \{d_W \circ f_n - (-1)^n f_n \circ d_V\}.$$

The tensor product of (V, d_V) and (W, d_W) is defined as $(V \otimes_k W)^n = \oplus_{i+j=n} V^i \otimes_k W^j$ with differential $d_V \otimes \mathrm{id} + \mathrm{id} \otimes d_W$.

Definition 2.4.1. A dg-category $\mathcal{A}$ is a category enriched over **Mod**-k, *i.e.*, for any $a_1, a_2 \in \mathcal{A}$ the hom space $\mathrm{Hom}_{\mathcal{A}}(a_1, a_2)$ is an object in **Mod**-k, and the composition maps

$$\mathrm{Hom}_{\mathcal{A}}(a_2, a_3) \otimes_k \mathrm{Hom}_{\mathcal{A}}(a_1, a_2) \to \mathrm{Hom}_{\mathcal{A}}(a_1, a_3)$$

are closed, degree zero morphism of dg-k-modules for any $a_1, a_2, a_3 \in \mathcal{A}$.

Remark 2.4.2. All the dg-categories considered in this thesis are assumed to be small, *i.e.*, the collection of objects and the collection of morphisms are sets. In the rest of the thesis, we will drop the adjective small most of the times.

Remark 2.4.3. In the literature, there are two different definitions of dg-categories. One requires $\mathcal{A}$ to be additive, the other does not. The drawback of requiring $\mathcal{A}$ to be additive is that then we cannot consider a dg-algebra as a dg-category with one object. For this reason, we do not require our dg-categories to be additive.

A dg-functor $F\colon \mathcal{A} \to \mathcal{C}$ between two dg-categories is a functor such that for any $a_1, a_2 \in \mathcal{A}$ the map $\mathrm{Hom}_{\mathcal{A}}(a_1, a_2) \xrightarrow{F} \mathrm{Hom}_{\mathcal{C}}(F(a_1), F(a_2))$ is a closed, degree zero morphism in **Mod**-k.

Remark 2.4.4. The reader might be worried that if $\mathcal{A}$ and $\mathcal{C}$ are additive, one should require dg-functors to be additive, *i.e.*, $F(a_1 \oplus a_2) \simeq F(a_1) \oplus F(a_2)$ for any $a_1, a_2 \in \mathcal{A}$. However, every dg-functor between additive dg-categories is automatically additive because it is k-linear.

A right dg-module over $\mathcal{A}$ is a dg-functor $\mathcal{A}^{\mathrm{op}} \to \mathbf{Mod}\text{-}k$. The category of right dg-modules over $\mathcal{A}$ is denoted by $\mathbf{Mod}\text{-}\mathcal{A}$, and for $M \in \mathbf{Mod}\text{-}\mathcal{A}$ we write M_a for the image of $a \in \mathcal{A}$. An example of a right dg-module is given for any $a \in \mathcal{A}$ by $h^a_{(-)} = \mathrm{Hom}_{\mathcal{A}}(-, a)$. We then get a dg-functor $a \mapsto h^a$ that turns out to be fully faithful and that is called the *Yoneda embedding*. In view of this embedding, it makes sense to write $\mathrm{Hom}_{\mathbf{Mod}\text{-}\mathcal{A}}(-,-) = \mathrm{Hom}_{\mathcal{A}}(-,-)$.

We now define the derived category of $\mathcal{A}$. The aim is to invert *quasi-isomorphisms*, *i.e.*, morphisms $f\colon M \to M'$ in $\mathbf{Mod}\text{-}\mathcal{A}$ such that f_a is a quasi-isomorphism in $\mathbf{Mod}\text{-}k$ for every $a \in \mathcal{A}$. To achieve this goal, we quotient by *acyclic* modules: a module $S \in \mathbf{Mod}\text{-}\mathcal{A}$ is called acyclic if for every $a \in \mathcal{A}$ the complex S_a is acyclic. We write $Acycl(\mathcal{A})$ for the full subcategory of acyclic modules. Then, we define the derived category of $\mathcal{A}$ as the Verdier quotient

$$\mathrm{D}(\mathcal{A}) = \mathrm{H}^0(\mathbf{Mod}\text{-}\mathcal{A}) \,/\mathrm{H}^0(Acycl(\mathcal{A}))$$

where, given a dg-category $\mathcal{C}$, the category $\mathrm{H}^0(\mathcal{C})$ is the category with the same objects as $\mathcal{C}$ and with morphisms

$$\mathrm{Hom}_{\mathrm{H}^0(\mathcal{C})}(c_1, c_2) = \mathrm{H}^0(\mathrm{Hom}_{\mathcal{C}}(c_1, c_2)) \qquad \forall\, c_1, c_2 \in \mathrm{H}^0(\mathcal{C}).$$

The category $\mathrm{D}(\mathcal{A})$ is a triangulated category with shift functor given by $M \mapsto M[1]$, $(M[1])_a = M_a[1]$.

Rather than $\mathcal{A}$, our main object of interest will be $\mathrm{D}(\mathcal{A})$. In particular, we are interested in constructing functors $\mathrm{D}(\mathcal{A}) \to \mathrm{D}(\mathcal{C})$. The way we do this is using bimodules.

Given two dg-categories $\mathcal{A}$ and $\mathcal{C}$, an $\mathcal{A}$-$\mathcal{C}$-bimodule is a dg-functor $\mathcal{A} \otimes_k \mathcal{C}^{\mathrm{op}} \to \mathbf{Mod}\text{-}k$, where $\mathcal{A} \otimes_k \mathcal{C}^{\mathrm{op}}$ is the dg-category whose objects are couples $(a, c) \in \mathcal{A} \times \mathcal{C}$ and whose morphisms are given by

$$\mathrm{Hom}_{\mathcal{A}\otimes_k\mathcal{C}^{\mathrm{op}}}((a_1, c_1), (a_2, c_2)) = \mathrm{Hom}_{\mathcal{A}}(a_1, a_2) \otimes_k \mathrm{Hom}_{\mathcal{C}}(c_2, c_1).$$

For M an $\mathcal{A}$-$\mathcal{C}$-bimodule we write ${}_aM_c$ for the image of $(a, c) \in \mathcal{A} \otimes_k \mathcal{C}^{\mathrm{op}}$. The category of $\mathcal{A}$-$\mathcal{C}$-bimodules is denoted by $\mathcal{A}\text{-}\mathbf{Mod}\text{-}\mathcal{C}$ and its derived category by $\mathrm{D}(\mathcal{A}\text{-}\mathcal{C})$.

Example 2.4.5. For a dg-category $\mathcal{A}$, the diagonal bimodule is denoted by $\mathcal{A} \in \mathcal{A}\text{-}\mathbf{Mod}\text{-}\mathcal{A}$ and it is given by

$${}_{a_1}\mathcal{A}_{a_2} = \mathrm{Hom}_{\mathcal{A}}(a_2, a_1) \qquad \forall a_1, a_2 \in \mathcal{A}.$$

Given a third dg-category $\mathcal{B}$, one can define a tensor product functor

$$- \otimes_{\mathcal{B}} -\colon \mathcal{A}\text{-}\mathbf{Mod}\text{-}\mathcal{B} \otimes_k \mathcal{B}\text{-}\mathbf{Mod}\text{-}\mathcal{C} \to \mathcal{A}\text{-}\mathbf{Mod}\text{-}\mathcal{C}.$$

This functor does not induce a functor between derived categories on the nose because

$-\otimes_{\mathcal{B}}-$ does not preserve acyclicity. To produce a functor between derived categories, we need to use h-projective bimodules.

Definition 2.4.6. A module $P \in \mathbf{Mod}\text{-}\mathcal{A}$ is called *h-projective* if for any $S \in Acycl(\mathcal{A})$ we have $\mathrm{Hom}_{\mathrm{H}^0(\mathbf{Mod}\text{-}\mathcal{A})}(P, S) = 0$. The subcategory of h-projective modules is denoted by $\mathcal{P}(\mathcal{A})$.

A bimodule $P \in \mathcal{A}\text{-}\mathbf{Mod}\text{-}\mathcal{C}$ is called $\mathcal{A}$-h-projective if for any $c \in \mathcal{C}$ the module P_c is h-projective. Similarly, we define $\mathcal{C}$-h-projectivity.

For any $\mathcal{A}$-h-projective bimodule $P \in \mathcal{A}\text{-}\mathbf{Mod}\text{-}\mathcal{C}$ the functor $-\otimes_{\mathcal{A}} P\colon \mathbf{Mod}\text{-}\mathcal{A} \to \mathbf{Mod}\text{-}\mathcal{C}$ preserves acyclicity, and therefore descends to a functor $\alpha_P(-) = -\otimes_{\mathcal{A}} P\colon \mathrm{D}(\mathcal{A}) \to \mathrm{D}(\mathcal{C})$. To generalise this to any bimodule $M \in \mathcal{A}\text{-}\mathbf{Mod}\text{-}\mathcal{C}$ we use *h-projective resolutions.*

Given $M \in \mathcal{A}\text{-}\mathbf{Mod}\text{-}\mathcal{C}$, an h-projective bimodule $P \in \mathcal{A}\text{-}\mathbf{Mod}\text{-}\mathcal{C}$ is called an h-projective resolution of M if there exists a quasi-isomorphism $f\colon P \to M$. Similarly, we define $\mathcal{A}$-h-projective and $\mathcal{C}$-h-projective resolutions.

As h-projective resolutions exist, see e.g. [AL17, Corollary 2.6], any bimodule $M \in \mathcal{A}\text{-}\mathbf{Mod}\text{-}\mathcal{C}$ induces a functor at the level of derived categories by choosing an h-projective resolution. This functor does not depend on the resolution, and it is denoted by

$$\alpha_M(-) = - \overset{L}{\otimes}_{\mathcal{A}} M\colon \mathrm{D}(\mathcal{A}) \to \mathrm{D}(\mathcal{C}).$$

We now want to describe the adjoint functor to tensor product. Given $M \in \mathbf{Mod}\text{-}\mathcal{A}$, the $\mathcal{A}$-dual of M is defined as the $\mathcal{A}^{\mathrm{op}}$-module that assigns to any $a \in \mathcal{A}$ the complex $\mathrm{Hom}_{\mathcal{A}}(M, {}_a\mathcal{A})$ and it is denoted by $M^{\mathcal{A}}$. Thus, we get the *dualising functor* $(\mathbf{Mod}\text{-}\mathcal{A})^{\mathrm{op}} \to \mathbf{Mod}\text{-}\mathcal{A}^{\mathrm{op}}$, $M \mapsto M^{\mathcal{A}}$. Using an h-projective resolution of M, we can derive the functor $(-)^{\mathcal{A}}$ and obtain the *derived dualising functor* $\mathrm{D}(\mathcal{A})^{\mathrm{op}} \to \mathrm{D}(\mathcal{A}^{\mathrm{op}})$, $M \mapsto M^{\tilde{\mathcal{A}}}$.

These constructions can also be performed with $\mathcal{A}$-$\mathcal{C}$-bimodules, resulting in bimodules rather than modules, e.g. for $M \in \mathcal{A}\text{-}\mathbf{Mod}\text{-}\mathcal{C}$, we have $M^{\mathcal{A}} \in \mathcal{C}\text{-}\mathbf{Mod}\text{-}\mathcal{A}$. Moreover, they can be extended to functors $\mathrm{Hom}_{\mathcal{C}}(M, -)$ and $\mathrm{Hom}_{\mathcal{A}^{\mathrm{op}}}(M, -)$, of which $M^{\mathcal{A}}$ is the particular case $\mathrm{Hom}_{\mathcal{A}^{\mathrm{op}}}(M, \mathcal{A})$, see [AL17, § 2.1.5].

Given $M \in \mathcal{A}\text{-}\mathbf{Mod}\text{-}\mathcal{C}$, exactly as in the standard theory of modules over rings, we have adjunctions

$$- \otimes_{\mathcal{A}} M \dashv \mathrm{Hom}_{\mathcal{C}}(M, -) \quad \text{and} \quad M \otimes_{\mathcal{C}} - \dashv \mathrm{Hom}_{\mathcal{A}^{\mathrm{op}}}(M, -) \tag{2.5}$$

which can be derived (using an h-projective resolution of M) to adjunctions

$$- \overset{L}{\otimes}_{\mathcal{A}} M \dashv \mathrm{RHom}_{\mathcal{C}}(M, -) \quad \text{and} \quad M \overset{L}{\otimes}_{\mathcal{C}} - \dashv \mathrm{RHom}_{\mathcal{A}^{\mathrm{op}}}(M, -).$$

Definition 2.4.7. The unit and counit of the adjoint pair $(- \otimes_{\mathcal{A}} M, \mathrm{Hom}_{\mathcal{C}}(M,-))$ evaluated at the diagonal bimodules are called *trace map* and *action map*

$$\mathrm{tr}\colon M^{\mathcal{C}} \otimes_{\mathcal{A}} M \xrightarrow{\xi \otimes m \mapsto \xi(m)} \mathcal{C} \qquad \mathrm{act}\colon \mathcal{A} \xrightarrow{a \mapsto (m \mapsto am)} \mathrm{Hom}_{\mathcal{C}}(M,M)$$

The unit and counit of the adjoint pair $(- \overset{L}{\otimes}_{\mathcal{A}} M, \mathrm{RHom}_{\mathcal{C}}(M,-))$ evaluated at the diagonal bimodules are called *derived trace map* and *derived action map*

$$\mathrm{tr}\colon M^{\widetilde{\mathcal{C}}} \overset{L}{\otimes}_{\mathcal{A}} M \to \mathcal{C} \qquad \mathrm{act}\colon \mathcal{A} \to \mathrm{RHom}_{\mathcal{C}}(M,M)$$

Similarly, we define the trace and action maps, and their derived counterparts, for the adjoint pair $(M \otimes_{\mathcal{C}} -, \mathrm{Hom}_{\mathcal{A}^{\mathrm{op}}}(M,-))$.

The next question we want to tackle is: for which bimodules M are the functors $\mathrm{RHom}_{\mathcal{C}}(M,-)$ and $\mathrm{RHom}_{\mathcal{A}^{\mathrm{op}}}(M,-)$ isomorphic to the tensor product with some bimodule? To answer this question, we introduce the notion of *perfect* module.

Definition 2.4.8. A module $M \in \mathbf{Mod}\text{-}\mathcal{A}$ is called *perfect* if M is a compact object in $\mathrm{D}(\mathcal{A})$.

A bimodule $M \in \mathcal{A}\text{-}\mathbf{Mod}\text{-}\mathcal{C}$ is called $\mathcal{A}$-perfect if for any $c \in \mathcal{C}$ the module M_c is $\mathcal{A}$-perfect. Similarly, we define $\mathcal{C}$-perfectness.

Theorem 2.4.9 ([AL21, Theorem 4.1])**.** *Take $M \in \mathcal{A}\text{-}\mathbf{Mod}\text{-}\mathcal{C}$ and consider the induced functor $\alpha_M\colon \mathrm{D}(\mathcal{A}) \to \mathrm{D}(\mathcal{C})$. Then, the following are equivalent:*

1. *The right adjoint of α_M is cocontinuous (resp. the left adjoint exists).*
2. *The right (resp. left) adjoint functor of α_M is given by $- \overset{L}{\otimes}_{\mathcal{C}} M^{\widetilde{\mathcal{C}}}$ (resp. $- \overset{L}{\otimes}_{\mathcal{C}} M^{\widetilde{\mathcal{A}}}$).*
3. *M is $\mathcal{C}$- (resp. $\mathcal{A}$-) perfect.*
4. *α_M (resp. $M \overset{L}{\otimes}_{\mathcal{C}} -$) preserves compactness.*

Proof. In [AL21, Theorem 4.1] the equivalence between (1), (2) and (3) is proved. Thus, we only have to show that (3) is equivalent to (4).

In [AL21, Theorem 4.1] Anno and Logvinenko also prove that M is $\mathcal{C}$-perfect if and only if α_M preserves compactness. Thus, we only have to prove that M is $\mathcal{A}$-perfect if and only if $M \overset{L}{\otimes}_{\mathcal{C}} -$ preserves compactness. This statement is proved in [AL17, § 2.1.6]. □

We conclude this section by defining a map that will be ubiquitous in § 3.

Definition 2.4.10. Given $M \in \mathbf{Mod}\text{-}\mathcal{A}$ and $N \in \mathbf{Mod}\text{-}\mathcal{A}$, we define the *evaluation map* as

$$\mathrm{ev}\colon N \otimes_{\mathcal{A}} M^{\mathcal{A}} \xrightarrow{n \otimes \xi \mapsto (m \mapsto n\xi(m))} \mathrm{Hom}_{\mathcal{A}}(M,N).$$

Remark 2.4.11. As it is explained in [AL17, § 2], if M is $\mathcal{A}$-h-projective and $\mathcal{A}$-perfect, then the evaluation map is a quasi-isomorphism for every $N \in \mathbf{Mod}\text{-}\mathcal{A}$.

2.4.1 Restriction and induction

We now introduce induction and restriction functors, and we prove Proposition 2.4.14 (that will be useful in § 2.4.7), which tells us when the adjoint to a restriction functor is still a restriction functor.

Definition 2.4.12. Given a dg-functor $F\colon \mathcal{A} \to \mathcal{C}$, we define

$$\begin{aligned} \operatorname{Ind}_F\colon \mathbf{Mod}\text{-}\mathcal{A} \to \mathbf{Mod}\text{-}\mathcal{C} \qquad & M \mapsto M \otimes_{\mathcal{A}} {}_F\mathcal{C} \\ \operatorname{Res}_F\colon \mathbf{Mod}\text{-}\mathcal{C} \to \mathbf{Mod}\text{-}\mathcal{A} \qquad & M \mapsto \big(a \mapsto M_{F(a)}\big) \end{aligned}$$

where ${}_a({}_F\mathcal{C})_c = \operatorname{Hom}_{\mathcal{C}}(c, F(a))$.

The tensor-hom adjunction (2.5) tells us that we have $\operatorname{Ind}_F \dashv \operatorname{Res}_F$. Moreover, an explicit computation shows $\operatorname{Ind}_F(h^a) \simeq h^{F(a)}$ for any $a \in \mathcal{A}$.

The functor Res_F clearly maps acyclic modules to acyclic modules, and therefore descends to a functor between derived categories. The functor Ind_F is given by tensor product with the bimodule ${}_F\mathcal{C}$, and thus descends to the derived category using h-projective resolutions. We will write $L\operatorname{Ind}_F$ for the derived functor of Ind_F, and $M_F = \operatorname{Res}_F(M)$.

We have[2]

Proposition 2.4.13 ([KL15, Proposition 3.9])**.** *The functor $L\operatorname{Ind}_F$ is left adjoint to the functor Res_F and both functors commute with arbitrary direct sums. Moreover, we have $L\operatorname{Ind}_F(h^a) \simeq h^{F(a)}$.*

If F induces a fully faithful functor $\mathrm{H}^0(F)\colon \mathrm{H}^0(\mathcal{A}) \to \mathrm{H}^0(\mathcal{C})$, then $L\operatorname{Ind}_F$ is fully faithful. Finally, if $\mathrm{H}^0(F)$ is an equivalence, so is $L\operatorname{Ind}_F$.

Proposition 2.4.14. *The functor Res_F has a right adjoint given by the functor*

$$\operatorname{Hom}_{\mathcal{A}}(\mathcal{C}_F, -)\colon \mathbf{Mod}\text{-}\mathcal{A} \to \mathbf{Mod}\text{-}\mathcal{C}.$$

Moreover, if F has right adjoint F^R, then $\operatorname{Hom}_{\mathcal{A}}(\mathcal{C}_F, -) \simeq \operatorname{Res}_{F^R}$.

Proof. Notice that the functor Res_F is given by tensor product with the bimodule $\mathcal{C}_F$, that is $\operatorname{Res}_F(M) = M \otimes_{\mathcal{C}} \mathcal{C}_F$ for any $M \in \mathbf{Mod}\text{-}\mathcal{C}$. Therefore, by tensor-hom adjunction we get the adjunction $\operatorname{Res}_F \dashv \operatorname{Hom}_{\mathcal{A}}(\mathcal{C}_F, -)$, as we claimed.

[2] In [KL15] Kuznetsov and Lunts work with additive dg-categories, but Proposition 2.4.13 holds for any small dg-category.

Now assume that F has a right adjoint F^R, that is there exists a dg-functor $F^R\colon \mathcal{C} \to \mathcal{A}$ together with natural isomorphisms of chain complexes

$$\mathrm{Hom}_{\mathcal{C}}(F(a), c) \simeq \mathrm{Hom}_{\mathcal{A}}(a, F^R(c)) \qquad \forall a \in \mathcal{A},\ c \in \mathcal{C} \tag{2.6}$$

Then, for any $c \in \mathcal{C}$ and any $M \in \textbf{Mod-}\mathcal{A}$ we have the following chain of functorial isomorphisms

$$\begin{aligned} \mathrm{Hom}_{\mathcal{A}}({}_c\mathcal{C}_F, M) &= \mathrm{Hom}_{\mathcal{A}}(\mathrm{Hom}_{\mathcal{C}}(F(-), c), M) \\ &\simeq \mathrm{Hom}_{\mathcal{A}}(\mathrm{Hom}_{\mathcal{A}}(-, F^R(c)), M) \\ &= \mathrm{Hom}_{\mathcal{A}}(h^{F^R(c)}_{(-)}, M) \\ &\simeq M_{F^R(c)} \\ &= \mathrm{Res}_{F^R}(M)_c \end{aligned}$$

where the isomorphism from the first to the second line follows from (2.6), and the one from the third to the fourth line is the dg-version of the Yoneda embedding. As the above chain of isomorphisms is functorial in $c \in \mathcal{C}$ and $M \in \textbf{Mod-}\mathcal{A}$, we get $\mathrm{Hom}_{\mathcal{A}}(\mathcal{C}_F, -) \simeq \mathrm{Res}_{F^R}$, as we wanted. □

Remark 2.4.15. Notice that both Res_F and Res_{F^R} preserve acyclity. Therefore, if F has a right adjoint functor F^R, then the adjunction $\mathrm{Res}_F \dashv \mathrm{Res}_{F^R}$ descends to an adjunction at the level of derived categories, and in particular we have $L\mathrm{Ind}_{F^R} \simeq \mathrm{Res}_F$.

2.4.2 Convolution of dg-modules

We now explain how to convolve morphisms of dg-modules. This operation is a lift to $\textbf{Mod-}\mathcal{A}$ of the cone construction in $\mathrm{D}(\mathcal{A})$, and it allows us to get rid of the functoriality issues that arise when using plain triangulated categories.

Take $M, N \in \textbf{Mod-}\mathcal{A}$ together with $f\colon M \to N$ a closed, degree 0 morphism in $\textbf{Mod-}\mathcal{A}$. Then, we define the *convolution* of f, and we denote it by $\{M \xrightarrow{f} N\}$, as the dg-module that sends $a \in \mathcal{A}$ to $M_a[1] \oplus N_a$ with differential given by

$$\begin{pmatrix} d_{M_a[1]} & 0 \\ f_a & d_{N_a} \end{pmatrix}$$

We have four obvious maps of degree zero

$$N \xrightarrow{i} \{M \xrightarrow{f} N\} \xrightarrow{p} M[1] \quad \text{and} \quad M[1] \xrightarrow{j} \{M \xrightarrow{f} N\} \xrightarrow{q} N$$

such that[3]

$$pj = \mathrm{id}_{M[1]} \qquad qi = \mathrm{id}_N \qquad jp + iq = \mathrm{id}_{\{M \xrightarrow{f} N\}}$$
$$d(i) = d(p) = 0 \qquad d(j) = if \qquad d(q) = -fp.$$

Notice that by construction $\{M \xrightarrow{f} N\} \simeq \mathrm{cone}(f)$ in $\mathrm{D}(\mathcal{A})$, and that to define a closed, degree 0 morphism $g\colon \{M \xrightarrow{f} N\} \to G$ it is enough to give a closed, degree 0 morphism $h\colon N \to G$ and a degree -1 morphism[4] $l\colon M \to G$ such that $hf = d(l)$. Then, $g = lp + hq$.

2.4.3 Twisted complexes

In this subsection we recall the formalism of twisted complexes of dg-modules, which we will use in § 3.3.

Twisted complexes of dg-modules were introduced in [BK89]. The treatment presented here is based on [AL17, § 3.1].

Let $\mathcal{A}$ be a dg-category, a *twisted complex* of $\mathcal{A}$-dg-modules is a collection $\{M_i, \alpha_{ij}\}$, $i, j \in \mathbb{Z}$, of $\mathcal{A}$-dg-modules M_i such that $M_i = 0$ for all but finitely many i's, and of morphisms of $\mathcal{A}$-dg-modules $\alpha_{ij}\colon M_i \to M_j$ such that

$$\deg(\alpha_{ij}) = i - j + 1 \quad \text{and} \quad (-1)^j d(\alpha_{ij}) + \sum_k \alpha_{kj}\alpha_{ik} = 0$$

A twisted complex is called *one-sided* if $\alpha_{ij} = 0$ for $i \geq j$.

The collection of twisted complexes of $\mathcal{A}$-dg-modules can be turned into a dg-category. For the general definition of a morphism of twisted complexes and of its differential, we refer the reader to [AL17, § 3.1].

Here, we simply notice that, given two one-sided twisted complexes $\{M_i, \alpha_{ij}\}$ and $\{N_i, \beta_{ij}\}$ such that $\alpha_{ij} = 0$ and $\beta_{ij} = 0$ for $|i - j| \geq 2$, a collection of closed, degree zero morphisms $f_i\colon M_i \to N_i$ of $\mathcal{A}$-dg-modules such that

$$\beta_{ij} f_i - f_j \alpha_{ij} = 0 \quad \forall\, i, j \in \mathbb{Z}$$

induces a closed, degree zero morphism of twisted complexes that we denote by

$$f = \{f_i\}\colon \{M_i, \alpha_{ij}\} \to \{N_i, \beta_{ij}\} \tag{2.7}$$

To any twisted complex $\{M_i, \alpha_{ij}\}$ we can associate an $\mathcal{A}$-dg-module, which we call its *convolution*, as follows: the underlying k-module is $M = \oplus_i M_i[-i]$, while the differential

[3] In the above equations we make use the following shorthand notation: when we write $d(q) = -fp$, we mean that $d(q)$ is equal to minus the composition $\{M \xrightarrow{f} N\} \xrightarrow{p} M[1] \xrightarrow{f[1]} N[1]$.

[4] Notice that a degree -1 morphisms $l\colon M \to G$ is the same thing as a degree zero morphism $l\colon M[1] \to G$.

is given by

$$d = \bigoplus_i d_i \oplus \bigoplus_{i,j} \alpha_{ij}$$

where d_i is the differential of $M_i[-i]$.

It is easy to check that in the situation we described above the morphism $\oplus_i f_i$ induces a morphism of $\mathcal{A}$-dg-modules $M \to N$, where we write N for the convolution of the twisted complex $\{N_i, \beta_{ij}\}$. We call $\oplus_i f_i \colon M \to N$ the convolution of the morphism (2.7).

Remark 2.4.16. Notice that the convolution of a morphism of dg-modules $f \colon M \to N$ we defined in § 2.4.2 can be interpreted as the convolution of the twisted complex $\{G_i, \alpha_{ij}\}$ where $G_{-1} = M$, $G_0 = N$, $G_i = 0$ for $i \neq -1, 0$, and $\alpha_{-10} = f$, $\alpha_{ij} = 0$ for $i \neq -1$, $j \neq 0$.

In the rest of this PhD thesis, we will be concerned with twisted complexes *concentrated in degree* 0 *and* -1, that is twisted complexes $\{M_i, \alpha_{ij}\}$ such that $M_i = 0$ for $i \neq -1, 0$. For this reason, we spend a few moments setting up notational conventions that will make the text more reader-friendly.

Given a twisted complex $\{M_i, \alpha_{ij}\}$ concentrated in degree 0 and -1, we will denote it by

$$M_{-1} \xrightarrow{\alpha_{-10}} M_0.$$

Given another twisted complex $\{N_i, \beta_{ij}\}$ concentrated in degree 0 and -1 together with two closed, degree zero morphism $f_i \colon M_i \to N_i$, $i = 0, -1$, we write

$$\begin{array}{ccc} M_{-1} & \xrightarrow{\alpha_{-10}} & M_0 \\ {\scriptstyle f_{-1}}\downarrow & & \downarrow{\scriptstyle f_0} \\ N_{-1} & \xrightarrow{\beta_{-10}} & N_0 \end{array} \tag{2.8}$$

for the morphism of twisted complexes $f \colon \{M_i, \alpha_{ij}\} \to \{N_i, \beta_{ij}\}$ induced by the f_i.

Remark 2.4.17. Starting from the morphism (2.8), we can first convolve the twisted complexes to obtain a morphism of $\mathcal{A}$-dg-modules, and then convolve the morphism of $\mathcal{A}$-dg-modules. The result of these operations is equal to the convolution of the twisted complex $\{G_i, \gamma_{ij}\}$ where

$$G_{-2} = M_{-1} \quad G_{-1} = M_0 \oplus N_{-1} \quad G_0 = N_0$$

and

$$\gamma_{-2-1} = (-\alpha_{-10}, f_{-1}) \quad \gamma_{-10} = f_0 + \beta_{-10}.$$

2.4.4 Bar categories

Dealing with morphisms in the derived category is rather complicated because they are formally defined as roofs. For this reason, we now recall the formalism of bar categories as defined in [AL21]. From our perspective, the use of the bar category of modules is that it gives a dg-enhancement of the derived category, and therefore it turns roofs of morphisms into morphisms of modules.

Let us stress that there is no new result in this subsection, its only purpose is to recall definitions and theorems. We define the bar category of (bi)modules and extend the constructions of § 2.4 (tensor products, homs, duals) to this new category. The main theorem is Theorem 2.4.19, which tells us that the bar category transforms quasi-isomorphisms of $\mathcal{A}$-modules into homotopy equivalences.

For the convenience of the reader, we recall that an *homotopy equivalence* in a dg-category $\mathcal{C}$ between two objects $c_1, c_2 \in \mathcal{C}$ is a closed, degree zero morphism $f\colon c_1 \to c_2$ such that $\mathrm{H}^0(f)$ is an isomorphism.

Let us fix $\mathcal{A}$ a small dg-category. Then, to $\mathcal{A}$ we can associate its *bar complex* $\overline{\mathcal{A}} \in \mathcal{A}\text{-}\mathbf{Mod}\text{-}\mathcal{A}$ as defined in [AL21, Definition 2.24]. The bar complex is an h-projective bimodule that can be equipped with the structure of a unital coalgebra in the monoidal category $(\mathcal{A}\text{-}\mathbf{Mod}\text{-}\mathcal{A}, \otimes_{\mathcal{A}}, \mathcal{A})$, see [AL21, Proposition 2.33]. The counit and the comultiplication are denoted by $\tau\colon \overline{\mathcal{A}} \to \mathcal{A}$ and $\Delta\colon \overline{\mathcal{A}} \to \overline{\mathcal{A}} \otimes_{\mathcal{A}} \overline{\mathcal{A}}$, respectively. It is well known that τ is a quasi-isomorphism, and therefore $\overline{\mathcal{A}}$ gives an h-projective resolution of the diagonal bimodule, see [Kel94].

Definition 2.4.18 ([AL21, Definition 3.2])**.** The *bar category of modules* $\overline{\mathbf{Mod}}\text{-}\mathcal{A}$ is defined as follows:

- The objects are given by dg-modules over $\mathcal{A}$
- For any $E, F \in \mathbf{Mod}\text{-}\mathcal{A}$ set

$$\overline{\mathrm{Hom}}_{\mathcal{A}}(E,F) := \mathrm{Hom}_{\overline{\mathbf{Mod}}\text{-}\mathcal{A}}(E,F) := \mathrm{Hom}_{\mathcal{A}}(E \otimes_{\mathcal{A}} \overline{\mathcal{A}}, F)$$

- For any $E \in \mathbf{Mod}\text{-}\mathcal{A}$ we set $\mathrm{id}_E \in \overline{\mathrm{Hom}}_{\mathcal{A}}(E,E)$ to be

$$E \otimes_{\mathcal{A}} \overline{\mathcal{A}} \xrightarrow{\mathrm{id}\otimes\tau} E \otimes_{\mathcal{A}} \mathcal{A} \xrightarrow{\simeq} E$$

- For any $E, F, G \in \mathbf{Mod}\text{-}\mathcal{A}$ the composition of $E \otimes_{\mathcal{A}} \overline{\mathcal{A}} \xrightarrow{f} F$ and $F \otimes_{\mathcal{A}} \overline{\mathcal{A}} \xrightarrow{g} G$ is the element given by

$$E \otimes_{\mathcal{A}} \overline{\mathcal{A}} \xrightarrow{\mathrm{id}\otimes\Delta} E \otimes_{\mathcal{A}} \overline{\mathcal{A}} \otimes_{\mathcal{A}} \overline{\mathcal{A}} \xrightarrow{f\otimes\mathrm{id}} F \otimes_{\mathcal{A}} \overline{\mathcal{A}} \xrightarrow{g} G$$

Theorem 2.4.19 ([AL21, Proposition 3.5], [AL21, Corollary 3.6])**.** *There exist a (non-full) inclusion*

$$\Upsilon\colon \mathbf{Mod}\text{-}\mathcal{A} \to \overline{\mathbf{Mod}}\text{-}\mathcal{A}$$

which is the identity on objects and that sends a morphism $f\colon M \to N$ *to the morphism*

$$M \otimes_{\mathcal{A}} \overline{\mathcal{A}} \xrightarrow{f\otimes\tau} N \otimes_{\mathcal{A}} \mathcal{A} \xrightarrow{\simeq} N.$$

Moreover, $\mathrm{H}^0(\Upsilon)$ *factors through* $\mathrm{D}(\mathcal{A})$ *and induces a canonical equivalence*

$$\Theta : \mathrm{D}(\mathcal{A}) \xrightarrow{\simeq} \mathrm{H}^0\left(\overline{\mathbf{Mod}}\text{-}\mathcal{A}\right)$$

giving $\overline{\mathbf{Mod}}\text{-}\mathcal{A}$ *the structure of a dg-enhancement of* $\mathrm{D}(\mathcal{A})$*.*

Similarly one can define the bar category of bimodules $\mathcal{A}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{B}$ for two small dg-categories $\mathcal{A}$, $\mathcal{B}$. In this case, the morphisms are given by

$$\overline{\mathrm{Hom}}_{\mathcal{A}\text{-}\mathcal{B}}(M, N) = \mathrm{Hom}_{\mathcal{A}\text{-}\mathbf{Mod}\text{-}\mathcal{B}}(\overline{\mathcal{A}} \otimes_{\mathcal{A}} M \otimes_{\mathcal{B}} \overline{\mathcal{B}}, N)$$

and we have $\mathrm{D}(\mathcal{A}\text{-}\mathcal{B}) \simeq \mathrm{H}^0\left(\mathcal{A}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{B}\right)$.

Let us now take three small dg-categories $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$. Then, one can define dg-functors

$$- \mathbin{\overline{\otimes}_{\mathcal{B}}} - : \mathcal{A}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{B} \otimes_k \mathcal{B}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{C} \to \mathcal{A}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{C}$$
$$\overline{\mathrm{Hom}}_{\mathcal{B}}(-,-) : \mathcal{A}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{B} \otimes_k \left(\mathcal{C}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{B}\right)^{\mathrm{op}} \to \mathcal{A}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{C}$$

as per [AL21, Definition 3.9, 3.10] by setting

$$M \mathbin{\overline{\otimes}_{\mathcal{B}}} N = M \otimes_{\mathcal{B}} \overline{\mathcal{B}} \otimes_{\mathcal{B}} N \quad \text{and} \quad \overline{\mathrm{Hom}}_{\mathcal{B}}(N', M) = \mathrm{Hom}_{\mathcal{B}}(N' \otimes_{\mathcal{B}} \overline{\mathcal{B}}, M)$$

for $M \in \mathcal{A}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{B}$, $N \in \mathcal{B}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{C}$, and $N' \in \mathcal{C}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{B}$.

For a fixed $M \in \mathcal{A}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{B}$ the functors $(- \mathbin{\overline{\otimes}_{\mathcal{A}}} M, \overline{\mathrm{Hom}}_{\mathcal{B}}(M, -))$ form an adjoint pair $\overline{\mathbf{Mod}}\text{-}\mathcal{A} \leftrightarrow \overline{\mathbf{Mod}}\text{-}\mathcal{B}$, and similarly for $(M \mathbin{\overline{\otimes}_{\mathcal{B}}} -, \overline{\mathrm{Hom}}_{\mathcal{A}^{\mathrm{op}}}(M, -))$, see [AL21, Proposition 3.14].

We define the *dualising functors* $\mathcal{A}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{B} \to \left(\mathcal{B}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{A}\right)^{\mathrm{op}}$ as

$$(-)^{\overline{\mathcal{A}}} := \overline{\mathrm{Hom}}_{\mathcal{A}^{\mathrm{op}}}(-, \mathcal{A}) \quad \text{and} \quad (-)^{\overline{\mathcal{B}}} := \overline{\mathrm{Hom}}_{\mathcal{B}}(-, \mathcal{B}),$$

see [AL21, Definition 3.31]. As showed in [AL21, Definition 3.35], one can construct natural transformations

$$M^{\overline{\mathcal{A}}} \mathbin{\overline{\otimes}_{\mathcal{A}}} (-) \to \overline{\mathrm{Hom}}_{\mathcal{A}^{\mathrm{op}}}(M, -) \tag{2.9}$$

$$(-) \mathbin{\overline{\otimes}_{\mathcal{B}}} M^{\overline{\mathcal{B}}} \to \overline{\mathrm{Hom}}_{\mathcal{B}}(M, -) \tag{2.10}$$

such that (2.9) is an homotopy equivalence if and only if M is $\mathcal{A}$-perfect, and (2.10) is an homotopy equivalence if and only if M is $\mathcal{B}$-perfect, see [AL21, Lemma 3.36]. Here, by homotopy equivalence we mean that for any dg-category $\mathcal{C}$ the natural transformations (2.9) and (2.10) become homotopy equivalences when evaluated at any object of $\mathcal{A}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{C}$ and $\mathcal{C}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{B}$, respectively.

2.4.5 Adjunctions

In this subsection, we describe some *homotopy adjoint pairs* of functors. We refer to [AL21] for a thorough treatment of this notion. In a nutshell, an homotopy adjoint pair is a pair of functors together with a unit and a counit that satisfy the usual relations but only up to homotopy.

One can construct natural trasformations $\mathrm{id} \to (-)^{\overline{\mathcal{A}}\overline{\mathcal{A}}}$ and $\mathrm{id} \to (-)^{\overline{\mathcal{B}}\overline{\mathcal{B}}}$ that are homotopy equivalences when evaluated at $\mathcal{A}$- and $\mathcal{B}$-perfect bimodules, respectively, see [AL21, Lemma 3.32]. Using these natural transformations together with (2.9) and (2.10) we get that for $M \in \mathcal{A}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{B}$ an $\mathcal{A}$- and $\mathcal{B}$-perfect bimodule the following form homotopy adjoint pair of functors

$$\left(-\,\overline{\otimes}_{\mathcal{A}}\, M, -\,\overline{\otimes}_{\mathcal{B}}\, M^{\overline{\mathcal{B}}}\right), \qquad \left(-\,\overline{\otimes}_{\mathcal{B}}\, M^{\overline{\mathcal{A}}}, -\,\overline{\otimes}_{\mathcal{A}}\, M\right). \tag{2.11}$$

Definition 2.4.20 ([AL21, Definition 4.2])**.** Fix $M \in \mathcal{A}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{B}$ an $\mathcal{A}$- and $\mathcal{B}$-perfect bimodule. Then, the *homotopy trace maps*

$$\mathrm{tr}\colon M\,\overline{\otimes}_{\mathcal{B}}\, M^{\overline{\mathcal{A}}} \to \mathcal{A} \quad \text{and} \quad \mathrm{tr}\colon M^{\overline{\mathcal{B}}}\,\overline{\otimes}_{\mathcal{A}}\, M \to \mathcal{B}$$

are the counit of the adjunctions (2.11) evaluated at the diagonal bimodules.

Remark 2.4.21. The reader might wonder why we do not define the homotopy action map, whose definition can be found in [AL21, § 4.2]. The reason is that we never need to use it as our functors will be defined using bimodules which are either h-projective, or h-projective on the right (and thus their action map induces the derived action map). The only thing we need to know is that the image of the homotopy action map in the derived category gives the derived action map as defined in Definition 2.4.7, which is true by construction, see [AL21, § 4.2].

2.4.6 Gluing of dg-categories

In this subsection, we describe the notion of gluing of dg-categories. There are two definitions of gluing in the literature: one for general dg-categories [Tab05], and one that works best when the dg-categories are additive [KL15].

As we are interested in the derived category of the gluing, we will see in § 2.4.6 that choosing either model does not make a difference for us. However, we have to balance two facts: that we want our results to hold for general dg-categories, and that working with additive dg-categories is sometimes easier.

The first gluing

Let us take $\mathcal{A}$ and $\mathcal{B}$ two small dg-categories, and let $\varphi \in \mathcal{A}\text{-}\mathbf{Mod}\text{-}\mathcal{B}$ be a bimodule.[5] Following [Tab07], we define the upper triangular dg-category associated to this datum as the dg-category $\mathcal{B} \sqcup_\varphi \mathcal{A}$ whose objects are

$$Obj(\mathcal{B} \sqcup_\varphi \mathcal{A}) = Obj(\mathcal{B}) \sqcup Obj(\mathcal{A}),$$

and with complexes of morphisms

$$\mathrm{Hom}_{\mathcal{B}\sqcup_\varphi\mathcal{A}}(x,y) = \begin{cases} \mathrm{Hom}_{\mathcal{B}}(x,y) & x,y \in \mathcal{B} \\ \mathrm{Hom}_{\mathcal{A}}(x,y) & x,y \in \mathcal{A} \\ {}_y\varphi_x & y \in \mathcal{A}, x \in \mathcal{B} \\ 0 & x \in \mathcal{A}, y \in \mathcal{B} \end{cases}$$

The grading, the differential, and the composition are defined in the obvious way.

In the following, we write

$$i_{\mathcal{B}} \colon \mathcal{B} \hookrightarrow \mathcal{B} \sqcup_\varphi \mathcal{A} \quad \text{and} \quad i_{\mathcal{A}} \colon \mathcal{A} \hookrightarrow \mathcal{B} \sqcup_\varphi \mathcal{A} \tag{2.12}$$

for the embedding functors.

Example 2.4.22. Let A and B two dg-algebras and V an A-B-bimodule. Then, the upper triangular dg-algebra associated to this datum is the dg-algebra

$$R = \begin{pmatrix} A & V \\ 0 & B \end{pmatrix}$$

with componentwise grading and differential, and composition law given by

$$\begin{pmatrix} a_1 & v_1 \\ 0 & b_1 \end{pmatrix} \cdot \begin{pmatrix} a_2 & v_2 \\ 0 & b_2 \end{pmatrix} = \begin{pmatrix} a_1a_2 & a_1v_2 + v_1b_2 \\ 0 & b_1b_2 \end{pmatrix}.$$

If we think of A and B as dg-categories $\star_A$ and $\star_B$ with one object and endomorphism dg-algebra A and B, respectively, then V is a $\star_A$-$\star_B$-bimodule, and we can form the gluing $\star_B \sqcup_V \star_A$. Unfortunately, the dg-categories $\star_R$ and $\star_B \sqcup_V \star_A$ are not equivalent: the latter dg-category has two objects, while the former has one. However, we can notice that R

[5] We will reserve the letter φ for the bimodule we use to glue two dg-categories.

has two idempotents

$$e_A = \begin{pmatrix} 1_A & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad e_B = \begin{pmatrix} 0 & 0 \\ 0 & 1_B \end{pmatrix}$$

such that $e_A + e_B = 1_R$. Thus, R has the structure of a bimodule over the ring $k \oplus k$, and we can associate to it a dg-category with two objects and morphisms dictated by the relations

$$e_A\, R\, e_A = A \qquad e_B\, R\, e_B = B \qquad e_A\, R\, e_B = V \qquad e_B\, R\, e_A = 0.$$

In other words, taking into account the $(k \oplus k)$-bimodule structure, from R we recover the gluing $\star_B \sqcup_V \star_A$.

The pre-triangulated gluing

Let us now take two small dg-categories $\mathcal{A}$ and $\mathcal{B}$, and an $\mathcal{A}$-$\mathcal{B}$-bimodule φ. Following [KL15, § 4] we define the gluing of $\mathcal{A}$ and $\mathcal{B}$ along φ, and we denote it by $\mathcal{B} \times_\varphi \mathcal{A}$, as follows: its objects are given by triples (b, a, μ) where

$$b \in \mathcal{B} \quad a \in \mathcal{A} \quad \text{and} \quad \mu \in {}_a\varphi_b \text{ is a closed, degree 0 element,}$$

and the morphisms are given by (here we set $r_1 = (b_1, a_1, \mu_1)$, $r_2 = (b_2, a_2, \mu_2)$)

$$\mathrm{Hom}_{\mathcal{B}\times_\varphi\mathcal{A}}(r_1, r_2) = \mathrm{Hom}_{\mathcal{B}}(b_1, b_2) \oplus \mathrm{Hom}_{\mathcal{A}}(a_1, a_2) \oplus {}_{a_2}\varphi_{b_1}[-1],$$

with a suitable choice of differential and composition law described in [KL15, § 4.1].

When $\mathcal{A}$ and $\mathcal{B}$ are additive, we write[6]

$$i_{\mathcal{B}} : \mathcal{B} \to \mathcal{R}, \quad b \mapsto (b, 0, 0) \quad \text{and} \quad i_{\mathcal{A}} : \mathcal{A} \to \mathcal{R}, \quad a \mapsto (0, a, 0) \tag{2.13}$$

for the embedding functors, and

$$i_{\mathcal{B}}^L : \mathcal{R} \longrightarrow \mathcal{B}, \quad (b, a, \mu) \mapsto b \quad \text{and} \quad i_{\mathcal{A}}^R : \mathcal{R} \longrightarrow \mathcal{A}, \quad (b, a, \mu) \mapsto a \tag{2.14}$$

for their left and right adjoint, respectively.

[6]If $\mathcal{A}$ and $\mathcal{B}$ were not addivite, we would not necessarily have a zero object.

Relationship between the two definitions

One might wonder what is the relationship between the two definitions of gluing we gave above. This relationship was described in [Efi20, Proposition 4.5] in terms of adjoint functors. More precisely, one can define the dg-category of upper triangular dg-categories and a natural functor from the category of dg-categories to the dg-category of upper triangular dg-categories. Then, $- \sqcup_\varphi -$ and $- \times_\varphi -$ are the left and right adjoint to this functor, respectively. Moreover, $\mathcal{B} \sqcup_\varphi \mathcal{A}$ and $\mathcal{B} \times_\varphi \mathcal{A}$ always have equivalent derived categories [Efi20, Proposition 4.2], and if $\mathcal{A}$ and $\mathcal{B}$ are pre-triangulated, see e.g. [AL17, § 3.2] for the definition of this notion, so is $\mathcal{B} \times_\varphi \mathcal{A}$ [KL15, Lemma 4.3].

The author was not aware of the results of [Efi20] when he wrote this chapter of the thesis. For this reason, we keep § 2.4.6 and the proof of § 2.4.7, but we will reference the reader to the relevant papers where these results first appeared.

When $\mathcal{B}$ and $\mathcal{A}$ are additive, the relationship between the two gluings that we outlined above can be reinterpreted as follows: the category $\mathcal{B} \sqcup_\varphi \mathcal{A}$ can be identified with the full subcategory of $\mathcal{B} \times_{\varphi[1]} \mathcal{A}$ of objects of the form $(b, 0, 0)$ or $(0, a, 0)$. Moreover, the fully faithful functor $F : \mathcal{B} \sqcup_\varphi \mathcal{A} \hookrightarrow \mathcal{B} \times_{\varphi[1]} \mathcal{A}$ induces an equivalence of derived categories $L\mathrm{Ind}_F \colon \mathrm{D}(\mathcal{B} \sqcup_\varphi \mathcal{A}) \xrightarrow{\simeq} \mathrm{D}(\mathcal{B} \times_{\varphi[1]} \mathcal{A})$. Indeed, we obtain fully faithfulness from Proposition 2.4.13, and then we get essential surjectivity by the fact that the essential image of $L\mathrm{Ind}_F$ is localising, see Remark 2.3.11, and contains the set of compact generators given by modules of the form $h^{(b,0,0)}$, $h^{(0,a,0)}$.

When the dg-categories are not additive

Consider $\mathcal{A}$ a small dg-category, we give the following

Definition 2.4.23. We define the additive envelope of $\mathcal{A}$ as the dg-category $\mathcal{A}^{\mathrm{add}}$ whose objects are formal expressions

$$a_1 \oplus \cdots \oplus a_n \qquad a_i \in \mathcal{A},$$

and whose morphisms are given by

$$\mathrm{Hom}_{\mathcal{A}^{\mathrm{add}}}\left(\bigoplus_{i=1}^{n} a_i, \bigoplus_{j=1}^{m} b_j\right) = \begin{pmatrix} \mathrm{Hom}_{\mathcal{A}}(a_1, b_1) & \dots & \mathrm{Hom}_{\mathcal{A}}(a_n, b_1) \\ \vdots & & \vdots \\ \mathrm{Hom}_{\mathcal{A}}(a_1, b_m) & \dots & \mathrm{Hom}_{\mathcal{A}}(a_n, b_m) \end{pmatrix}$$

with degreewise graded decomposition, termwise differential, and composition given by matrix multiplication.

Remark 2.4.24. Notice that the additive envelope of $\mathcal{A}$ can be also equivalently defined as the smallest additive subcategory of $\textbf{Mod-}\mathcal{A}$ containing the image of $\mathcal{A}$ via the Yoneda embedding.

The category $\mathcal{A}^{\text{add}}$ is an additive dg-category, and we have a fully faithful embedding $\mathcal{A} \hookrightarrow \mathcal{A}^{\text{add}}$. By Remark 2.4.4, restriction along the previous embedding gives an equivalence $\textbf{Mod-}\mathcal{A} \simeq \textbf{Mod-}\mathcal{A}^{\text{add}}$ whose inverse is the induction, sending $M \in \textbf{Mod-}\mathcal{A}$ to $M^{\text{add}} := M \otimes_{\mathcal{A}} \mathcal{A}^{\text{add}}$.

Now consider $\mathcal{A}$, $\mathcal{B}$, and $\varphi \in \mathcal{A}\textbf{-Mod-}\mathcal{B}$, and write $\varphi^{\text{add}} = \mathcal{A}^{\text{add}} \otimes_{\mathcal{A}} \varphi \otimes_{\mathcal{B}} \mathcal{B}^{\text{add}}$. Then, we have a fully faithful functor of dg-categories

$$\mathcal{B} \sqcup_{\varphi} \mathcal{A} \hookrightarrow \mathcal{B}^{\text{add}} \sqcup_{\varphi^{\text{add}}} \mathcal{A}^{\text{add}} \hookrightarrow \mathcal{B}^{\text{add}} \times_{\varphi^{\text{add}}[1]} \mathcal{A}^{\text{add}} \tag{2.15}$$

that, as in § 2.4.6, induces inverse equivalences (recall the adjunction of Proposition 2.4.13)

$$L\text{Ind}_{(2.15)} \colon \mathrm{D}(\mathcal{B} \sqcup_{\varphi} \mathcal{A}) \rightleftarrows \mathrm{D}(\mathcal{B}^{\text{add}} \times_{\varphi^{\text{add}}[1]} \mathcal{A}^{\text{add}}) \colon \text{Res}_{(2.15)} \tag{2.16}$$

2.4.7 Semiorthogonal decompositions for glued dg-categories

Let $\mathcal{A}$ and $\mathcal{B}$ be two small dg-categories, and $\varphi \in \mathcal{A}\textbf{-Mod-}\mathcal{B}$. We now wish to describe two SODs of the category $\mathrm{D}(\mathcal{R})$, where either $\mathcal{R} = \mathcal{B} \sqcup_{\varphi} \mathcal{A}$ or $\mathcal{R} = \mathcal{B} \times_{\varphi[1]} \mathcal{A}$. We will write

$$\alpha_1 = L\text{Ind}_{i_{\mathcal{B}}}, \quad \alpha_2 = L\text{Ind}_{i_{\mathcal{A}}} \quad \text{and} \quad \alpha_1^R = \text{Res}_{i_{\mathcal{B}}}, \quad \alpha_2^R = \text{Res}_{i_{\mathcal{A}}}$$

where $i_{\mathcal{A}}$ and $i_{\mathcal{B}}$ are defined in (2.12) and (2.13) in the respective cases.

By Proposition 2.4.13, the functors α_1 and α_2 are fully faithful. We use them to embed the categories $\mathrm{D}(\mathcal{B})$ and $\mathrm{D}(\mathcal{A})$ in $\mathrm{D}(\mathcal{R})$. We have

Proposition 2.4.25 ([Efi20, Lemma 5.10], [KL15, Proposition 4.6])**.** *Let $\mathcal{R}$ be either $\mathcal{B} \sqcup_{\varphi} \mathcal{A}$ or $\mathcal{B} \times_{\varphi[1]} \mathcal{A}$. Then, there exists an SOD*

$$\mathrm{D}(\mathcal{R}) = \langle \alpha_1(\mathrm{D}(\mathcal{B})), \alpha_2(\mathrm{D}(\mathcal{A})) \rangle \tag{2.17}$$

with right gluing functor given by $- \overset{L}{\otimes}_{\mathcal{A}} \varphi[1] \colon \mathrm{D}(\mathcal{A}) \to \mathrm{D}(\mathcal{B})$. Moreover, for any $F \in \mathrm{D}(\mathcal{R})$ there is a distinguished triangle

$$\alpha_1^R(F) \to \alpha_1^L(F) \to \alpha_2^R(F) \overset{L}{\otimes}_{\mathcal{A}} \varphi[1], \tag{2.18}$$

where α_1^L is the left adjoint of α_1.

Proof. When $\mathcal{R} = \mathcal{B} \times_{\varphi[1]} \mathcal{A}$, this is [Efi20, Lemma 5.10] for two general dg-categories, and [KL15, Proposition 4.6] when the dg-categories are additive. We now deduce from the additive case the non-additive one.

Thus, from now on $\mathcal{A}$ and $\mathcal{B}$ are just small dg-categories. Write $\mathcal{S} = \mathcal{B}^{\mathrm{add}} \times_{\varphi^{\mathrm{add}}[1]} \mathcal{A}^{\mathrm{add}}$. Then, by (2.16) we have

$$\mathrm{D}(\mathcal{B} \sqcup_\varphi \mathcal{A}) \simeq \mathrm{Res}_{(2.15)}\,(\mathrm{D}(\mathcal{S})) \simeq \langle \mathrm{Res}_{(2.15)}(\mathrm{D}(\mathcal{B}^{\mathrm{add}}) \otimes^L_{\mathcal{B}^{\mathrm{add}}} \mathcal{S}), \mathrm{Res}_{(2.15)}(\mathrm{D}(\mathcal{A}^{\mathrm{add}}) \otimes^L_{\mathcal{A}^{\mathrm{add}}} \mathcal{S})\rangle$$

Recall that induction along the inclusions $\mathcal{A} \hookrightarrow \mathcal{A}^{\mathrm{add}}$ and $\mathcal{B} \hookrightarrow \mathcal{B}^{\mathrm{add}}$ gives equivalences between the respective categories of modules. Thus, we obtain

$$\mathrm{D}(\mathcal{B}^{\mathrm{add}}) \otimes^L_{\mathcal{B}^{\mathrm{add}}} \mathcal{S} \simeq \mathrm{D}(\mathcal{B}) \otimes^L_{\mathcal{B}} \mathcal{B}^{\mathrm{add}} \otimes^L_{\mathcal{B}^{\mathrm{add}}} \mathcal{S} \simeq \mathrm{D}(\mathcal{B}) \otimes^L_{\mathcal{B}} \mathcal{S}$$

and therefore

$$\mathrm{Res}_{(2.15)}(\mathrm{D}(\mathcal{B}^{\mathrm{add}}) \otimes^L_{\mathcal{B}^{\mathrm{add}}} \mathcal{S}) \simeq \mathrm{D}(\mathcal{B}) \otimes^L_{\mathcal{B}} \mathrm{Res}_{(2.15)}(\mathcal{S}) \simeq \mathrm{D}(\mathcal{B}) \otimes^L_{\mathcal{B}} (\mathcal{B} \sqcup_\varphi \mathcal{A}).$$

Similarly, one proves

$$\mathrm{Res}_{(2.15)}(\mathrm{D}(\mathcal{A}^{\mathrm{add}}) \otimes^L_{\mathcal{A}^{\mathrm{add}}} \mathcal{S}) \simeq \mathrm{D}(\mathcal{A}) \otimes^L_{\mathcal{A}} (\mathcal{B} \sqcup_\varphi \mathcal{A})$$

and therefore we get

$$\mathrm{D}(\mathcal{B} \sqcup_\varphi \mathcal{A}) = \langle \mathrm{D}(\mathcal{B}) \otimes^L_{\mathcal{B}} (\mathcal{B} \sqcup_\varphi \mathcal{A}), \mathrm{D}(\mathcal{A}) \otimes^L_{\mathcal{A}} (\mathcal{B} \sqcup_\varphi \mathcal{A})\rangle$$

which is (2.17) when $\mathcal{A}$ and $\mathcal{B}$ are not additive. To prove the statements about the gluing functor and the distinguished triangle (2.18), one can proceed as in [KL15, Proposition 4.6]. □

Remark 2.4.26. Notice that, as the functor $- \overset{L}{\otimes}_{\mathcal{A}} \varphi$ is cocontinuous, the hypotheses of Lemma 2.3.19 are satisfied, and therefore we get $\mathrm{D}(\mathcal{R})^{\mathrm{c}} = \langle \alpha_1(\mathrm{D}(\mathcal{B})^{\mathrm{c}}), \alpha_2(\mathrm{D}(\mathcal{A})^{\mathrm{c}})\rangle$.

The second SOD of $\mathrm{D}(\mathcal{R})$ we want to construct is a mutation of the SOD (2.17). Namely, we will move $\alpha_1(\mathrm{D}(\mathcal{B}))$ to the right and we will identify the left piece of the SOD. The following lemma is the key step in doing so

Lemma 2.4.27. *The functor α_2^R has a right adjoint $\alpha_2^{RR}\colon \mathrm{D}(\mathcal{A}) \to \mathrm{D}(\mathcal{R})$ that is described as follows.*

1. *If $\mathcal{R} = \mathcal{B} \times_{\varphi[1]} \mathcal{A}$ for two additive dg-categories $\mathcal{A}$ and $\mathcal{B}$, then $\alpha_2^{RR} = \mathrm{Res}_{i^R_{\mathcal{A}}}$, where $i^R_{\mathcal{A}}$ is defined in (2.14).*

2. *If $\mathcal{R} = \mathcal{B} \sqcup_\varphi \mathcal{A}$, then α_2^{RR} sends $M \in \mathrm{D}(\mathcal{A})$ to the module $\alpha_2^{RR}(M) \in \mathrm{D}(\mathcal{R})$ such that*

$$\alpha_2^{RR}(M)_{i_{\mathcal{A}}(a)} = M_a \quad \text{and} \quad \alpha_2^{RR}(M)_{i_{\mathcal{B}}(b)} = 0$$

for any $a \in \mathcal{A}$ *and* $b \in \mathcal{B}$.

Proof. Case (1) follows from Proposition 2.4.14 applied to $F = i_{\mathcal{A}}$ and $F^R = i^R_{\mathcal{A}}$.

Case (2) is proved as follows. The functor α_2^R commutes with arbitrary direct sums, thus it has a right adjoint by Brown representability [Nee96]. Then, for $a \in \mathcal{A}$ we have[7]

$$\begin{aligned} \alpha_2^{RR}(N)_{i_{\mathcal{A}}(a)} &\simeq \mathrm{Hom}_{\mathrm{D}(\mathcal{R})}(h^{i_{\mathcal{A}}(a)}, \alpha_2^{RR}(N)) \\ &\simeq \mathrm{Hom}_{\mathrm{D}(\mathcal{A})}(\alpha_2^R(h^{i_{\mathcal{A}}(a)}), N) \\ &\simeq \mathrm{Hom}_{\mathrm{D}(\mathcal{A})}(h^a, N) \simeq N_a \end{aligned}$$

and for $b \in \mathcal{B}$

$$\alpha_2^{RR}(N)_{i_{\mathcal{B}}(b)} \simeq \mathrm{Hom}_{\mathrm{D}(\mathcal{R})}(h^{i_{\mathcal{B}}(b)}, \alpha_2^{RR}(N)) \simeq \mathrm{Hom}_{\mathrm{D}(\mathcal{A})}(\alpha_2^R(h^{i_{\mathcal{B}}(b)}), N) \simeq 0.$$

□

Proposition 2.4.28 ([Efi20, Lemma 5.1])**.** *Let* $\mathcal{R}$ *be either* $\mathcal{B} \sqcup_\varphi \mathcal{A}$ *or* $\mathcal{B} \times_{\varphi[1]} \mathcal{A}$*. Then, there exists an SOD*

$$\mathrm{D}(\mathcal{R}) = \langle \alpha_2^{RR}(\mathrm{D}(\mathcal{A})), \alpha_1(\mathrm{D}(\mathcal{B}))\rangle \tag{2.19}$$

with left gluing functor given by $- \overset{L}{\otimes}_{\mathcal{A}} \varphi : \mathrm{D}(\mathcal{A}) \to \mathrm{D}(\mathcal{B})$.

Proof. We give a proof that works both for $\mathcal{R} = \mathcal{B} \times_{\varphi[1]} \mathcal{A}$ (where $\mathcal{A}$ and $\mathcal{B}$ are additive) and for $\mathcal{R} = \mathcal{B} \sqcup_\varphi \mathcal{A}$.

By Lemma 2.4.27 we have the adjunction $\alpha_2^R \dashv \alpha_2^{RR}$. Moreover, the descriptions of α_2^{RR} given in the lemma imply

$$\alpha_2^R \alpha_2^{RR} = \mathrm{id}_{\mathrm{D}(\mathcal{A})}.$$

Thus, α_2^{RR} is fully faithful, and by Lemma 2.3.24 we have the SOD

$$\mathrm{D}(\mathcal{R}) = \langle \alpha_2^{RR}(\mathrm{D}(\mathcal{A})), {}^\perp(\alpha_2^{RR}(\mathrm{D}(\mathcal{A})))\rangle.$$

However, ${}^\perp(\alpha_2^{RR}(\mathrm{D}(\mathcal{A}))) = \ker \alpha_2^R$, and by (2.17) we know that $\ker \alpha_2^R = \alpha_2(\mathrm{D}(\mathcal{A}))^\perp = \alpha_1(\mathrm{D}(\mathcal{B}))$. Thus, (2.19) follows.

We are left to prove the claim for the left gluing functor. Take $F \in \mathrm{D}(\mathcal{A})$ and consider the distinguished triangle (2.18) for the module $\alpha_2^{RR}(F)$. The relations $\alpha_2^R\alpha_2^{RR} = \mathrm{id}_{\mathrm{D}(\mathcal{A})}$ and $\alpha_1^R \alpha_2^{RR} = 0$ show

$$\alpha_1^L \alpha_2^{RR}(F) = F \overset{L}{\otimes}_{\mathcal{A}} \varphi[1]$$

which proves the claim for the left gluing functor by Remark 2.3.6. □

[7] Recall that for any dg-category $\mathcal{C}$ and any $c_1 \in \mathcal{C}$ the module h^{c_1} is defined as $h^{c_1}_{c_2} = \mathrm{Hom}_{\mathcal{C}}(c_2, c_1)$ for any $c_2 \in \mathcal{C}$.

Remark 2.4.29. If the bimodule φ is $\mathcal{B}$-perfect, Theorem 2.4.9 implies that the left gluing functor of the SOD constructed in Proposition 2.4.28 preserves compactness, and therefore we can apply Lemma 2.3.17 to get, from (2.19), an SOD of compact objects $\mathrm{D}(\mathcal{R})^{\mathrm{c}} = \langle \alpha_2^{RR}(\mathrm{D}(\mathcal{A})^{\mathrm{c}}), \alpha_2(\mathrm{D}(\mathcal{B})^{\mathrm{c}})\rangle$.

Example 2.4.30. Let us consider an upper triangular dg-algebra R as in Example 2.4.22 with $A = B = k$ and V a vector space concentrated in degree 0 such that $\dim_k V < \infty$.

Such (trivial) dg-algebra can be obtained as the path algebra of a quiver with two vertices and morphisms from the second to the first vertex indexed by elements of V:

$$2 \xrightarrow{V} 1. \tag{2.20}$$

Therefore, modules over R correspond to representations of the quiver.

In this setup, Proposition 2.4.25 and Proposition 2.4.28 recover the well known full exceptional collections given by the projective and simple modules, respectively.

More precisely, the constant paths at the two vertices of the quiver (2.20) give rise to two simple modules S_1, S_2, while paths out of the vertices give rise to two projective modules P_1, P_2 such that $R = P_1 \oplus P_2$ as a right R-module. Then, the SOD (2.19) reads $\mathrm{D}(R) = \langle S_2, S_1\rangle$, while the SOD (2.17) reads $\mathrm{D}(R) = \langle P_1, P_2\rangle$.

Notice that to match the SOD $\mathrm{D}(R) = \langle P_1, P_2\rangle$ with the SOD (2.17) we perform tensor products along non-unital maps of rings. Namely, if we write e_i for the constant path at the vertex $i = 1, 2$ of the quiver (2.20), then we have a non-unital map of rings $f_i \colon k \to R$ sending 1 to e_i, and we have $P_i = k \otimes_k R$, where k acts on R via f_i.

Example 2.4.31. We now show that the hypotheses of Lemma 2.3.17 are not redundant. Consider the upper triangular dg-algebra

$$R = \begin{pmatrix} k & V \\ 0 & k \end{pmatrix}$$

where $V = \oplus_{n\geq 0} k$ is concentrated in degree 0. To make things clear, we will denote the top left k as k_1, and the bottom right k as k_2. From Proposition 2.4.28 and Example 2.4.22 we know that there exists an SOD[8] $\mathrm{D}(R) = \langle \mathrm{D}(k_1), \mathrm{D}(k_2)\rangle$ with left gluing functor given by $- \overset{L}{\otimes}_k V : \mathrm{D}(k_1) \to \mathrm{D}(k_2)$. As V is not perfect, Remark 2.4.29 does not apply and we cannot deduce an SOD for compact objects. Indeed, such a decomposition cannot exist because the inclusion of $\mathrm{D}(k_1)$ does not preserve compactness. To see this, consider the module $k_1[-1] \in \mathrm{D}(k_1)$ as a module over R via the projection map $R \to k_1$. As a module over k_1, this module is compact. Let us consider the module $\bigoplus_{n\geq 0} k_2 \in \mathrm{D}(k_2)$ as

[8]Notice that we necessarily need to use this SOD, as by Remark 2.4.26 the one of Proposition 2.4.25 always induces an SOD of compact objects.

a module over R. Then, we have

$$\mathrm{Hom}_{\mathrm{D}(R)}(k_1[-1], \bigoplus_{n\geq 0} k_2) \simeq \mathrm{Hom}_{\mathrm{D}(k_2)}(\bigoplus_{n\geq 0} k_2, \bigoplus_{n\geq 0} k_2) \simeq \prod_{n\geq 0}\left(\bigoplus_{n\geq 0} k_2\right),$$

whereas

$$\bigoplus_{n\geq 0} \mathrm{Hom}_{\mathrm{D}(R)}(k_1[-1], k_2) \simeq \bigoplus_{n\geq 0} \mathrm{Hom}_{\mathrm{D}(k_2)}(\bigoplus_{n\geq 0} k_2, k_2) \simeq \bigoplus_{n\geq 0}\left(\prod_{n\geq 0} k_2\right),$$

proving that $k_1[-1]$ is not compact in $\mathrm{D}(R)$.

Remark 2.4.32. The existence of the SOD in Proposition 2.4.28 is motivated by [HLS16, Theorem 3.15] and the discussion preceding it. Let us explain the difference between (2.17) and (2.19). This is best understood by looking at modules over rings. Assume we have two rings R and A, and two morphisms $i : A \to R$, $g : R \to A$ such that $gi = \mathrm{id}_A$. Starting from an A-module N_A we can produce two different R-modules. Namely, we can either consider N_A with the structure of R-module given by g, *i.e.* we *restrict* the action, or we can consider the R-module $N_A \otimes_A R$ (in this case we are *inducing* the action via i). The first construction corresponds to the functor Res_g, while the second one corresponds to Ind_i. Hence, in (2.17) we are *inducing* the $\mathcal{R}$-module structure, whereas in (2.19) we are *restricting* it.

We conclude this subsection by showing that SODs similar to (2.17) and (2.19) exists for left modules, *i.e.*, for $\mathrm{D}(\mathcal{R}^{\mathrm{op}})$. Indeed, it is clear that $(\mathcal{B} \sqcup_\varphi \mathcal{A})^{\mathrm{op}} \simeq \mathcal{A}^{\mathrm{op}} \sqcup_\varphi \mathcal{B}^{\mathrm{op}}$, where $\varphi \in \mathcal{A}\text{-}\mathbf{Mod}\text{-}\mathcal{B} = \mathcal{B}^{\mathrm{op}}\text{-}\mathbf{Mod}\text{-}\mathcal{A}^{\mathrm{op}}$, and by [KL15, Lemma A.1] we have $(\mathcal{B} \times_{\varphi[1]} \mathcal{A})^{\mathrm{op}} \simeq \mathcal{A}^{\mathrm{op}} \times_{\varphi[1]} \mathcal{B}^{\mathrm{op}}$.

Therefore, if we write

$$\beta_1 = L\mathrm{Ind}_{i_{\mathcal{A}}^{\mathrm{op}}}, \quad \beta_2 = L\mathrm{Ind}_{i_{\mathcal{B}^{\mathrm{op}}}} \quad \text{and} \quad \beta_1^R = \mathrm{Res}_{i_{\mathcal{A}}^{\mathrm{op}}}, \quad \beta_2^R = \mathrm{Res}_{i_{\mathcal{B}}^{\mathrm{op}}}$$

and β_2^{RR} for the right adjoint to β_2^R, we have

Proposition 2.4.33. *Let $\mathcal{R}$ be either $\mathcal{B} \sqcup_\varphi \mathcal{A}$ or $\mathcal{B} \times_{\varphi[1]} \mathcal{A}$. Then, there exists an SOD*

$$\mathrm{D}(\mathcal{R}^{\mathrm{op}}) = \langle \beta_1(\mathrm{D}(\mathcal{A}^{\mathrm{op}})), \beta_2(\mathrm{D}(\mathcal{B}^{\mathrm{op}}))\rangle$$

with right gluing functor given by $\varphi[1] \overset{L}{\otimes}_{\mathcal{B}} - \colon \mathrm{D}(\mathcal{B}^{\mathrm{op}}) \to \mathrm{D}(\mathcal{A}^{\mathrm{op}})$. *Moreover, we have an SOD*

$$\mathrm{D}(\mathcal{R}^{\mathrm{op}}) = \langle \beta_2^{RR}(\mathrm{D}(\mathcal{B}^{\mathrm{op}})), \beta_1(\mathrm{D}(\mathcal{A}^{\mathrm{op}}))\rangle$$

with left gluing functor given by $\varphi \overset{L}{\otimes}_{\mathcal{B}} - \colon \mathrm{D}(\mathcal{B}^{\mathrm{op}}) \to \mathrm{D}(\mathcal{A}^{\mathrm{op}})$.

2.4.8 Modules and bimodules on glued categories

Let $\mathcal{A}$, $\mathcal{B}$ be two small dg-categories and consider $\varphi \in \mathcal{A}\text{-}\mathbf{Mod}\text{-}\mathcal{B}$ a bimodule. For the rest of this section, $\mathcal{R}$ will denote either $\mathcal{B} \sqcup_{\varphi} \mathcal{A}$ or $\mathcal{B} \times_{\varphi[1]} \mathcal{A}$.

We now describe the notation we will use to work with modules over $\mathcal{R}$. We follow [AL19, § 7.2].

A module $F \in \mathbf{Mod}\text{-}\mathcal{R}$ can be described as a couple

$$\begin{pmatrix} F_{\mathcal{A}} & F_{\mathcal{B}} \end{pmatrix}$$

of an $\mathcal{A}$-module $F_{\mathcal{A}}$ and a $\mathcal{B}$-module $F_{\mathcal{B}}$ together with a closed, degree 0 morphism $\rho_F \in \operatorname{Hom}_{\mathcal{B}}(F_{\mathcal{A}} \otimes_{\mathcal{A}} \varphi, F_{\mathcal{B}})$ that we call the *structure morphism*. A morphism $f\colon F \to G$ of degree i is given by a couple of degree i morphisms $(f_{\mathcal{A}}\colon F_{\mathcal{A}} \to G_{\mathcal{A}}, f_{\mathcal{B}}\colon F_{\mathcal{B}} \to G_{\mathcal{B}})$ such that the following diagram commutes

$$\begin{array}{ccc} F_{\mathcal{A}} \otimes_{\mathcal{A}} \varphi & \xrightarrow{\rho_F} & F_{\mathcal{B}} \\ {\scriptstyle f_{\mathcal{A}}\otimes \mathrm{id}}\downarrow & & \downarrow{\scriptstyle f_{\mathcal{B}}} \\ G_{\mathcal{A}} \otimes_{\mathcal{A}} \varphi & \xrightarrow{\rho_G} & G_{\mathcal{B}} \end{array} \tag{2.21}$$

Differential and composition are computed componentwise. Notice that this description of the category $\mathbf{Mod}\text{-}\mathcal{R}$ mirrors the SOD of Proposition 2.4.28.

The category $\mathbf{Mod}\text{-}\mathcal{R}^{\mathrm{op}}$ admits a similar description: objects are couples[9]

$$\begin{pmatrix} F_{\mathcal{A}} & F_{\mathcal{B}} \end{pmatrix}^t$$

together with a structure morphism $\rho_F\colon \varphi \otimes_{\mathcal{B}} F_{\mathcal{B}} \to F_{\mathcal{A}}$, and morphisms are couples making an analogue of (2.21) commute.

The category $\mathcal{R}\text{-}\mathbf{Mod}\text{-}\mathcal{R}$ can be described as follows: a bimodule is given by a matrix of bimodules

$$\begin{pmatrix} {}_{\mathcal{A}}F_{\mathcal{A}} & {}_{\mathcal{A}}F_{\mathcal{B}} \\ {}_{\mathcal{B}}F_{\mathcal{A}} & {}_{\mathcal{B}}F_{\mathcal{B}} \end{pmatrix}$$

together with a closed, degree 0 structure morphism

$$\begin{aligned} \rho_F \in\, & \operatorname{Hom}_{\mathcal{A}\text{-}\mathcal{B}}({}_{\mathcal{A}}F_{\mathcal{A}} \otimes_{\mathcal{A}} \varphi, {}_{\mathcal{A}}F_{\mathcal{B}}) \oplus \operatorname{Hom}_{\mathcal{B}\text{-}\mathcal{B}}({}_{\mathcal{B}}F_{\mathcal{A}} \otimes_{\mathcal{A}} \varphi, {}_{\mathcal{B}}F_{\mathcal{B}}) \oplus \\ & \oplus \operatorname{Hom}_{\mathcal{A}\text{-}\mathcal{A}}(\varphi \otimes_{\mathcal{B}} ({}_{\mathcal{B}}F_{\mathcal{A}}), {}_{\mathcal{A}}F_{\mathcal{A}}) \oplus \operatorname{Hom}_{\mathcal{A}\text{-}\mathcal{B}}(\varphi \otimes_{\mathcal{B}} ({}_{\mathcal{B}}F_{\mathcal{B}}), {}_{\mathcal{A}}F_{\mathcal{B}}) \end{aligned}$$

[9] We think of them as column vectors.

whose components make the following diagram commute

$$\begin{array}{ccc} \varphi \otimes_{\mathcal{B}} ({}_{\mathcal{B}}F_{\mathcal{A}}) \otimes_{\mathcal{A}} \varphi & \longrightarrow & \varphi \otimes_{\mathcal{B}} ({}_{\mathcal{B}}F_{\mathcal{B}}) \\ \downarrow & & \downarrow \\ {}_{\mathcal{A}}F_{\mathcal{A}} \otimes_{\mathcal{A}} \varphi & \longrightarrow & {}_{\mathcal{A}}F_{\mathcal{B}}. \end{array}$$

Morphisms of degree i are matrices of morphisms of degree i that commute with the components of the structure morphism in way similar to (2.21). Differential and composition are computed componentwise.

The diagonal bimodule $\mathcal{R}$ is given by the matrix

$$\begin{pmatrix} \mathcal{A} & \varphi \\ 0 & \mathcal{B} \end{pmatrix}$$

with the obvious structure morphisms, whereas the bar complex $\overline{\mathcal{R}}$ is described[10] by the matrix

$$\begin{pmatrix} \overline{\mathcal{A}} & \left\{ \overline{\mathcal{A}} \otimes_{\mathcal{A}} \varphi \otimes_{\mathcal{B}} \overline{\mathcal{B}} \xrightarrow{(-\mathrm{id}^{\otimes 2} \otimes \tau, \tau \otimes \mathrm{id}^{\otimes 2})} (\overline{\mathcal{A}} \otimes_{\mathcal{A}} \varphi) \oplus (\varphi \otimes_{\mathcal{B}} \overline{\mathcal{B}}) \right\} \\ 0 & \overline{\mathcal{B}} \end{pmatrix} \tag{2.22}$$

whose structure morphism components are given in [AL19, (7.16)].

Using (2.22) one can give a description of $\overline{\mathbf{Mod}}$-$\mathcal{R}$. Objects are the same as before. A morphism $f\colon F \to G$ of degree i in $\overline{\mathbf{Mod}}$-$\mathcal{R}$ is a triple $(f_{\mathcal{A}}, f_{\mathcal{B}}, f_{\mathcal{AB}})$ of degree $(i, i, i-1)$ morphisms

$$F_{\mathcal{A}} \xrightarrow{f_{\mathcal{A}}} G_{\mathcal{A}} \qquad F_{\mathcal{B}} \xrightarrow{f_{\mathcal{B}}} G_{\mathcal{B}} \qquad F_{\mathcal{A}} \overline{\otimes}_{\mathcal{A}} \varphi \overset{f_{\mathcal{AB}}}{\dashrightarrow} G_{\mathcal{B}}$$

in $\overline{\mathbf{Mod}}$-$\mathcal{A}$, $\overline{\mathbf{Mod}}$-$\mathcal{B}$, and $\overline{\mathbf{Mod}}$-$\mathcal{B}$, respectively. The composition law can be found in [AL19, § 7.2]. The differential is given by

$$F_{\mathcal{A}} \xrightarrow{d(f_{\mathcal{A}})} G_{\mathcal{A}} \qquad F_{\mathcal{B}} \xrightarrow{d(f_{\mathcal{B}})} G_{\mathcal{B}} \qquad F_{\mathcal{A}} \overline{\otimes}_{\mathcal{A}} \varphi \overset{d(f_{\mathcal{AB}}) - (-1)^i (f_{\mathcal{B}} \circ \rho_F - \rho_G \circ (f_{\mathcal{A}} \overline{\otimes}\, \mathrm{id}))}{\dashrightarrow} G_{\mathcal{B}} \tag{2.23}$$

2.4.9 Compatibilities: induction, restrictions, and SODs

We now explain how one switches between an $\mathcal{R}$-module and its matrix description, and how the matrix notation is related to induction and restriction along the functors (2.13) and the SODs of § 2.4.7. We consider only the case $\mathcal{R} = \mathcal{B} \sqcup_{\varphi} \mathcal{A}$ because this is what we

[10]The word described here should be read as *homotopy equivalent*. Indeed, the bar complex $\overline{\mathcal{R}}$ is not equal to the bimodule described by the matrix (2.22) but merely homotopy equivalent to it, as Lemma 3.2.1 shows (notice that both (2.22) and $\overline{\mathcal{R}}$ are h-projective bimodules). The point is that (2.22) is the bar complex of $\mathcal{R}$ when we consider $\mathcal{R}$ as a $k \oplus k$ bimodule, where the two copies of k act via the identity of $\mathcal{A}$ and $\mathcal{B}$.

will use, but the case $\mathcal{R} = \mathcal{B} \times_{\varphi[1]} \mathcal{A}$ is analogous.

First, let us take $F \in \textbf{Mod-}\mathcal{R}$, then $F_{\mathcal{B}}$ and $F_{\mathcal{A}}$ are given by restricting along the functors (2.13), *i.e.*, we have

$$\mathrm{Res}_{i_{\mathcal{A}}}(F) = F_{\mathcal{A}} \qquad \mathrm{Res}_{i_{\mathcal{B}}}(F) = F_{\mathcal{B}}. \tag{2.24}$$

The structure morphism is given by the right action of $\mathcal{R}$ on $F_{\mathcal{A}}$, namely

$$\rho_F \colon F_{\mathcal{A}} \otimes_{\mathcal{A}} \varphi \simeq \mathrm{Res}_{i_{\mathcal{B}}}(F_{\mathcal{A}} \otimes_{\mathcal{A}} \mathcal{R}) \to F_{\mathcal{B}}.$$

Conversely, given $\begin{pmatrix} F_{\mathcal{A}} & F_{\mathcal{B}} \end{pmatrix}$ together with a structure morphism ρ_F, the associated $\mathcal{R}$-module sends

$$\mathcal{A} \ni a \mapsto (F_{\mathcal{A}})_a \quad \text{and} \quad \mathcal{B} \ni b \mapsto (F_{\mathcal{B}})_b.$$

Similarly, if $F \in \textbf{Mod-}\mathcal{R}^{\mathrm{op}}$, then we have

$$\mathrm{Res}_{i_{\mathcal{A}^{\mathrm{op}}}}(F) = F_{\mathcal{A}} \qquad \mathrm{Res}_{i_{\mathcal{B}^{\mathrm{op}}}}(F) = F_{\mathcal{B}}$$

and the module associated to $\begin{pmatrix} F_{\mathcal{A}} & F_{\mathcal{B}} \end{pmatrix}^t$ sends

$$\mathcal{A} \ni a \mapsto {}_aF_{\mathcal{A}} \quad \text{and} \quad \mathcal{B} \ni b \mapsto {}_bF_{\mathcal{B}}. \tag{2.25}$$

From (2.24) we see that the projection functors of the SOD of Proposition 2.4.28 send a module $F \in \mathrm{D}(\mathcal{R})$ to its components $F_{\mathcal{A}} \in \mathrm{D}(\mathcal{A})$ and $F_{\mathcal{B}} \in \mathrm{D}(\mathcal{B})$. The left adjoint to $L\mathrm{Ind}_{i_{\mathcal{B}}}$, which is the remaining projection functor of the SOD of Proposition 2.4.25, sends a module $F \in \mathrm{D}(\mathcal{R})$ to $L\mathrm{Ind}^L_{i_{\mathcal{B}}}(F) \simeq \mathrm{cone}(F_{\mathcal{A}} \otimes_{\mathcal{A}} \varphi \xrightarrow{\rho_F} F_{\mathcal{B}})$.

Finally, if $M \in \overline{\textbf{Mod}}\textbf{-}\mathcal{A}$ and $N \in \overline{\textbf{Mod}}\textbf{-}\mathcal{B}$, then by (2.24) we have

$$\begin{gathered} M \mathbin{\overline{\otimes}}_{\mathcal{A}} \mathcal{R} = \begin{pmatrix} M \mathbin{\overline{\otimes}}_{\mathcal{A}} \mathcal{A} & M \mathbin{\overline{\otimes}}_{\mathcal{A}} \varphi \end{pmatrix} \\ N \mathbin{\overline{\otimes}}_{\mathcal{B}} \mathcal{R} = \begin{pmatrix} 0 & N \mathbin{\overline{\otimes}}_{\mathcal{B}} \mathcal{B} \end{pmatrix} \\ \mathrm{Res}^R_{i_{\mathcal{A}}}(M) = \begin{pmatrix} M & 0 \end{pmatrix} \end{gathered} \tag{2.26}$$

where the structure morphism of $M \mathbin{\overline{\otimes}}_{\mathcal{A}} \mathcal{R}$ is the identity morphism.

2.4.10 Compatibilities: taking cones and homotopy equivalences

Let $\mathcal{R}$ be either $\mathcal{B} \sqcup_{\varphi} \mathcal{A}$ or $\mathcal{B} \times_{\varphi[1]} \mathcal{A}$. We now want to explain how to take cones and detect quasi-isomorphisms in $\mathcal{R}\textbf{-Mod-}\mathcal{R}$. Everything relies on the following simple

Lemma 2.4.34. *With the notation as above, we have*

(1) If we have a distinguished triangle $M_1 \to M_2 \to M_3$ in $\mathrm{D}(\mathcal{R}\text{-}\mathcal{R})$, then taking the four components of M_1, M_2 and M_3 we obtain distinguished triangles in the respective derived categories.

(2) A bimodule $M \in \mathcal{R}\text{-}\mathbf{Mod}\text{-}\mathcal{R}$ is quasi-isomorphic to zero if and only if its components are quasi-isomorphic to zero.

Proof. The statement (1) follows from the fact that taking components means applying a restriction functor, which is a triangulated functor.

Let us prove (2) in the case $\mathcal{R} = \mathcal{B} \sqcup_\varphi \mathcal{A}$. Notice that we have

$$\mathcal{R}^{\mathrm{op}} \otimes_k \mathcal{R} \simeq (\mathcal{R}^{\mathrm{op}} \otimes_k \mathcal{B}) \sqcup_{\overline{\varphi}} (\mathcal{R}^{\mathrm{op}} \otimes_k \mathcal{A})$$

where $\overline{\varphi} = \mathcal{R} \otimes_k \varphi \in (\mathcal{R}^{\mathrm{op}} \otimes_k \mathcal{A})\text{-}\mathbf{Mod}\text{-}(\mathcal{R}^{\mathrm{op}} \otimes_k \mathcal{B})$. Therefore, we get

$$\mathrm{D}(\mathcal{R}\text{-}\mathcal{R}) = \mathrm{D}(\mathcal{R}^{\mathrm{op}} \otimes_k \mathcal{R}) \simeq \mathrm{D}((\mathcal{R}^{\mathrm{op}} \otimes_k \mathcal{B}) \sqcup_{\overline{\varphi}} (\mathcal{R}^{\mathrm{op}} \otimes_k \mathcal{A})).$$

Applying Proposition 2.4.28 to the above equivalence, we obtain the SOD

$$\mathrm{D}(\mathcal{R}\text{-}\mathcal{R}) = \langle \mathrm{Res}^{R}_{j_{\mathcal{A}}}(\mathrm{D}(\mathcal{R}^{\mathrm{op}} \otimes_k \mathcal{A})), L\mathrm{Ind}_{j_{\mathcal{B}}}(\mathrm{D}(\mathcal{R}^{\mathrm{op}} \otimes_k \mathcal{B}))\rangle$$

where

$$j_{\mathcal{A}} \colon \mathcal{R}^{\mathrm{op}} \otimes_k \mathcal{A} \hookrightarrow (\mathcal{R}^{\mathrm{op}} \otimes_k \mathcal{B}) \sqcup_{\overline{\varphi}} (\mathcal{R}^{\mathrm{op}} \otimes_k \mathcal{A})$$
$$j_{\mathcal{B}} \colon \mathcal{R}^{\mathrm{op}} \otimes_k \mathcal{B} \hookrightarrow (\mathcal{R}^{\mathrm{op}} \otimes_k \mathcal{B}) \sqcup_{\overline{\varphi}} (\mathcal{R}^{\mathrm{op}} \otimes_k \mathcal{A})$$

are the embeddings defined in (2.12). Notice that the projection functors of the above SOD are given by $N \mapsto N_{\mathcal{A}}$ and $N \mapsto N_{\mathcal{B}}$ for $N \in \mathrm{D}(\mathcal{R}\text{-}\mathcal{R})$. Hence, $M \in \mathcal{R}\text{-}\mathbf{Mod}\text{-}\mathcal{R}$ is quasi-isomorphic to zero if and only if $M_{\mathcal{A}} \in \mathcal{R}\text{-}\mathbf{Mod}\text{-}\mathcal{A}$ and $M_{\mathcal{B}} \in \mathcal{R}\text{-}\mathbf{Mod}\text{-}\mathcal{B}$ are quasi-isomorphic to zero. To conclude, we notice that we have $\mathcal{R}^{\mathrm{op}} \simeq \mathcal{A}^{\mathrm{op}} \sqcup_\varphi \mathcal{B}^{\mathrm{op}}$, where $\varphi \in \mathcal{A}\text{-}\mathbf{Mod}\text{-}\mathcal{B} \simeq \mathcal{B}^{\mathrm{op}}\text{-}\mathbf{Mod}\text{-}\mathcal{A}^{\mathrm{op}}$, and therefore we can apply Proposition 2.4.28 to

$$\mathrm{D}(\mathcal{R}\text{-}\mathcal{A}) = \mathrm{D}(\mathcal{R}^{\mathrm{op}} \otimes_k \mathcal{A}) = \mathrm{D}(\mathcal{A} \otimes_k \mathcal{R}^{\mathrm{op}}) = \mathrm{D}((\mathcal{A} \otimes_k \mathcal{A}^{\mathrm{op}}) \sqcup_{\mathcal{A} \otimes_k \varphi} (\mathcal{A} \otimes_k \mathcal{B}^{\mathrm{op}}))$$

to show that $M_{\mathcal{A}}$ is quasi-isomorphic to zero if and only if ${}_{\mathcal{A}}M_{\mathcal{A}}$ and ${}_{\mathcal{B}}M_{\mathcal{A}}$ are. Similarly we apply Proposition 2.4.28 to $\mathrm{D}(\mathcal{R}\text{-}\mathcal{B}) = \mathrm{D}(\mathcal{B} \otimes_k \mathcal{R}^{\mathrm{op}})$ to show that $M_{\mathcal{B}}$ is quasi-isomorphic to zero if and only if ${}_{\mathcal{A}}M_{\mathcal{B}}$ and ${}_{\mathcal{B}}M_{\mathcal{B}}$ are quasi-isomorphic to zero.

The case $\mathcal{R} = \mathcal{B} \times_{\varphi[1]} \mathcal{A}$ can be proven similarly using [KL15, Proposition A.2]. □

Corollary 2.4.35. *A morphism $f \colon M_1 \to M_2$ in $\mathcal{R}\text{-}\mathbf{Mod}\text{-}\mathcal{R}$ is a quasi-isomorphism if and only if its components are.*

Proof. f is a quasi-isomorphism if and only $\mathrm{cone}(f) \simeq 0$ in $\mathrm{D}(\mathcal{R}\text{-}\mathcal{R})$. However, statement (2) of Lemma 2.4.34 shows that this is the same thing as requiring that the components of $\mathrm{cone}(f)$ are zero in the respective derived categories. In turn, by Lemma 2.4.34 (1) this is equivalent to say that the components of f are isomorphisms in the respective derived categories, and this is the same as saying that the components of f are quasi-isomorphisms. □

2.5 Spherical functors

In this section we introduce spherical functors, which will be the main player of § 3. We split this section in two parts.

In the first one, we give the definition of spherical functors in the framework of dg-categories. This is a personal choice as spherical functors can be defined using different types of enhancements, see [DKSS21] for the $(\infty, 1)$-categorical setup.

In the second part of this section, we prove some statements about spherical functors that are independent of the framework one chooses to define them. Let us explain what we mean more clearly. As being spherical is a property of the functor, it should not matter in which framework one spells out this property, and any statement about spherical functors that is true in one framework must be true in any framework one chooses to work with them. However, there are statements about spherical functors that can be proved without fixing a framework in the first place. These are the kind of statements that we prove in § 2.5.2.

For an introduction to spherical functors and their role in algebraic geometry, the reader is referred to § 1.

2.5.1 Definitions

Let $\mathcal{A}$ and $\mathcal{C}$ be two small dg-categories. We begin by giving the following

Definition 2.5.1. Let $M \in \mathrm{D}(\mathcal{A}\text{-}\mathcal{C})$ be an $\mathcal{A}$- and $\mathcal{C}$-perfect bimodule. Then, we define the *twist bimodule* as the cone of the derived trace map

$$T_M := \mathrm{cone}(M^{\tilde{\mathcal{C}}} \overset{L}{\otimes}_{\mathcal{A}} M \xrightarrow{\mathrm{tr}} \mathcal{C}) \in \mathrm{D}(\mathcal{C}\text{-}\mathcal{C})$$

and the *cotwist bimodule* as the shifted cone of the derived action map

$$C_M := \mathrm{cone}(\mathcal{A} \xrightarrow{\mathrm{act}} \mathrm{RHom}_{\mathcal{C}}(M, M))[-1] \in \mathrm{D}(\mathcal{A}\text{-}\mathcal{A}).$$

Let us denote T_{α_M} and C_{α_M} the endofunctors of $\mathrm{D}(\mathcal{C})$ and $\mathrm{D}(\mathcal{A})$ induced by T_M and

C_M. They are called the *twist* and *cotwist* functor, respectively.

The following is the definition of a spherical functor. In this form, it is due to Rina Anno and Timothy Logvinenko, see [AL17, Definition 5.2].

Definition 2.5.2. Let $M \in \mathrm{D}(\mathcal{A}\text{-}\mathcal{C})$ be an $\mathcal{A}$- and $\mathcal{C}$-perfect bimodule. The bimodule M is called a *spherical bimodule*, and the induced functor $\alpha_M(-) = - \overset{L}{\otimes}_{\mathcal{A}} M$ is called a *spherical functor*, if all the following hold

(i) T_{α_M} is an autoequivalence of $\mathrm{D}(\mathcal{C})$

(ii) C_{α_M} is an autoequivalence of $\mathrm{D}(\mathcal{A})$

(iii) The natural morphism $\alpha_M^L T_{\alpha_M}[-1] \to \alpha_M^L \alpha_M \alpha_M^R \to \alpha_M^R$ is an isomorphism

(iv) The natural morphism $\alpha_M^R \to \alpha_M^R \alpha_M \alpha_M^L \to C_{\alpha_M} \alpha_M^L[1]$ is an isomorphism

Remark 2.5.3. The natural transformations of (iii) and (iv) come from the definition of the functors T_{α_M} and C_{α_M}.

Namely, by definition T_{α_M} sits in the distinguished triangle $\alpha_M \alpha_M^R \to \mathrm{id}_{\mathrm{D}(\mathcal{C})} \to T_{\alpha_M} \to \alpha_M \alpha_M^R[1]$. Thereofore, we get a morphism $\alpha_M^L T_{\alpha_M}[-1] \to \alpha_M^L \alpha_M \alpha_M^R$, and then we get to α_M^R by evaluating on α_M^R the counit $\alpha_M^L \alpha_M \to \mathrm{id}_{\mathrm{D}(\mathcal{A})}$ coming from the adjunction $\alpha_M^L \dashv \alpha_M$. Similarly, one constructs the natural morphism in (iv).

The following is one of the central theorems of the theory of spherical functors[11]

Theorem 2.5.4 ([AL17, Theorem 5.1])**.** *If any two conditions of Definition 2.5.2 are satisfied, then all four are satisfied.*

Notice that by definition, we have a distinguished triangle[12]

$$M \overset{L}{\otimes}_{\mathcal{C}} M^{\tilde{\mathcal{C}}}[-1] \to C_M \to \mathcal{A} \xrightarrow{\text{act}} \mathrm{RHom}_{\mathcal{C}}(M, M) \simeq M \overset{L}{\otimes}_{\mathcal{C}} M^{\tilde{\mathcal{C}}}$$

in $\mathrm{D}(\mathcal{A}\text{-}\mathcal{A})$. The morphism

$$M \overset{L}{\otimes}_{\mathcal{C}} M^{\tilde{\mathcal{C}}}[-1] \to C_M \tag{2.27}$$

will play an important role in § 3. Let us remark that the couple $(C_M, (2.27))$ is defined only up to (non-unique) isomorphism in the derived category.

Remark 2.5.5. Notice that under the isomorphism $\alpha_M^R \simeq C_{\alpha_M} \alpha_M^L[1]$ the counit $\alpha_M^L \alpha_M \to \mathrm{id}_{\mathrm{D}(\mathcal{A})}$ is identified with the morphism $\alpha_M^R \alpha_M \to C_{\alpha_M}[1]$ induced by (2.27). Indeed, as

[11] Theorem 2.5.4 provides a great example of the shortcomings of the theory of triangulated categories. Indeed, the same result cannot be proved in the realm of triangulated categories, even though it is still conceptually correct, see also the introduction to [AL17].

[12] The isomorphism follows from the fact that the evaluation map for M is a quasi-isomorphism, see Remark 2.4.11.

the units and counits of the adjunctions $\alpha_M^L \dashv \alpha_M \dashv \alpha_M^R$ are induced by the derived trace and action maps, see e.g. [AL17, § 2.3], the morphism

$$M \xrightarrow{\mathrm{id}\otimes \mathrm{act}} M \overset{L}{\otimes}_{\mathcal{C}} M^{\tilde{\mathcal{A}}} \overset{L}{\otimes}_{\mathcal{A}} M \xrightarrow{\mathrm{tr}\otimes \mathrm{id}} M$$

is the identity in $\mathrm{D}(\mathcal{A}\text{-}\mathcal{A})$, and therefore the following diagram commutes in $\mathrm{D}(\mathcal{A}\text{-}\mathcal{A})$

$$\begin{array}{ccccc} M \overset{L}{\otimes}_{\mathcal{C}} M^{\tilde{\mathcal{C}}} & \xrightarrow{\mathrm{id}\otimes\mathrm{act}\otimes\mathrm{id}} & M \overset{L}{\otimes}_{\mathcal{C}} M^{\tilde{\mathcal{A}}} \overset{L}{\otimes}_{\mathcal{A}} M \overset{L}{\otimes}_{\mathcal{C}} M^{\tilde{\mathcal{C}}} & \xrightarrow{\mathrm{id}^{\otimes 2}\otimes(2.27)} & M \overset{L}{\otimes}_{\mathcal{C}} M^{\tilde{\mathcal{A}}} \overset{L}{\otimes}_{\mathcal{A}} C_M[1] \\ \downarrow{\scriptstyle \mathrm{id}} & & & & \downarrow{\scriptstyle \mathrm{tr}\otimes\mathrm{id}} \\ M \overset{L}{\otimes}_{\mathcal{C}} M^{\tilde{\mathcal{C}}} & & \xrightarrow{\qquad (2.27) \qquad} & & C_M[1] \end{array}$$

2.5.2 Model independent statements

As explained at the beginning of § 2.5, in this subsection we prove statements about spherical functors that do not require us to fix a framework to work with them. For this reason, the reader can think of the triangulated categories of this subsection as being enhanced via either dg-categories or $(\infty, 1)$-categories.

When working with spherical functors in this more relaxed way, Definition 2.5.2 takes the following equivalent form.

Definition 2.5.6. Let $\mathcal{A}$ and $\mathcal{C}$ be two (cocomplete) enhanced triangulated categories and $\Psi\colon \mathcal{A} \to \mathcal{C}$ be a (cocontinuous) functor. We say that $\Psi\colon \mathcal{A} \to \mathcal{C}$ is *spherical* if it is has (cocontinuous) left and right adjoints Ψ^L, Ψ^R, and the functors

$$T_\Psi := \mathrm{cone}(\Psi\Psi^R \to \mathrm{id}_{\mathcal{C}}) \quad \text{and} \quad C_\Psi := \mathrm{cone}(\mathrm{id}_{\mathcal{A}} \to \Psi^R\Psi)[-1]$$

are equivalences. We will call T_Ψ the *twist* and C_Ψ the *cotwist* around Ψ, respectively.

The following lemma explains the relation between four periodic SODs and spherical functors.

Lemma 2.5.7 ([BB15],[HLS16]). *Let $\mathcal{A}$ be a (cocomplete) triangulated category and assume we have a four periodic SOD*

$$\mathcal{A} = \langle \mathcal{S}_1, \mathcal{S}_2 \rangle = \langle \mathcal{S}_2, \mathcal{S}_3 \rangle = \langle \mathcal{S}_3, \mathcal{S}_4 \rangle = \langle \mathcal{S}_4, \mathcal{S}_1 \rangle$$

where $\mathcal{S}_j$ is a (localising) triangulated subcategory for every $j = 1, 2, 3, 4$. Then, the functor $\Psi := i_{\mathcal{S}_4}^R i_{\mathcal{S}_1}$ is spherical and we have

$$T_\Psi^{-1} = i_{\mathcal{S}_4}^R i_{\mathcal{S}_2} i_{\mathcal{S}_2}^R i_{\mathcal{S}_4}$$

Conversely, any spherical functor arises from a four periodic SOD as above.

Proof. The adjoints to Ψ are constructed in [BB15, Proposition B.3]. Moreover, Bodzenta and Bondal also prove that the twist and the cotwist around Ψ are equivalences, and the formula for T_Ψ^{-1} stated above. Therefore, according to Definition 2.5.6, we only have to show that when $\mathcal{A}$ is cocomplete and the subcategories $\mathcal{S}_j$ are localising, then the adjoints to Ψ are cocontinuous.

The left adjoint Ψ^L is cocontinuous because every left adjoint is cocontinuous. To show that Ψ^R is cocontinuous it is enough to show that $\Psi = i_{\mathcal{S}_4}^R i_{\mathcal{S}_1}$ preserves compactness. By Lemma 2.3.22 we know that $\mathcal{S}_1^c = \mathcal{S}_1 \cap \mathcal{A}^c$, and therefore $i_{\mathcal{S}_1}$ preserves compactness. Moreover, again by Lemma 2.3.22, we know that the projection functors of the SOD $\mathcal{A} = \langle \mathcal{S}_3, \mathcal{S}_4 \rangle$ preserve compactness. However, by Remark 2.3.3 $i_{\mathcal{S}_4}^R$ is a projection functor of this SOD, and thus it preserves compactness. Hence, $\Psi = i_{\mathcal{S}_4}^R i_{\mathcal{S}_1}$ preserves compactness because the functors $i_{\mathcal{S}_4}^R$ and $i_{\mathcal{S}_1}$ do.

The converse statement that any spherical functor arises from a four periodic SOD is [HLS16, Theorem 3.11] in the framework of dg-categories, and [DKSS21, Proposition 2.5.12] in the framework of $(\infty, 1)$-categories. □

Chapter 3

Composition of spherical twists

While § 2 dealt with preliminaries and definitions, in this chapter we begin to explore the mathematical advances presented in this thesis.

The focus of this chapter is on spherical functors. We will answer the following question, which we stated in § 1: how do we represent the composition of two spherical twists around two spherical functors as the twist around a single spherical functor?

Let us briefly recall the setup. We have a triangulated category $\mathcal{C}$ and two autoequivalences $\Phi_{\mathcal{A}}, \Phi_{\mathcal{B}} \in \operatorname{Aut}(\mathcal{C})$ realised as the spherical twists around the spherical functors $\alpha_{\mathcal{A}} \colon \mathcal{A} \to \mathcal{C}$ and $\alpha_{\mathcal{B}} \colon \mathcal{B} \to \mathcal{C}$, respectively, and we ask how can we represent the autoequivalence $\Phi_{\mathcal{B}}\Phi_{\mathcal{A}} \in \operatorname{Aut}(\mathcal{C})$ as a spherical twist around a spherical functor.

In a nutshell, the answer is that we glue the source categories and the spherical functors. Let us give an heuristic explanation of why this is the correct answer. Recall that by definition

$$T_{\alpha_{\mathcal{A}}} = \operatorname{cone}(\eta_{\mathcal{A}} \colon \alpha_{\mathcal{A}}\alpha_{\mathcal{A}}^R \to \operatorname{id}_{\mathcal{C}}) \quad \text{and} \quad T_{\alpha_{\mathcal{B}}} = \operatorname{cone}(\eta_{\mathcal{B}} \colon \alpha_{\mathcal{B}}\alpha_{\mathcal{B}}^R \to \operatorname{id}_{\mathcal{C}}),$$

where $\eta_{\mathcal{A}}$ and $\eta_{\mathcal{B}}$ are the counit of the adjunctions $\alpha_{\mathcal{A}} \dashv \alpha_{\mathcal{A}}^R$ and $\alpha_{\mathcal{B}} \dashv \alpha_{\mathcal{B}}^R$, respectively.

Then, we have the commutative diagram

$$\begin{array}{ccc} \alpha_{\mathcal{B}}\alpha_{\mathcal{B}}^R\alpha_{\mathcal{A}}\alpha_{\mathcal{A}}^R & \xrightarrow{\eta_{\mathcal{B}}} & \alpha_{\mathcal{A}}\alpha_{\mathcal{A}}^R \\ {\scriptstyle \eta_{\mathcal{A}}}\downarrow & & \downarrow{\scriptstyle \eta_{\mathcal{A}}} \\ \alpha_{\mathcal{B}}\alpha_{\mathcal{B}}^R & \xrightarrow{\eta_{\mathcal{B}}} & \operatorname{id}_{\mathcal{C}} \end{array} \tag{3.1}$$

whose rows have cones isomorphic to $T_{\alpha_{\mathcal{B}}}\alpha_{\mathcal{A}}\alpha_{\mathcal{A}}^R$ and $T_{\alpha_{\mathcal{B}}}$, respectively, and whose columns have cones isomorphic to $\alpha_{\mathcal{B}}\alpha_{\mathcal{B}}^R T_{\alpha_{\mathcal{A}}}$ and $T_{\alpha_{\mathcal{A}}}$, respectively.

At this point we would like to apply the octahedral axiom, but if we do so in the realm of triangulated categories we do not have the control we need to bring our proofs to conclusion. For this reason, we lift the above commutative square to a diagram of dg-bimodules (we always work in an enhanced framework, so we can perfom this lift), and then we apply [AL17, Lemma 3.6], which is a dg-version of the octahedron axiom.

We will spell out all the details in the rest of this chapter, but to conclude the heuristic explanation we began, we present the reader with the final outcome. That is, from the commutative square (3.1) we obtain the commutative diagram

$$\begin{array}{ccccc}
\alpha_{\mathcal{B}}\alpha_{\mathcal{B}}^R\alpha_{\mathcal{A}}\alpha_{\mathcal{A}}^R & \xrightarrow{\eta_{\mathcal{B}}} & \alpha_{\mathcal{A}}\alpha_{\mathcal{A}}^R & \longrightarrow & T_{\alpha_{\mathcal{B}}}\alpha_{\mathcal{A}}\alpha_{\mathcal{A}}^R \\
\big\downarrow{\scriptstyle \eta_{\mathcal{A}}} & & \big\downarrow{\scriptstyle \eta_{\mathcal{A}}} & & \big\downarrow{\scriptstyle T_{\alpha_{\mathcal{B}}}(\eta_{\mathcal{A}})} \\
\alpha_{\mathcal{B}}\alpha_{\mathcal{B}}^R & \xrightarrow{\eta_{\mathcal{B}}} & \mathrm{id}_{\mathcal{C}} & \longrightarrow & T_{\alpha_{\mathcal{B}}} \\
\big\downarrow & & \big\downarrow & & \big\downarrow \\
\alpha_{\mathcal{B}}\alpha_{\mathcal{B}}^R T_{\alpha_{\mathcal{A}}} & \xrightarrow{\eta_{\mathcal{B}}} & T_{\alpha_{\mathcal{A}}} & \longrightarrow & *
\end{array} \tag{3.2}$$

whose rows and columns are distinguished triangles.

Therefore, we see that in (3.2) we have $* \simeq T_{\alpha_{\mathcal{B}}}T_{\alpha_{\mathcal{A}}}$, and applying once more [AL17, Lemma 3.6] to the fixed dg-lift of (3.1) we obtain that a double cone[1] of the complex

$$\alpha_{\mathcal{B}}\alpha_{\mathcal{B}}^R\alpha_{\mathcal{A}}\alpha_{\mathcal{A}}^R \xrightarrow{(-\eta_{\mathcal{B}},\eta_{\mathcal{A}})} \alpha_{\mathcal{B}}\alpha_{\mathcal{B}}^R \oplus \alpha_{\mathcal{A}}\alpha_{\mathcal{A}}^R \xrightarrow{\eta_{\mathcal{B}}+\eta_{\mathcal{A}}} \mathrm{id}_{\mathcal{C}} \tag{3.3}$$

is isomorphic to $T_{\alpha_{\mathcal{B}}}T_{\alpha_{\mathcal{A}}}$.

This conclusion is useful for us because, using Lemma 3.2.1, we recognise that the first two terms from the left represent the composition $\beta\beta^R$, where

$$\beta\colon \mathcal{B} \sqcup_{\varphi} \mathcal{A} \to \mathcal{C} \qquad \varphi = \alpha_{\mathcal{B}}^R\alpha_{\mathcal{A}}$$

is the functor defined in Theorem 1.3.1, and $\mathcal{B} \sqcup_{\varphi} \mathcal{A}$ is the gluing of $\mathcal{B}$ and $\mathcal{A}$ along φ as defined in § 2.4.6. Therefore, we get the distinguished triangle

$$\beta\beta^R \to \mathrm{id}_{\mathcal{C}} \to T_{\alpha_{\mathcal{B}}}T_{\alpha_{\mathcal{A}}}$$

that we take as a hint that we are moving in the right direction.

Even though the strategy outlined above is indeed the one we will use to prove the validity of Theorem 1.3.1, there are many technicalities we skipped over. On top of other things, such as giving a complete description of the cotwist around β in full generality, and describing its properties in some examples, we will take care of all the relevant technical points in the present chapter.

In § 3.4 and § 3.5, we will specialise our results to the case of spherical twists around spherical objects and and $\mathbb{P}$-twist around $\mathbb{P}$-objects. In the former case, we prove that the

[1] Notice the article: double cones are not unique.

cotwist around the glued spherical functor is, up to shift, the Serre duality functor, see Theorem 3.4.11.

Arguably, Theorem 3.1.4 below, which is the formal version of Theorem 1.3.1, is a very abstract result. Nevertheless, we will provide plenty of examples that show that gluing of spherical functors appear naturally in many geometric situations, see e.g. Example 3.4.8, Example 3.4.10, and § 4.

3.1 The setup and the main result

First of all, let us state Theorem 1.3.1 in a more precise way by making use of the formalism for spherical functors that we introduced in § 2.5.1.

Let $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$ be three small dg-categories over a field k, and let $M \in \mathrm{D}(\mathcal{A}\text{-}\mathcal{C})$ and $N \in \mathrm{D}(\mathcal{B}\text{-}\mathcal{C})$ be two spherical dg-modules as per Definition 2.5.6. Let us fix $M_h \in \mathcal{A}\text{-}\mathbf{Mod}\text{-}\mathcal{C}$ and $N_h \in \mathcal{B}\text{-}\mathbf{Mod}\text{-}\mathcal{C}$ two h-projective resolutions of M and N, respectively. Then, we define:

(1) $\mathcal{R}_{M_h,N_h} = \mathcal{B} \sqcup_{\varphi_{M_h,N_h}} \mathcal{A}$ as the gluing of $\mathcal{B}$ and $\mathcal{A}$ along the bimodule $\varphi_{M_h,N_h} := M_h \otimes_{\mathcal{C}} N_h^{\mathcal{C}}$

(2) P_{M_h,N_h} as the $\mathcal{R}_{M_h,N_h}$-$\mathcal{C}$-bimodule given by the matrix $\begin{pmatrix} M_h & N_h \end{pmatrix}^t$ and the structure morphism

$$\rho_{P_{M_h,N_h}} := \mathrm{id} \otimes \mathrm{tr} \colon M_h \otimes_{\mathcal{C}} N_h^{\mathcal{C}} \otimes_{\mathcal{B}} N_h \to M_h$$

(3) $C_{P_{M_h,N_h}}$ as the $\mathcal{R}_{M_h,N_h}$-$\mathcal{R}_{M_h,N_h}$-bimodule given by the matrix

$$\begin{pmatrix} \{\mathcal{A} \xrightarrow{\mathrm{act}} \mathrm{Hom}_{\mathcal{A}}(M_h, M_h)\}[-1] & 0 \\ N_h \otimes_{\mathcal{C}} M_h^{\mathcal{C}}[-1] & \{\mathcal{B} \xrightarrow{\mathrm{act}} \mathrm{Hom}_{\mathcal{B}}(N_h, N_h)\}[-1] \end{pmatrix}$$

and the structure morphisms

$$N_h \otimes_{\mathcal{C}} M_h^{\mathcal{C}}[-1] \otimes_{\mathcal{A}} M_h \otimes_{\mathcal{C}} N_h^{\mathcal{C}} \xrightarrow{i_{N_h} \circ \mathrm{ev} \circ (\mathrm{id} \otimes \mathrm{tr} \otimes \mathrm{id})} \{\mathcal{B} \xrightarrow{\mathrm{act}} \mathrm{Hom}_{\mathcal{B}}(N_h, N_h)\}[-1]$$
$$M_h \otimes_{\mathcal{C}} N_h^{\mathcal{C}} \otimes_{\mathcal{B}} N_h \otimes_{\mathcal{C}} M_h^{\mathcal{C}}[-1] \xrightarrow{i_{M_h} \circ \mathrm{ev} \circ (\mathrm{id} \otimes \mathrm{tr} \otimes \mathrm{id})} \{\mathcal{A} \xrightarrow{\mathrm{act}} \mathrm{Hom}_{\mathcal{A}}(M_h, M_h)\}[-1]$$

where i_{M_h} and i_{N_h} are defined in § 2.4.2.

The following series of lemmas show that $\mathcal{R}_{M_h,N_h}$, P_{M_h,N_h} and $C_{P_{M_h,N_h}}$ only depend on M and N, and not on the chosen h-projective resolutions.

Lemma 3.1.1. *Let* $M_h, M'_h \in \mathcal{A}$-**Mod**-$\mathcal{C}$ *be two h-projective resolutions of* M*, and* $N_h, N'_h \in \mathcal{B}$-**Mod**-$\mathcal{C}$ *be two h-projective resolutions of* N*. Then,* $\mathcal{R}_{M_h,N_h}$ *is quasi-equivalent to* $\mathcal{R}_{M'_h,N'_h}$*.*

Proof. Both M_h and M'_h are h-projective resolutions of M. Thus, there exists morphism $f\colon M'_h \to M_h$ in **Mod**-$\mathcal{A}$ that is a quasi-isomorphism . Similarly, there exists a quasi-isomorphism $g\colon N'_h \to N_h$.

As the dualisation functor $(-)^{\mathcal{C}}$ induces a controvariant quasi-equivalence of the category of $\mathcal{C}$-h-projective, $\mathcal{C}$-perfect bimodules, see [AL17, pag. 2591] for an explanation of this fact, the dual map $g\colon (N_h)^{\mathcal{C}} \to (N'_h)^{\mathcal{C}}$ is a quasi-isomorphism between $\mathcal{C}$-perfect, $\mathcal{C}$-h-projective bimodules.

Therefore, we have a roof of quasi isomorphisms

$$M_h \otimes_{\mathcal{C}} N_h^{\mathcal{C}} \xleftarrow{f\otimes \mathrm{id}} M'_h \otimes_{\mathcal{C}} (N_h)^{\mathcal{C}} \xrightarrow{\mathrm{id}\otimes g} M'_h \otimes_{\mathcal{C}} (N'_h)^{\mathcal{C}}$$

that induces a roof of quasi-equivalences

$$\mathcal{R}_{M_h,N_h} \xleftarrow{F} \mathcal{R}_{M'_h,N_h} \xrightarrow{G} \mathcal{R}_{M'_h,N'_h}. \tag{3.4}$$

Thus, the invariance of the category $\mathcal{R}_{M_h,N_h}$ up to quasi-equivalence is proved. □

Now recall that given a quasi-equivalence between two dg-categories, induction and restriction functors induce equivalences between the respective derived categories of modules, see § 2.4.1.

We now prove that under the equivalence of derived categories induced by the quasi-equivalence of Lemma 3.1.1 the bimodules P_{M_h,N_h} and $P_{M'_h,N'_h}$ correspond to one another.

Lemma 3.1.2. *Under the equivalence* $\mathrm{D}(\mathcal{R}_{M_h,N_h}\text{-}\mathcal{C}) \simeq \mathrm{D}(\mathcal{R}_{M'_h,N'_h}\text{-}\mathcal{C})$ *induced by the quasi-equivalence of Lemma 3.1.1, the bimodule* P_{M_h,N_h} *corresponds to the bimodule* $P_{M'_h,N'_h}$*.*

Proof. The roof of quasi-equivalences (3.4) induces a roof of equivalences

$$\mathrm{D}(\mathcal{R}_{M_h,N_h}\text{-}\mathcal{C}) \xrightarrow{\mathrm{Res}_F} \mathrm{D}(\mathcal{R}_{M'_h,N_h}\text{-}\mathcal{C}) \xleftarrow{\mathrm{Res}_G} \mathrm{D}(\mathcal{R}_{M'_h,N'_h}\text{-}\mathcal{C})$$

and therefore we get the equivalence $\mathrm{Res}_G^{-1}\mathrm{Res}_F\colon \mathrm{D}(\mathcal{R}_{M_h,N_h}\text{-}\mathcal{C}) \to \mathrm{D}(\mathcal{R}_{M'_h,N'_h}\text{-}\mathcal{C})$ that is the induced equivalence we speak of in the statement of the lemma.

Thus, we want to prove that $\mathrm{Res}_G^{-1}\mathrm{Res}_F(P_{M_h,N_h}) \simeq P_{M'_h,N'_h}$. To do so, we will prove that $\mathrm{Res}_F(P_{M_h,N_h}) \simeq \mathrm{Res}_G(P_{M'_h,N'_h})$.

We keep employing the notation established in the proof of Lemma 3.1.1.

First of all, notice that restricting a module along the quasi-equivalence F amounts to send the left $\mathcal{R}_{M_h,N_h}$-module with components $\begin{pmatrix} S_{\mathcal{A}} & S_{\mathcal{B}} \end{pmatrix}^t$ and structure morphism

$\rho\colon M_h \otimes N_h^{\mathcal{C}} \otimes_{\mathcal{B}} S_{\mathcal{B}} \to S_{\mathcal{A}}$ to the $\mathcal{R}_{M'_h,N_h}$-module with components $\begin{pmatrix} S_{\mathcal{A}} & S_{\mathcal{B}} \end{pmatrix}^t$ and structure morphism

$$\rho \circ (f \otimes \mathrm{id}^{\otimes 2})\colon M'_h \otimes_{\mathcal{C}} (N_h)^{\mathcal{C}} \otimes_{\mathcal{B}} S_{\mathcal{B}} \to M_h \otimes N_h^{\mathcal{C}} \otimes_{\mathcal{B}} S_{\mathcal{B}} \to S_{\mathcal{A}}.$$

Let us write $\widetilde{S}$ for the $\mathcal{R}_{M'_h,N_h}$-bimodule with components $\begin{pmatrix} M'_h & N_h \end{pmatrix}^t$ and structure morphism $\mathrm{id} \otimes \mathrm{tr}\colon M'_h \otimes_{\mathcal{C}} (N_h)^{\mathcal{C}} \otimes_{\mathcal{B}} N_h \to M'_h$. We now show that the morphisms $f\colon M'_h \to M_h$ and $\mathrm{id}\colon N_h \to N_h$ induce a quasi-isomorphism between $\widetilde{S}$ and the bimodule $\mathrm{Res}_F(P_{M_h,N_h})$. Indeed, it is clear that the following diagram commutes

$$\begin{array}{ccc} M'_h \otimes_{\mathcal{C}} (N_h)^{\mathcal{C}} \otimes_{\mathcal{B}} N_h & \xrightarrow{\mathrm{id} \otimes \mathrm{tr}} & M'_h \\ \mathrm{id}\downarrow & & \downarrow f \\ M'_h \otimes_{\mathcal{C}} (N_h)^{\mathcal{C}} \otimes_{\mathcal{B}} N_h & \xrightarrow{(\mathrm{id} \otimes \mathrm{tr}) \circ (f \otimes \mathrm{id}^{\otimes 2})} & M_h, \end{array}$$

where the bottom row is the structure morphism for $\mathrm{Res}_F(P_{M_h,N_h})$, and therefore the morphisms f and id induce a morphism of bimodules $\widetilde{S} \to \mathrm{Res}_F(P_{M_h,N_h})$. As the components of this morphism are quasi-isomorphisms, we obtain $\widetilde{S} \simeq \mathrm{Res}_F(P_{M_h,N_h})$ in $\mathrm{D}(\mathcal{R}_{M'_h,N_h}\text{-}\mathcal{C})$.

Similarly, one proves that the maps $\mathrm{id}\colon M'_h \to M'_h$ and $g\colon N'_h \to N_h$ induce a quasi-isomorphism between the bimodule $\mathrm{Res}_G(P_{M'_h,N'_h})$ and $\widetilde{S}$.

Thus, $\mathrm{Res}_F(P_{M_h,N_h}) \simeq \mathrm{Res}_G(P_{M'_h,N'_h})$ in $\mathrm{D}(\mathcal{R}_{M_h,N'_h}\text{-}\mathcal{C})$, and the proof of the lemma is complete. □

Finally, we also prove that $C_{P_{M_h,N_h}}$ and $C_{P_{M'_h,N'_h}}$ correspond to one another under the equivalence $\mathrm{D}(\mathcal{R}_{M_h,N_h}\text{-}\mathcal{R}_{M_h,N_h}) \simeq \mathrm{D}(\mathcal{R}_{M'_h,N'_h}\text{-}\mathcal{R}_{M'_h,N'_h})$ induced by the quasi-equivalence of Lemma 3.1.1.

Lemma 3.1.3. *Under the equivalence* $\mathrm{D}(\mathcal{R}_{M_h,N_h}\text{-}\mathcal{R}_{M_h,N_h}) \simeq \mathrm{D}(\mathcal{R}_{M'_h,N'_h}\text{-}\mathcal{R}_{M'_h,N'_h})$ *induced by the quasi-equivalence of Lemma 3.1.1, the bimodule* $C_{P_{M_h,N_h}}$ *corresponds to the bimodule* $C_{P_{M'_h,N'_h}}$.

Proof. The proof works similarly to that of Lemma 3.1.2. As in that proof, we keep employing the notation introduced in the proof of Lemma 3.1.1.

Consider $\tilde{f}\colon M'_h \to M_h$ and $\tilde{g}\colon N'_h \to N_h$ two homotopy inverses to f and g, that is $f\tilde{f}$ and $\tilde{f}f$ are equal to the identity up to homotopy, and similarly for $g\tilde{g}$ and $\tilde{g}g$. Then, define the morphisms

$$e_{\mathcal{A}}\colon \mathcal{A} \xrightarrow{\mathrm{act}} \mathrm{Hom}_{\mathcal{A}}(M_h, M_h) \xrightarrow{\tilde{f}\circ -} \mathrm{Hom}_{\mathcal{A}}(M_h, M'_h)$$

$$e_{\mathcal{B}}\colon \mathcal{B} \xrightarrow{\mathrm{act}} \mathrm{Hom}_{\mathcal{B}}(N_h, N_h) \xrightarrow{\tilde{g}\circ -} \mathrm{Hom}_{\mathcal{B}}(N_h, N'_h)$$

Notice that as M_h and N_h are h-projective, the morphisms $\tilde{f} \circ -$ and $\tilde{g} \circ -$ are quasi-isomorphisms.

Now consider the $\mathcal{R}_{M'_h,N_h}$-$\mathcal{R}_{M'_h,N_h}$ bimodule defined by the matrix

$$\begin{pmatrix} \{\mathcal{A} \xrightarrow{e_{\mathcal{A}}} \mathrm{Hom}_{\mathcal{A}}(M_h, M'_h)\}[-1] & 0 \\ N'_h \otimes_{\mathcal{C}} M_h^{\mathcal{C}}[-1] & \{\mathcal{B} \xrightarrow{e_{\mathcal{B}}} \mathrm{Hom}_{\mathcal{B}}(N_h, N'_h)\}[-1] \end{pmatrix} \tag{3.5}$$

and the structure morphisms

$$\begin{aligned} N'_h \otimes_{\mathcal{C}} M_h^{\mathcal{C}}[-1] \otimes_{\mathcal{A}} M'_h \otimes_{\mathcal{C}} N_h^{\mathcal{C}} &\xrightarrow{\mathrm{id}^{\otimes 2} \otimes f \otimes \mathrm{id}} N'_h \otimes_{\mathcal{C}} M_h^{\mathcal{C}}[-1] \otimes_{\mathcal{A}} M_h \otimes_{\mathcal{C}} N_h^{\mathcal{C}} \to \\ &\xrightarrow{\mathrm{ev} \circ (\mathrm{id} \otimes \mathrm{tr} \otimes \mathrm{id})} \mathrm{Hom}_{\mathcal{C}}(N_h, N'_h) \to \{\mathcal{B} \xrightarrow{e_{\mathcal{B}}} \mathrm{Hom}_{\mathcal{C}}(N_h, N'_h)\}[-1] \end{aligned}$$

and

$$\begin{aligned} M'_h \otimes_{\mathcal{C}} N_h^{\mathcal{C}} \otimes_{\mathcal{B}} N'_h \otimes_{\mathcal{C}} M_h^{\mathcal{C}}[-1] &\xrightarrow{\mathrm{id}^{\otimes 2} \otimes g \otimes \mathrm{id}} M'_h \otimes_{\mathcal{C}} N_h^{\mathcal{C}} \otimes_{\mathcal{B}} N_h \otimes_{\mathcal{C}} M_h^{\mathcal{C}}[-1] \to \\ &\xrightarrow{\mathrm{ev} \circ (\mathrm{id} \otimes \mathrm{tr} \otimes \mathrm{id})} \mathrm{Hom}_{\mathcal{A}}(M_h, M'_h) \to \{\mathcal{A} \xrightarrow{e_{\mathcal{A}}} \mathrm{Hom}_{\mathcal{C}}(M_h, M'_h)\}[-1] \end{aligned}$$

Then, we define the following morphisms from the components of $\mathrm{Res}_F(C_{P_{M_h,N_h}})$ to the components of (3.5)

$$\begin{gathered} \{\mathcal{A} \xrightarrow{\mathrm{act}} \mathrm{Hom}_{\mathcal{A}}(M_h, M_h)\} \xrightarrow{(\mathrm{id}, \tilde{f})} \{\mathcal{A} \xrightarrow{e_{\mathcal{A}}} \mathrm{Hom}_{\mathcal{A}}(M_h, M'_h)\} \\ \{\mathcal{B} \xrightarrow{\mathrm{act}} \mathrm{Hom}_{\mathcal{B}}(N_h, N_h)\} \xrightarrow{(\mathrm{id}, \tilde{g})} \{\mathcal{B} \xrightarrow{e_{\mathcal{B}}} \mathrm{Hom}_{\mathcal{B}}(N_h, N'_h)\} \\ N_h \otimes_{\mathcal{C}} M_h^{\mathcal{C}} \xrightarrow{\tilde{g} \otimes \mathrm{id}} N'_h \otimes_{\mathcal{C}} M_h^{\mathcal{C}} \end{gathered} \tag{3.6}$$

Above, we wrote $(\mathrm{id}, \tilde{f})$ for the morphism induced by the commutative diagram

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\mathrm{act}} & \mathrm{Hom}_{\mathcal{A}}(M_h, M_h) \\ \big\downarrow \mathrm{id} & & \big\downarrow \tilde{f} \circ - \\ \mathcal{A} & \xrightarrow{e_{\mathcal{A}}} & \mathrm{Hom}_{\mathcal{A}}(M_h, M'_h) \end{array}$$

between the convolution of the top and bottom row, see also § 2.4.2. A similar remark applies for $(\mathrm{id}, \tilde{g})$.

Notice that if we prove that the morphisms (3.6) induce a morphism of bimodules $\mathrm{Res}_F(C_{P_{M_h,N_h}}) \to$ (3.5), then we have proved that these two bimodules are quasi-isomorphic by Corollary 2.4.35. Indeed, the first two morphisms in (3.6) are quasi-isomorphisms because $\tilde{f} \circ -$ and $\tilde{g} \circ -$ are, and the third morphism is a quasi-isomorphism because $\tilde{g}$ is so and $M_h^{\mathcal{C}}$ is $\mathcal{C}$-h-projective being the $\mathcal{C}$-dual of a $\mathcal{C}$-h-projective, $\mathcal{C}$-perfect bimodule.

Unfortunately, the morphisms (3.6) do not induce a morphism of bimodules. Indeed, let us write ρ and ρ' for the structure morphisms of $C_{P_{M_h,N_h}}$ and (3.5), respectively, and ${}_{\mathcal{A}}\rho_{\mathcal{A}}$, ${}_{\mathcal{B}}\rho_{\mathcal{B}}$, ${}_{\mathcal{A}}(\rho')_{\mathcal{A}}$ and ${}_{\mathcal{B}}(\rho')_{\mathcal{B}}$ for their components. Then, it is easy to see that the diagram

$$\begin{array}{ccc} N_h \otimes_{\mathcal{C}} M_h^{\mathcal{C}} \otimes_{\mathcal{A}} M_h' \otimes_{\mathcal{C}} N_h^{\mathcal{C}}[-1] & \xrightarrow{{}_{\mathcal{B}}\rho_{\mathcal{B}} \circ (\mathrm{id}^{\otimes 2} \otimes f \otimes \mathrm{id})} & \{\mathcal{B} \xrightarrow{\mathrm{act}} \mathrm{Hom}_{\mathcal{B}}(N_h, N_h)\} \\ \downarrow{\scriptstyle \tilde{g} \otimes \mathrm{id}^{\otimes 3}} & & \downarrow{\scriptstyle (\mathrm{id}, \tilde{g})} \\ N_h' \otimes_{\mathcal{C}} M_h^{\mathcal{C}} \otimes_{\mathcal{A}} M_h' \otimes_{\mathcal{C}} N_h^{\mathcal{C}}[-1] & \xrightarrow{{}_{\mathcal{B}}(\rho')_{\mathcal{B}}} & \{\mathcal{B} \xrightarrow{e_{\mathcal{B}}} \mathrm{Hom}_{\mathcal{B}}(N_h, N_h')\} \end{array} \tag{3.7}$$

commutes on the nose, but the diagram

$$\begin{array}{ccc} M_h' \otimes_{\mathcal{C}} N_h^{\mathcal{C}}[-1] \otimes_{\mathcal{B}} N_h \otimes_{\mathcal{C}} M_h^{\mathcal{C}} & \xrightarrow{{}_{\mathcal{B}}\rho_{\mathcal{B}} \circ (f \otimes \mathrm{id}^{\otimes 3})} & \{\mathcal{A} \xrightarrow{\mathrm{act}} \mathrm{Hom}_{\mathcal{A}}(M_h, M_h)\} \\ \downarrow{\scriptstyle \mathrm{id}^{\otimes 2} \otimes \tilde{g} \otimes \mathrm{id}} & & \downarrow{\scriptstyle (\mathrm{id}, \tilde{f})} \\ M_h' \otimes_{\mathcal{C}} N_h^{\mathcal{C}}[-1] \otimes_{\mathcal{B}} N_h' \otimes_{\mathcal{C}} M_h^{\mathcal{C}} & \xrightarrow{{}_{\mathcal{B}}(\rho')_{\mathcal{B}}} & \{\mathcal{A} \xrightarrow{e_{\mathcal{A}}} \mathrm{Hom}_{\mathcal{A}}(M_h, M_h')\} \end{array} \tag{3.8}$$

only commutes up to homotopy.

Even though we encounter this unfortunate hurdle, we are able to overcome it because both $\mathrm{Res}_F(C_{P_{M_h,N_h}})$ and (3.5) have $\mathcal{A}$-$\mathcal{B}$ component equal to zero. Because of this property we can apply [AL19, Lemma 7.3] (in its equivalent version for bimodules with $\mathcal{A}$-$\mathcal{B}$ components equal to zero) and we know that to show that $\mathrm{Res}_F(C_{P_{M_h,N_h}}) \simeq$ (3.5) in $\mathrm{D}(\mathcal{R}_{M_h',N_h}$-$\mathcal{R}_{M_h',N_h})$ it is enough to find three quasi-isomorphism for which the diagrams (3.7) and (3.8) commute up to homotopy. We defined such quasi-isomorphisms in (3.6), and thus we get $\mathrm{Res}_F(C_{P_{M_h,N_h}}) \simeq$ (3.5) in $\mathrm{D}(\mathcal{R}_{M_h',N_h}$-$\mathcal{R}_{M_h',N_h})$.

Similarly, one proves that (3.5) $\simeq \mathrm{Res}_G(C_{P_{M_h',N_h'}})$ in the derived category, and we get $\mathrm{Res}_G^{-1}\mathrm{Res}_F(C_{P_{M_h,N_h}}) \simeq C_{P_{M_h',N_h'}}$ in $\mathrm{D}(\mathcal{R}_{M_h',N_h'}$-$\mathcal{R}_{M_h',N_h'})$, as we wanted. □

In light of Lemma 3.1.1, Lemma 3.1.2, and Lemma 3.1.3, we can drop the letter "h" from the notation for $\mathcal{R}_{M_h,N_h}$, P_{M_h,N_h} and $C_{P_{M_h,N_h}}$. Furthermore, as we proved that the construction of $\mathcal{R}_{M,N}$, $P_{M,N}$ and $C_{P_{M,N}}$ does not depend on the chosen h-projective resolutions of M and N, from now we can assume that we lifted our spherical bimodules M and N to two h-projective bimodules.

Let us remark that Lemma 3.1.1, Lemma 3.1.2, and Lemma 3.1.3 are a sign that our construction is correct. Indeed, the twist around a spherical bimodule only depends on the equivalence class of the bimodule in the derived category, and thus also our construction should only depend on M and N as elements of their respective derived categories.

We can now state the precise version of Theorem 1.3.1.

Theorem 3.1.4. *Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be three small dg-categories over a field k, $M \in \mathrm{D}(\mathcal{A}$-$\mathcal{C})$, $N \in \mathrm{D}(\mathcal{B}$-$\mathcal{C})$ be two spherical bimodules, and $\mathcal{R}_{M,N}$, $P_{M,N}$ and $C_{P_{M,N}}$ be defined as in (1), (2), and (3), respectively.*

Then, the $\mathcal{R}_{M,N}$-$\mathcal{C}$ bimodule $P_{M,N}$ is spherical, the twist around it is given by the composition $T_{\alpha_N}T_{\alpha_M}$, and its cotwist is given by the bimodule $C_{P_{M,N}}$.

Proof. By Theorem 2.5.4 we know that, if we prove that $P_{M,N}$ is perfect on both sides, then to show that it is a spherical bimodule it is enough to check any two of the four conditions of Definition 2.5.2. Thus, the result follows from Lemma 3.1.6, Proposition 3.2.3, Proposition 3.3.1, and Proposition 3.3.9. □

Remark 3.1.5. Theorem 3.1.4 appeared in the author's published work [Bar22, Theorem 3.0.1]. However, it appeared in a different and slightly incorrect form. More precisely, in *ibidem* the statement of Theorem 3.1.4 allowed not only to choose lifts of M and N, but also to choose a lift of $\mathrm{RHom}_{\mathcal{C}}(N, M)$, independently of the chosen lifts of M and N. This last degree of freedom goes one step too far and does not allow to prove that the dg-category $\mathcal{R}_{M,N}$ and the bimodules $P_{M,N}$ and $C_{P_{M,N}}$ are independent of the chosen lifts of M and N in the sense proved by Lemma 3.1.1, Lemma 3.1.2, and Lemma 3.1.3. This mistake was pointed out to the author by the examiners of his PhD thesis, whom he would like to thank.

Once again let us remark that from now on we assume that we fixed two h-projective resolutions of the bimodules M and N and that we replaced them with these resolutions. We are able to do so because we proved that our constructions do not depend on the chosen resolutions up to quasi-equivalences.

Thus, from now on $M \in \mathcal{A}$-**Mod**-$\mathcal{C}$ and $N \in \mathcal{B}$-**Mod**-$\mathcal{C}$ are two h-projective, spherical bimodules.

Before dealing with the technical proofs, let us point the attention of the reader to the following remarks.

- Notice that as M and N are h-projective, their underived trace maps

$$\mathrm{tr}\colon M \otimes_{\mathcal{C}} M^{\mathcal{C}} \to \mathcal{A} \quad \text{and} \quad \mathrm{tr}\colon N \otimes_{\mathcal{C}} N^{\mathcal{C}} \to \mathcal{B}$$

  induce the derived trace maps in the derived category. Therefore, the structure morphism for the bimodule $P_{M,N}$ is identified, in the derived category, with the evaluation morphism

$$M \otimes_{\mathcal{C}} N^{\mathcal{C}} \otimes_{\mathcal{B}} N \simeq \mathrm{RHom}_{\mathcal{C}}(N, M) \overset{L}{\otimes}_{\mathcal{B}} N \xrightarrow{\mathrm{ev}} M,$$

  where $\simeq$ denotes an isomorphism in the derived category.

- M and N being h-projective, their actions maps

$$\mathrm{act}\colon \mathcal{A} \to \mathrm{Hom}_{\mathcal{A}}(M, M) \quad \text{and} \quad \mathrm{act}\colon \mathcal{B} \to \mathrm{Hom}_{\mathcal{B}}(N, N)$$

induce the derived action maps in the derived category. Thus, the $\mathcal{A}$-$\mathcal{A}$ and $\mathcal{B}$-$\mathcal{B}$ component of $C_{P_{M,N}}$ are quasi-isomorphic to C_M and C_N, respectively.

Now that we have set up the scene, we can begin the proof of Theorem 3.1.4. As we will reserve the letter $\mathcal{R}$ and P for the dg-category and the bimodule constructed in (1) and (2), respectively, from now on we will drop the subscripts M and N from their notation. Thus, $\mathcal{R} = \mathcal{R}_{M,N}$ and $P = P_{M,N}$. Similarly, we write $C_P = C_{P_{M,N}}$ for the bimodule defined in (3).

For future reference, we write

$$\alpha_P \colon \mathrm{D}(\mathcal{R}) \to \mathrm{D}(\mathcal{C}) \tag{3.9}$$

for the functor induced by P.

To use Theorem 2.5.4, we need to show that the bimodule P of Theorem 3.1.4 is perfect on both sides, so this is the first thing we prove. On top of this, we also prove that P is $\mathcal{C}$-h-projective.

Lemma 3.1.6. *The $\mathcal{R}$-$\mathcal{C}$-bimodule P of Theorem 3.1.4 is $\mathcal{R}$- and $\mathcal{C}$-perfect. Moreover, with the choices performed above, P is also $\mathcal{C}$-h-projective.*

Proof. By (2.25), the bimodule P sends $a \in \mathcal{A}$ to ${}_aM$ and $b \in \mathcal{B}$ to ${}_bN$. Thus, P is $\mathcal{C}$-perfect because M and N are. Moreover, M and N are $\mathcal{C}$-h-projective, and thus P is too.

We now show that $\mathcal{R}$-perfectness of P follows from $\mathcal{A}$-perfectness of M and $\mathcal{B}$-perfectness of N. Recall that $\mathcal{R}$ is defined as the gluing of $\mathcal{B}$ and $\mathcal{A}$ along $M \otimes_{\mathcal{C}} N^{\mathcal{C}}$. Thus, by Proposition 2.4.33 we have the SOD

$$\mathrm{D}(\mathcal{R}^{\mathrm{op}}) = \langle \mathrm{Res}^{R}_{i_{\mathcal{B}^{\mathrm{op}}}}(\mathrm{D}(\mathcal{B}^{\mathrm{op}})), L\mathrm{Ind}_{i_{\mathcal{A}^{\mathrm{op}}}}(\mathrm{D}(\mathcal{A}^{\mathrm{op}}))\rangle$$

and moreover the above SOD has a left gluing functor given by $(M \otimes_{\mathcal{C}} N^{\mathcal{C}}) \overset{L}{\otimes}_{\mathcal{B}} - \colon \mathrm{D}(\mathcal{B}^{\mathrm{op}}) \to \mathrm{D}(\mathcal{A}^{\mathrm{op}})$.

If we assume that the bimodule $M \otimes_{\mathcal{C}} N^{\mathcal{C}}$ is $\mathcal{A}$-perfect, then the gluing bimodule of the SOD above preserves compactness by Theorem 2.4.9, and therefore Lemma 2.3.17 implies that P is $\mathcal{R}$-perfect if and only if its projections to $\mathrm{D}(\mathcal{B}^{\mathrm{op}})$ and $\mathrm{D}(\mathcal{A}^{\mathrm{op}})$ are. In § 2.4.9 we explained that such projections are given by the components of P, *i.e.*, by M and N. Thus, if $M \otimes_{\mathcal{C}} N^{\mathcal{C}}$ is $\mathcal{A}$-perfect, then P is $\mathcal{R}$-perfect because its components are.

We now show that $M \otimes_{\mathcal{C}} N^{\mathcal{C}}$ is $\mathcal{A}$-perfect. First, notice that we have $M \otimes_{\mathcal{C}} N^{\mathcal{C}} \simeq M \overset{L}{\otimes}_{\mathcal{C}} N^{\mathcal{C}}$ in $\mathrm{D}(\mathcal{A}\text{-}\mathcal{B})$ because M is h-projective. Then, by Theorem 2.4.9 and $\mathcal{A}$-perfectness of M to prove that $M \otimes_{\mathcal{C}} N^{\mathcal{C}}$ is $\mathcal{A}$-perfect it is enough to prove that $N^{\mathcal{C}}$ is $\mathcal{C}$-perfect. However, $N^{\mathcal{C}}$

is the dual of a $\mathcal{C}$-h-projective, $\mathcal{C}$-perfect bimodule, and therefore it is still $\mathcal{C}$-h-projective and $\mathcal{C}$-perfect, see [AL17, pag. 2591] for an explanation of this fact. □

We conclude this subsection by explicitly describing the components and structure morphisms of two bimodules that will be important in the following.

Lemma 3.1.7. *The $\mathcal{C}$-$\mathcal{R}$-bimodule $P^{\mathcal{C}}$ is given by the matrix $\begin{pmatrix} M^{\mathcal{C}} & N^{\mathcal{C}} \end{pmatrix}$ and the structure morphism*

$$\rho_{P^{\mathcal{C}}} \colon M^{\mathcal{C}} \otimes_{\mathcal{A}} M \otimes_{\mathcal{C}} N^{\mathcal{C}} \xrightarrow{\mathrm{tr} \otimes \mathrm{id}} N^{\mathcal{C}}.$$

Proof. Direct verification using what we explained in § 2.4.9. □

Lemma 3.1.8. *The $\mathcal{R}$-$\mathcal{R}$-bimodule $\mathrm{Hom}_{\mathcal{C}}(P, P)$ is given by the matrix*

$$\mathrm{Hom}_{\mathcal{C}}(P, P) = \begin{pmatrix} \mathrm{Hom}_{\mathcal{C}}(M, M) & \mathrm{Hom}_{\mathcal{C}}(N, M) \\ \mathrm{Hom}_{\mathcal{C}}(M, N) & \mathrm{Hom}_{\mathcal{C}}(N, N) \end{pmatrix}$$

with structure morphisms

$$\mathrm{Hom}_{\mathcal{C}}(M, N) \otimes_{\mathcal{A}} M \otimes_{\mathcal{C}} N^{\mathcal{C}} \xrightarrow{\mathrm{ev} \circ (\mathrm{ev} \otimes \mathrm{id})} \mathrm{Hom}_{\mathcal{C}}(N, N)$$
$$\mathrm{Hom}_{\mathcal{C}}(M, M) \otimes_{\mathcal{A}} M \otimes_{\mathcal{C}} N^{\mathcal{C}} \xrightarrow{\mathrm{ev} \circ (\mathrm{ev} \otimes \mathrm{id})} \mathrm{Hom}_{\mathcal{C}}(N, M)$$
$$M \otimes_{\mathcal{C}} N^{\mathcal{C}} \otimes_{\mathcal{B}} \mathrm{Hom}_{\mathcal{C}}(M, N) \xrightarrow{\mathrm{ev} \circ (\mathrm{id} \otimes \mathrm{cmp})} \mathrm{Hom}_{\mathcal{C}}(M, M)$$
$$M \otimes_{\mathcal{C}} N^{\mathcal{C}} \otimes_{\mathcal{B}} \mathrm{Hom}_{\mathcal{C}}(N, N) \xrightarrow{\mathrm{ev} \circ (\mathrm{id} \otimes \mathrm{cmp})} \mathrm{Hom}_{\mathcal{C}}(N, M)$$

Proof. Direct verification using what we explained in § 2.4.9. □

3.2 The twist

We now wish to give a description of the twist around the functor (3.9). First, we prove the following lemma, which is a lift to the category of bimodules of [KL15, Proposition 4.9].

Lemma 3.2.1. *There exist a commutative diagram*

$$\begin{array}{ccc} \mathcal{R} \overline{\otimes}_{\mathcal{A}} \varphi \overline{\otimes}_{\mathcal{B}} \mathcal{R} & \xrightarrow{f_{\mathcal{A}}} & \mathcal{R} \overline{\otimes}_{\mathcal{A}} \mathcal{R} \\ {\scriptstyle f_{\mathcal{B}}} \downarrow & & \downarrow {\scriptstyle g_{\mathcal{A}}} \\ \mathcal{R} \overline{\otimes}_{\mathcal{B}} \mathcal{R} & \xrightarrow{g_{\mathcal{B}}} & \mathcal{R} \end{array} \tag{3.10}$$

*in $\mathcal{R}$-**Mod**-$\mathcal{R}$ such that the induced map*

$$\left\{ \mathcal{R} \overline{\otimes}_{\mathcal{A}} \varphi \overline{\otimes}_{\mathcal{B}} \mathcal{R} \xrightarrow{(-f_{\mathcal{A}}, f_{\mathcal{B}})} \mathcal{R} \overline{\otimes}_{\mathcal{A}} \mathcal{R} \oplus \mathcal{R} \overline{\otimes}_{\mathcal{B}} \mathcal{R} \right\} \xrightarrow{g_{\mathcal{A}} + g_{\mathcal{B}}} \mathcal{R} \tag{3.11}$$

is a quasi-isomorphism in $\mathcal{R}$-**Mod**-$\mathcal{R}$.

Proof. We define $f_{\mathcal{A}}$ as the morphism

$$\mathcal{R} \otimes_{\mathcal{A}} \overline{\mathcal{A}} \otimes_{\mathcal{A}} \varphi \otimes_{\mathcal{B}} \overline{\mathcal{B}} \otimes_{\mathcal{B}} \mathcal{R} \xrightarrow{(\mathrm{id}^{\otimes 2} \otimes \mathrm{cmp}) \circ (\mathrm{id}^{\otimes 3} \otimes \tau \otimes \mathrm{id})} \mathcal{R} \otimes_{\mathcal{A}} \overline{\mathcal{A}} \otimes_{\mathcal{A}} \mathcal{R},$$

where cmp is the composition of morphisms in the dg-category $\mathcal{R}$, and $g_{\mathcal{A}}$ as the morphism

$$\mathcal{R} \otimes_{\mathcal{A}} \overline{\mathcal{A}} \otimes_{\mathcal{A}} \mathcal{R} \xrightarrow{\mathrm{cmp} \circ (\mathrm{id} \otimes \tau \otimes \mathrm{id})} \mathcal{R}.$$

Similarly, we define $f_{\mathcal{B}}$ and $g_{\mathcal{B}}$.

Then, for any $r \otimes a \otimes v \otimes b \otimes r' \in \mathcal{R} \otimes_{\mathcal{A}} \overline{\mathcal{A}} \otimes_{\mathcal{A}} \varphi \otimes_{\mathcal{B}} \overline{\mathcal{B}} \otimes_{\mathcal{B}} \mathcal{R}$ we have

$$\begin{aligned} g_{\mathcal{A}}(f_{\mathcal{A}}(r \otimes a \otimes v \otimes b \otimes r')) &= g_{\mathcal{A}}(r \otimes a \otimes v \cdot \tau(b) \cdot r') \\ &= r \cdot \tau(a) \cdot v \cdot \tau(b) \cdot r' = g_{\mathcal{B}}(r \cdot \tau(a) \cdot v \otimes b \otimes r') = g_{\mathcal{B}}(f_{\mathcal{B}}(r \otimes a \otimes v \otimes b \otimes r')). \end{aligned}$$

Thus, we get the commutative diagram (3.10) and the induced morphism (3.11).

To prove that (3.11) is a quasi-isomorphism, we use Corollary 2.4.35. The $\mathcal{A}$-$\mathcal{A}$ and $\mathcal{B}$-$\mathcal{B}$ component of (3.11) are given by the maps $\tau \colon \overline{\mathcal{A}} \to \mathcal{A}$ and $\tau \colon \overline{\mathcal{B}} \to \mathcal{B}$, respectively. Thus, they are quasi-isomorphisms. The $\mathcal{B}$-$\mathcal{A}$ component of (3.11) is given by $0 \to 0$, and thus it is a quasi-isomorphism. Finally, the $\mathcal{A}$-$\mathcal{B}$ component of (3.11) is given by the morphism

$$\{\overline{\mathcal{A}} \otimes_{\mathcal{A}} \varphi \otimes_{\mathcal{B}} \overline{\mathcal{B}} \xrightarrow{(-\tau \otimes \mathrm{id}^{\otimes 2}, \mathrm{id}^{\otimes 2} \otimes \tau)} \varphi \otimes_{\mathcal{B}} \overline{\mathcal{B}} \oplus \overline{\mathcal{A}} \otimes_{\mathcal{A}} \varphi\} \xrightarrow{\tau \otimes \mathrm{id} + \mathrm{id} \otimes \tau} \varphi$$

which is a quasi-isomorphim because $\tau \otimes \mathrm{id}^{\otimes 2}$, $\mathrm{id}^{\otimes 2} \otimes \tau$, $\tau \otimes \mathrm{id}$, and $\mathrm{id} \otimes \tau$ are.

Thus, the proof of the lemma is complete. □

Remark 3.2.2. Notice that Lemma 3.2.1 applies to *any* gluing of a dg-category $\mathcal{B}$ and a dg-category $\mathcal{A}$ along an $\mathcal{A}$-$\mathcal{B}$ bimodule φ. Moreover, the bimodules appearing in the brackets of equation (3.11) are h-projective, and thus the quasi-isomorphism (3.11) gives an h-projective resolution of the diagonal bimodule of the gluing. Indeed, the bimodules appearing in the brackets of equation (3.11) are $\mathcal{R} \otimes_{\mathcal{A}} \overline{\mathcal{A}} \otimes_{\mathcal{A}} \mathcal{R}$ and $\mathcal{R} \otimes_{\mathcal{B}} \overline{\mathcal{B}} \otimes_{\mathcal{B}} \mathcal{R}$, which are h-projective because $\overline{\mathcal{A}}$ and $\overline{\mathcal{B}}$ are h-projective bimodules, and $\mathcal{R} \otimes_{\mathcal{A}} \overline{\mathcal{A}} \otimes_{\mathcal{A}} \varphi \otimes_{\mathcal{B}} \overline{\mathcal{B}} \otimes_{\mathcal{B}} \mathcal{R}$, which is h-projective because $\overline{\mathcal{A}} \otimes_{\mathcal{A}} \varphi \otimes_{\mathcal{B}} \overline{\mathcal{B}}$ is h-projective by [AL17, Proposition 2.5].

Proposition 3.2.3. *The twist around the functor* (3.9) *is given by* $T_{\alpha_N} T_{\alpha_M}$.

Proof. By Definition 2.5.2, to identify the twist around the functor (3.9) we have to identify the cone of the derived trace map for P, see Definition 2.4.7. To do so, we are free to choose any lift of the derived trace map to a morphism in **Mod**-$\mathcal{C}$.

To lift the derived trace map, we have to to choose an h-projective resolution of P, so to fix a lift of the functor (3.9) to a functor $\mathcal{P}(\mathcal{R}) \to \mathcal{P}(\mathcal{C})$ (lift which is unique up to homotopy).

Let us write $\widetilde{\mathcal{R}}$ for the h-projective resolution of $\mathcal{R}$ constructed in Lemma 3.2.1, that is for the convolution of the morphism in the brackets of (3.11), and $\widetilde{\tau}\colon \widetilde{\mathcal{R}} \to \mathcal{R}$ for the morphism $g_{\mathcal{A}}+g_{\mathcal{B}}$ in (3.11). As $\widetilde{\mathcal{R}}$ is h-projective and P is $\mathcal{C}$-h-projective, see Lemma 3.1.6, the bimodule $\widetilde{\mathcal{R}} \otimes_{\mathcal{R}} P$ is h-projective by [AL17, Proposition 2.5], and we take $\widetilde{\mathcal{R}} \otimes_{\mathcal{R}} P$ as an h-projective resolution of P.

Given this h-projective resolution, the derived trace map of P is lifted to the morphism

$$\mathrm{tr}\colon (\widetilde{\mathcal{R}} \otimes_{\mathcal{R}} P)^{\mathcal{C}} \otimes_{\mathcal{R}} \widetilde{\mathcal{R}} \otimes_{\mathcal{R}} P \to \mathcal{C}. \tag{3.12}$$

However, notice that the quasi-isomorphism $\widetilde{\tau}\colon \widetilde{\mathcal{R}} \otimes_{\mathcal{R}} P \to P$ remains a quasi-isomorphism upon dualisation because $\widetilde{\mathcal{R}} \otimes_{\mathcal{R}} P$ is a $\mathcal{C}$-h-projective, $\mathcal{C}$-perfect bimodule. Therefore, the convolution of (3.12) is quasi-isomorphic to the convolution of the morphism

$$\mathrm{tr}\colon P^{\mathcal{C}} \otimes_{\mathcal{R}} \widetilde{\mathcal{R}} \otimes_{\mathcal{R}} P \xrightarrow{(-\circ\widetilde{\tau})\otimes \mathrm{id}} (\widetilde{\mathcal{R}} \otimes_{\mathcal{R}} P)^{\mathcal{C}} \otimes_{\mathcal{R}} \widetilde{\mathcal{R}} \otimes_{\mathcal{R}} P \xrightarrow{(3.12)} \mathcal{C}, \tag{3.13}$$

and thus the functor T_{α_P} is the functor induced by the bimodule given by the convolution of (3.13).

Let us now study the convolution of (3.13). First of all, notice that (3.13) is equal to the morphism

$$P^{\mathcal{C}} \otimes_{\mathcal{R}} \widetilde{\mathcal{R}} \otimes_{\mathcal{R}} P \xrightarrow{\mathrm{id}\otimes\widetilde{\tau}\otimes\mathrm{id}} P^{\mathcal{C}} \otimes_{\mathcal{R}} P \xrightarrow{\mathrm{tr}} \mathcal{C}, \tag{3.14}$$

and thus we can study (3.14) in place of (3.13).

Then, using the definition of $\widetilde{\mathcal{R}}$, we see that $P^{\mathcal{C}} \otimes_{\mathcal{R}} \widetilde{\mathcal{R}} \otimes_{\mathcal{R}} P$ is equal to

$$\{M^{\mathcal{C}} \overline{\otimes}_{\mathcal{A}} M \otimes_{\mathcal{C}} N^{\mathcal{C}} \overline{\otimes}_{\mathcal{B}} N \xrightarrow{(-\mathrm{id}\otimes\mathrm{tr},\mathrm{tr}\otimes\mathrm{id})} M^{\mathcal{C}} \overline{\otimes}_{\mathcal{A}} M \oplus N^{\mathcal{C}} \overline{\otimes}_{\mathcal{B}} N\} \tag{3.15}$$

where we write $\mathrm{tr}\colon M^{\mathcal{C}} \overline{\otimes}_{\mathcal{A}} M \to \mathcal{C}$ and $\mathrm{tr}\colon N^{\mathcal{C}} \overline{\otimes}_{\mathcal{B}} N \to \mathcal{C}$ for the maps

$$M^{\mathcal{C}} \otimes_{\mathcal{A}} \overline{\mathcal{A}} \otimes_{\mathcal{A}} M \xrightarrow{\mathrm{tr}\circ(\mathrm{id}\otimes\tau\otimes\mathrm{id})} \mathcal{C} \quad \text{and} \quad N^{\mathcal{C}} \otimes_{\mathcal{B}} \overline{\mathcal{B}} \otimes_{\mathcal{B}} N \xrightarrow{\mathrm{tr}\circ(\mathrm{id}\otimes\tau\otimes\mathrm{id})} \mathcal{C}.$$

Plugging (3.15) into (3.14), the latter takes the form

$$\{M^{\mathcal{C}} \overline{\otimes}_{\mathcal{A}} M \otimes_{\mathcal{C}} N^{\mathcal{C}} \overline{\otimes}_{\mathcal{B}} N \xrightarrow{(-\mathrm{id}\otimes\mathrm{tr},\mathrm{tr}\otimes\mathrm{id})} M^{\mathcal{C}} \overline{\otimes}_{\mathcal{A}} M \oplus N^{\mathcal{C}} \overline{\otimes}_{\mathcal{B}} N\} \xrightarrow{\mathrm{tr}+\mathrm{tr}} \mathcal{C}. \tag{3.16}$$

Indeed, if $\psi \otimes a \otimes m \in M^{\mathcal{C}} \otimes_{\mathcal{A}} \overline{\mathcal{A}} \otimes_{\mathcal{A}} M$, then its image via (3.14) is

$$\begin{aligned}\psi \otimes a \otimes m \overset{(3.14)}{\mapsto} \operatorname{tr}((-1)^{\deg(\psi)}(\psi \cdot \tau(a)) \otimes m) &= (-1)^{\deg(\psi)}(\psi \cdot \tau(a))(m)\\ &= (-1)^{\deg(\psi)}\psi(\tau(a) \cdot m)\\ &= \operatorname{tr}(\psi \otimes a \otimes m),\end{aligned}$$

and similarly one shows the statement for elements of $N^{\mathcal{C}} \overline{\otimes}_{\mathcal{B}} N$.

By [AL17, Lemma 3.4], the convolution of (3.16), and thus of (3.14), is equal to

$$\left\{M^{\mathcal{C}} \overline{\otimes}_{\mathcal{A}} M \xrightarrow{\operatorname{tr}} \mathcal{C}\right\} \otimes_{\mathcal{C}} \left\{N^{\mathcal{C}} \overline{\otimes}_{\mathcal{B}} N \xrightarrow{\operatorname{tr}} \mathcal{C}\right\}.$$

As M and N are h-projective bimodules, the functor induced by the above tensor product is the composition $T_{\alpha_N} T_{\alpha_M}$, and therefore we get $T_{\alpha_P} \simeq T_{\alpha_N} T_{\alpha_M}$, as we wanted.

□

3.3 The cotwist

We now wish to describe the cotwist around the functor (3.9).

Proposition 3.3.1. *The cotwist around the functor* (3.9) *is described by bimodule* C_P *defined in* (3)*, that is by the bimodule given by matrix*

$$\begin{pmatrix} \{\mathcal{A} \xrightarrow{\text{act}} \operatorname{Hom}_{\mathcal{C}}(M, M)\}[-1] & 0 \\ N \otimes_{\mathcal{C}} M^{\mathcal{C}}[-1] & \{\mathcal{B} \xrightarrow{\text{act}} \operatorname{Hom}_{\mathcal{C}}(N, N)\}[-1] \end{pmatrix}$$

and the structure morphisms

$$N \otimes_{\mathcal{C}} M^{\mathcal{C}}[-1] \otimes_{\mathcal{A}} M \otimes_{\mathcal{C}} N^{\mathcal{C}} \xrightarrow{i_N \circ \text{ev} \circ (\text{id} \otimes \text{tr} \otimes \text{id})} \{\mathcal{B} \xrightarrow{\text{act}} \operatorname{Hom}_{\mathcal{C}}(N, N)\}[-1]$$

$$M \otimes_{\mathcal{C}} N^{\mathcal{C}} \otimes_{\mathcal{B}} N \otimes_{\mathcal{C}} M^{\mathcal{C}}[-1] \xrightarrow{i_M \circ \text{ev} \circ (\text{id} \otimes \text{tr} \otimes \text{id})} \{\mathcal{A} \xrightarrow{\text{act}} \operatorname{Hom}_{\mathcal{C}}(M, M)\}[-1]$$

where i_M *and* i_N *are defined in* § 2.4.2*.*

Proof. Our aim is to give a description of the shifted cone of the derived action map for P, see Definition 2.4.7. To calculate such cone we are free to choose any map of $\mathcal{R}$-$\mathcal{R}$-bimodules whose image in $\mathrm{D}(\mathcal{R}\text{-}\mathcal{R})$ is the derived action map. As by Lemma 3.1.6 P is $\mathcal{C}$-h-projective, we have $\operatorname{Hom}_{\mathcal{C}}(P, P) = \operatorname{RHom}_{\mathcal{C}}(P, P)$, and as a lift of the derived action map we can take the action map $\text{act}\colon \mathcal{R} \to \operatorname{Hom}_{\mathcal{C}}(P, P)$. By Lemma 3.1.8, the matrix description of act is given by

$$\begin{pmatrix} \mathcal{A} & M \otimes_{\mathcal{C}} N^{\mathcal{C}} \\ 0 & \mathcal{B} \end{pmatrix} \xrightarrow{\begin{pmatrix} \mathrm{act} & \mathrm{ev} \\ 0 & \mathrm{act} \end{pmatrix}} \begin{pmatrix} \mathrm{Hom}_{\mathcal{C}}(M,M) & \mathrm{Hom}_{\mathcal{C}}(N,M) \\ \mathrm{Hom}_{\mathcal{C}}(M,N) & \mathrm{Hom}_{\mathcal{C}}(N,N) \end{pmatrix}.$$

Now consider the bimodule[2]

$$C'_P[1] := \begin{pmatrix} \{\mathcal{A} \xrightarrow{\mathrm{act}} \mathrm{Hom}_{\mathcal{C}}(M,M)\} & 0 \\ \mathrm{Hom}_{\mathcal{C}}(M,N) & \{\mathcal{B} \xrightarrow{\mathrm{act}} \mathrm{Hom}_{\mathcal{C}}(N,N)\} \end{pmatrix}$$

with structure morphisms

$$\mathrm{Hom}_{\mathcal{C}}(M,N) \otimes_{\mathcal{A}} M \otimes_{\mathcal{C}} N^{\mathcal{C}} \xrightarrow{\mathrm{ev}\otimes\mathrm{id}} N \otimes_{\mathcal{C}} N^{\mathcal{C}} \to \xrightarrow{\mathrm{ev}} \mathrm{Hom}_{\mathcal{C}}(N,N) \xrightarrow{i_N} \{\mathcal{B} \xrightarrow{\mathrm{act}} \mathrm{Hom}_{\mathcal{C}}(N,N)\}$$

and

$$M \otimes_{\mathcal{C}} N^{\mathcal{C}} \otimes_{\mathcal{B}} \mathrm{Hom}_{\mathcal{C}}(M,N) \xrightarrow{\mathrm{id}\otimes\mathrm{cmp}} M \otimes_{\mathcal{A}} M^{\mathcal{C}} \to \xrightarrow{\mathrm{ev}} \mathrm{Hom}_{\mathcal{C}}(M,M) \xrightarrow{i_M} \{\mathcal{A} \xrightarrow{\mathrm{act}} \mathrm{Hom}_{\mathcal{C}}(M,M)\}.$$

Then, we have a morphism of $\mathcal{R}$-$\mathcal{R}$-bimodules

$$\begin{pmatrix} \mathrm{Hom}_{\mathcal{C}}(M,M) & \mathrm{Hom}_{\mathcal{C}}(N,M) \\ \mathrm{Hom}_{\mathcal{C}}(M,N) & \mathrm{Hom}_{\mathcal{C}}(N,N) \end{pmatrix} \xrightarrow{\begin{pmatrix} i_M & 0 \\ \mathrm{id} & i_N \end{pmatrix}} C'_P[1]$$

whose composition with $\mathrm{act}\colon \mathcal{R} \to \mathrm{Hom}_{\mathcal{C}}(P,P)$ is by construction the differential of the morphism

$$\begin{pmatrix} \mathcal{A} & M \otimes_{\mathcal{C}} N^{\mathcal{C}} \\ 0 & \mathcal{B} \end{pmatrix} \xrightarrow{\begin{pmatrix} j_M & 0 \\ 0 & j_N \end{pmatrix}} C'_P[1].$$

Thus, we obtain a morphism

$$\left\{\mathcal{R} \xrightarrow{\mathrm{act}} \mathrm{Hom}_{\mathcal{C}}(P,P)\right\} \to C'_P[1]$$

that by Lemma 2.4.34 is a quasi-isomorphism. Indeed, the $\mathcal{A}$-$\mathcal{A}$, $\mathcal{B}$-$\mathcal{B}$ and $\mathcal{B}$-$\mathcal{A}$ components are quasi-isomorphisms by definition, while the $\mathcal{A}$-$\mathcal{B}$ component is given by $\mathrm{ev}\colon M \otimes_{\mathcal{C}}$

[2]For the notation $\{-\}$ and the definition of the morphisms i and j, see § 2.4.2.

$N^{\mathcal{C}} \to \mathrm{Hom}_{\mathcal{C}}(N, M)$, which is a quasi-isomorphism because N is $\mathcal{C}$-h-projective and $\mathcal{C}$-perfect, see Remark 2.4.11.

Thus, C'_P is quasi-isomorphic to the cotwist bimodule for P. To conclude the proof of the proposition, we now consider the morphism of bimodules $C_P \to C'_P$ induced by the maps

$$\mathrm{id}\colon \{\mathcal{A} \xrightarrow{\mathrm{act}} \mathrm{Hom}_{\mathcal{C}}(M, M)\} \to \{\mathcal{A} \xrightarrow{\mathrm{act}} \mathrm{Hom}_{\mathcal{C}}(M, M)\}$$

$$\mathrm{id}\colon \{\mathcal{B} \xrightarrow{\mathrm{act}} \mathrm{Hom}_{\mathcal{C}}(N, N)\} \to \{\mathcal{B} \xrightarrow{\mathrm{act}} \mathrm{Hom}_{\mathcal{C}}(N, N)\}$$

$$\mathrm{ev}\colon N \otimes_{\mathcal{C}} M^{\mathcal{C}}[-1] \to \mathrm{Hom}_{\mathcal{C}}(M, N)[-1].$$

The resulting morphism $C_P \to C'_P$ is a quasi-isomorphism by Lemma 2.4.34 because its components are. Indeed, M is $\mathcal{C}$-h-projective and $\mathcal{C}$-perfect, and thus the evaluation map $\mathrm{ev}\colon N \otimes_{\mathcal{C}} M^{\mathcal{C}} \to \mathrm{Hom}_{\mathcal{C}}(M, N)$ is a quasi-isomorphism, see Remark 2.4.11.

Thus, the proof of the proposition is now complete. □

Having proved Proposition 3.3.1, we now wish to describe how the functor C_{α_P} interacts with the SOD $\mathrm{D}(\mathcal{R}) = \langle \mathrm{D}(\mathcal{B}), \mathrm{D}(\mathcal{A})\rangle$ of Proposition 2.4.25. To do so, we fix a lift of C_{α_P} to the dg-level. That is, we consider the functor

$$- \otimes_{\mathcal{R}} \widetilde{\mathcal{R}} \otimes_{\mathcal{R}} C_P \colon \mathbf{Mod}\text{-}\mathcal{R} \to \mathbf{Mod}\text{-}\mathcal{R} \tag{3.17}$$

where C_P is the bimodule described in Proposition 3.3.1, and $\widetilde{\mathcal{R}}$ is the h-projective resolution of $\mathcal{R}$ constructed in Lemma 3.2.1.

During the course of the proofs of the propositions and lemmas below, we will make use of the following notation; we will write

$$C_M = \{\mathcal{A} \xrightarrow{\mathrm{act}} \mathrm{Hom}_{\mathcal{C}}(M, M)\}[-1] \quad \text{and} \quad C_N = \{\mathcal{B} \xrightarrow{\mathrm{act}} \mathrm{Hom}_{\mathcal{C}}(N, N)\}[-1]$$

for the $\mathcal{A}$-$\mathcal{A}$ and $\mathcal{B}$-$\mathcal{B}$ component of C_P, respectively. Moreover, we will write σ_M and σ_N for the morphisms

$$M \otimes_{\mathcal{C}} M^{\mathcal{C}}[-1] \xrightarrow{\mathrm{ev}} \mathrm{Hom}_{\mathcal{C}}(M, M)[-1] \xrightarrow{i_M} C_M$$

and

$$N \otimes_{\mathcal{C}} N^{\mathcal{C}}[-1] \xrightarrow{\mathrm{ev}} \mathrm{Hom}_{\mathcal{C}}(N, N)[-1] \xrightarrow{i_N} C_N$$

respectively.

We kindly advise the reader to glance through § 2.4.3 before reading the following proofs, as in *ibidem* we set up the necessary vocabulary and notation. Moreover, for the reader's convenience, we recall that we write $\tau\colon \overline{\mathcal{A}} \to \mathcal{A}$ and $\tau\colon \overline{\mathcal{B}} \to \mathcal{B}$ for the quasiisomorphims between the bar complex and the diagonal bimodule of a dg-category, see § 2.4.4.

Proposition 3.3.2. *Let $F \in \mathbf{Mod}\text{-}\mathcal{R}$ be a right $\mathcal{R}$-dg-module with components $F_{\mathcal{A}}$, $F_{\mathcal{B}}$ and structure morphism $\rho\colon F_{\mathcal{A}} \otimes_{\mathcal{A}} M \otimes_{\mathcal{C}} N^{\mathcal{C}} \to F_{\mathcal{B}}$. Write G for the image of F via the functor (3.17).*

Then, $G_{\mathcal{A}}$ is given by the convolution of the morphism of twisted complexes

$$F_{\mathcal{A}} \overline{\otimes}_{\mathcal{A}} M \otimes_{\mathcal{C}} N^{\mathcal{C}} \overline{\otimes}_{\mathcal{B}} N \otimes_{\mathcal{B}} M^{\mathcal{C}}[-1] \xrightarrow{(-\operatorname{id}\otimes(\sigma_M\circ(\operatorname{id}\otimes\operatorname{tr}\otimes\operatorname{id})),(\rho\circ(\operatorname{id}\otimes\tau\otimes\operatorname{id}))\otimes\operatorname{id})} \begin{matrix} F_{\mathcal{A}} \overline{\otimes}_{\mathcal{A}} C_M \\ \oplus \\ F_{\mathcal{B}} \overline{\otimes}_{\mathcal{B}} N \otimes_{\mathcal{C}} M^{\mathcal{C}}[-1] \end{matrix}$$

$G_{\mathcal{B}}$ is given by the convolution of the twisted complex

$$F_{\mathcal{A}} \overline{\otimes}_{\mathcal{A}} M \otimes_{\mathcal{C}} N^{\mathcal{C}} \overline{\otimes}_{\mathcal{B}} C_N \xrightarrow{(\rho\circ(\operatorname{id}\otimes\tau\otimes\operatorname{id}))\otimes\operatorname{id}} F_{\mathcal{B}} \overline{\otimes}_{\mathcal{B}} C_N$$

and the structure morphism $G_{\mathcal{A}} \otimes_{\mathcal{A}} M \otimes_{\mathcal{C}} N^{\mathcal{C}} \to G_{\mathcal{B}}$ is given by the convolution of the morphism of twisted complexes induced by the morphisms

$$F_{\mathcal{A}} \overline{\otimes}_{\mathcal{A}} M \otimes_{\mathcal{C}} N^{\mathcal{C}} \overline{\otimes}_{\mathcal{B}} N \otimes_{\mathcal{C}} M^{\mathcal{C}}[-1] \otimes_{\mathcal{A}} M \otimes_{\mathcal{C}} N^{\mathcal{C}} \xrightarrow{\operatorname{id}\otimes(\sigma_N\circ(\operatorname{id}\otimes\operatorname{tr}\otimes\operatorname{id}))} F_{\mathcal{A}} \overline{\otimes}_{\mathcal{A}} M \otimes_{\mathcal{C}} N^{\mathcal{C}} \overline{\otimes}_{\mathcal{B}} C_N$$

and

$$\begin{matrix} F_{\mathcal{A}} \overline{\otimes}_{\mathcal{A}} C_M \otimes_{\mathcal{A}} M \otimes_{\mathcal{C}} N^{\mathcal{C}} \\ \oplus \\ F_{\mathcal{B}} \overline{\otimes}_{\mathcal{B}} N \otimes_{\mathcal{C}} M^{\mathcal{C}}[-1] \otimes_{\mathcal{A}} M \otimes_{\mathcal{B}} N^{\mathcal{C}} \end{matrix} \xrightarrow{0+\operatorname{id}\otimes(\sigma_N\circ(\operatorname{id}\otimes\operatorname{tr}\otimes\operatorname{id}))} F_{\mathcal{B}} \overline{\otimes}_{\mathcal{B}} C_N$$

Proof. Notice that by Remark 2.4.17 the $\mathcal{R}$-dg-bimodule $\widetilde{\mathcal{R}}$ is the convolution, shifted by 1, of the morphism of twisted complexes

$$\begin{array}{ccc} \mathcal{R} \overline{\otimes}_{\mathcal{A}} M \otimes_{\mathcal{C}} N^{\mathcal{C}} \overline{\otimes}_{\mathcal{B}} \mathcal{R} & \xrightarrow{f_{\mathcal{A}}} & \mathcal{R} \overline{\otimes}_{\mathcal{A}} \mathcal{R} \\ {\scriptstyle f_{\mathcal{B}}}\downarrow & & \downarrow \\ \mathcal{R} \overline{\otimes}_{\mathcal{B}} \mathcal{R} & \longrightarrow & 0 \end{array}$$

where $f_{\mathcal{A}}$ and $f_{\mathcal{B}}$ are defined in Lemma 3.2.1.

It follows from the definition of convolution given in § 2.4.3 that tensor product commutes with convolution, and therefore $F \otimes_{\mathcal{R}} \widetilde{\mathcal{R}} \otimes_{\mathcal{R}} C_P$ is given by the convolution, shifted by 1, of the morphism of twisted complexes

$$\begin{array}{ccc} F \overline{\otimes}_{\mathcal{A}} M \otimes_{\mathcal{C}} N^{\mathcal{C}} \overline{\otimes}_{\mathcal{B}} C_P & \xrightarrow{f_{\mathcal{A}}} & F \overline{\otimes}_{\mathcal{A}} C_P \\ {\scriptstyle f_{\mathcal{B}}}\downarrow & & \downarrow \\ F \overline{\otimes}_{\mathcal{B}} C_P & \longrightarrow & 0 \end{array} \tag{3.18}$$

Then, the statement of the propositions follows from the description of C_P given in Proposition 3.3.1. $\square$

Remark 3.3.3. In Proposition 3.3.2 we did not explain at length how (3.17) acts on morphisms, but it is clear from the proof of the proposition that given a morphism $f\colon F \to G$ of $\mathcal{R}$-dg-modules with components $(f_{\mathcal{A}}, f_{\mathcal{B}})$ the image of f via (3.17) is the morphism that maps each copy of $F_{\mathcal{A}}$ and $F_{\mathcal{B}}$ to each copy of $G_{\mathcal{A}}$ and $G_{\mathcal{B}}$, respectively, appearing in the equations of the statement of Proposition 3.3.2.

We now deal with a few special cases of the previous proposition that we will need during the course of the proof of Proposition 3.3.9.

Lemma 3.3.4. *Let $G_{\mathcal{A}}$ a $\mathcal{A}$-dg-module, and write*

$$G = G_{\mathcal{A}} \overline{\otimes}_{\mathcal{A}} \mathcal{R} = \begin{pmatrix} G_{\mathcal{A}} \otimes_{\mathcal{A}} \overline{\mathcal{A}} & G_{\mathcal{A}} \overline{\otimes}_{\mathcal{A}} M \otimes_{\mathcal{C}} N^{\mathcal{C}} \end{pmatrix}.$$

Then, the image of G via the functor (3.17) is quasi-isomorphic to the $\mathcal{R}$-dg-module $\begin{pmatrix} G_{\mathcal{A}} \overline{\otimes}_{\mathcal{A}} C_M & 0 \end{pmatrix}$.

Proof. By Proposition 3.3.2 and (2.26) we know that the image of G via the functor (3.17) is the $\mathcal{R}$-dg-module whose $\mathcal{A}$ component is given by the convolution of the morphism

$$G_{\mathcal{A}} \otimes_{\mathcal{A}} \overline{\mathcal{A}} \overline{\otimes}_{\mathcal{A}} M \otimes_{\mathcal{C}} N^{\mathcal{C}} \overline{\otimes}_{\mathcal{B}} N \otimes_{\mathcal{B}} M^{\mathcal{C}}[-1] \xrightarrow{\alpha} \begin{matrix} G_{\mathcal{A}} \otimes_{\mathcal{A}} \overline{\mathcal{A}} \overline{\otimes}_{\mathcal{A}} C_M \\ \oplus \\ G_{\mathcal{A}} \overline{\otimes}_{\mathcal{A}} M \otimes_{\mathcal{C}} N^{\mathcal{C}} \overline{\otimes}_{\mathcal{B}} N \otimes_{\mathcal{C}} M^{\mathcal{C}}[-1] \end{matrix} \tag{3.19}$$

where

$$\alpha = (-\operatorname{id} \otimes (\sigma_M \circ (\operatorname{id} \otimes \operatorname{tr} \otimes \operatorname{id})), \operatorname{id} \otimes \tau \otimes \operatorname{id})$$

and whose $\mathcal{B}$-component is given by the convolution of the morphism

$$G_{\mathcal{A}} \otimes_{\mathcal{A}} \overline{\mathcal{A}} \overline{\otimes}_{\mathcal{A}} M \otimes_{\mathcal{C}} N^{\mathcal{C}} \overline{\otimes}_{\mathcal{B}} C_N \xrightarrow{\operatorname{id} \otimes \tau \otimes \operatorname{id}} G_{\mathcal{A}} \overline{\otimes}_{\mathcal{A}} M \otimes_{\mathcal{C}} N^{\mathcal{C}} \overline{\otimes}_{\mathcal{B}} C_N.$$

We now want to construct a quasi-isomorphism of $\mathcal{R}$-dg-modules $g\colon G \otimes_{\mathcal{R}} \widetilde{\mathcal{R}} \otimes_{\mathcal{R}} C_P \to \begin{pmatrix} G_{\mathcal{A}} \overline{\otimes}_{\mathcal{A}} C_M & 0 \end{pmatrix}$. Notice that, as the target $\mathcal{R}$-dg-module has only the $\mathcal{A}$-component, any morphism of $\mathcal{A}$-dg-modules between the $\mathcal{A}$-component of $G \otimes_{\mathcal{R}} \widetilde{\mathcal{R}} \otimes_{\mathcal{R}} C_P$ and the $\mathcal{A}$-component of $\begin{pmatrix} G_{\mathcal{A}} \overline{\otimes}_{\mathcal{A}} C_M & 0 \end{pmatrix}$ induces a morphism of $\mathcal{R}$-dg-modules between $G \otimes_{\mathcal{R}} \widetilde{\mathcal{R}} \otimes_{\mathcal{R}} C_P$ and $\begin{pmatrix} G_{\mathcal{A}} \overline{\otimes}_{\mathcal{A}} C_M & 0 \end{pmatrix}$. Moreover, as a morphism of $\mathcal{R}$-dg-modules is a quasi-isomorphism if and only if its components are,[3] to define the quasi-isomorphism g it is enough to give a quasi-isomorphism between (3.19) and $G_{\mathcal{A}} \overline{\otimes}_{\mathcal{A}} C_M$.

[3]Recall that quasi-isomorphism of dg-modules are defined fibrewise.

We define g as the morphism of $\mathcal{R}$-dg-modules associated to the quasi-isomorphism of $\mathcal{A}$-dg-modules given by the convolution of the morphism of twisted complexes

$$\begin{array}{ccc} G_{\mathcal{A}} \otimes_{\mathcal{A}} \overline{\mathcal{A}} \,\overline{\otimes}_{\mathcal{A}} M \otimes_{\mathcal{C}} N^{\mathcal{C}} \,\overline{\otimes}_{\mathcal{B}} N \otimes_{\mathcal{B}} M^{\mathcal{C}}[-1] & \xrightarrow{(3.19)} & \begin{array}{c} G_{\mathcal{A}} \otimes_{\mathcal{A}} \overline{\mathcal{A}} \,\overline{\otimes}_{\mathcal{A}} C_M \\ \oplus \\ G_{\mathcal{A}} \,\overline{\otimes}_{\mathcal{A}} M \otimes_{\mathcal{C}} N^{\mathcal{C}} \,\overline{\otimes}_{\mathcal{B}} N \otimes_{\mathcal{C}} M^{\mathcal{C}}[-1] \end{array} \\ \downarrow & & \downarrow{\scriptstyle \gamma} \\ 0 & \longrightarrow & G_{\mathcal{A}} \,\overline{\otimes}_{\mathcal{A}} C_M \end{array}$$

where

$$\gamma = (\mathrm{id} \otimes \tau \otimes \mathrm{id}, \mathrm{id} \otimes (\sigma_M \circ (\mathrm{id} \otimes \mathrm{tr} \otimes \mathrm{id})))$$

and $\tau \colon \overline{\mathcal{A}} \to \mathcal{A}$ is the quasi-isomorphism that turns $\overline{\mathcal{A}}$ into an h-projective resolution of $\mathcal{A}$, see § 2.4.4. □

Lemma 3.3.5. *Let* $F = \begin{pmatrix} 0 & F_{\mathcal{B}} \end{pmatrix}$ *and* $G = \begin{pmatrix} 0 & G_{\mathcal{B}} \end{pmatrix}$ *be two* $\mathcal{R}$*-dg-modules and* $f \colon F \to G$ *a morphism of in* **Mod**-$\mathcal{R}$ *given by a morphism* $f_{\mathcal{B}} \colon F_{\mathcal{B}} \to G_{\mathcal{B}}$ *in* **Mod**-$\mathcal{B}$*.*

Then, the images of F *and* G *via (3.17) are equal to*

$$\begin{pmatrix} F_{\mathcal{B}} \,\overline{\otimes}_{\mathcal{B}} N \otimes_{\mathcal{C}} M^{\mathcal{C}}[-1] & F_{\mathcal{B}} \,\overline{\otimes}_{\mathcal{B}} C_N \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} G_{\mathcal{B}} \,\overline{\otimes}_{\mathcal{B}} N \otimes_{\mathcal{C}} M^{\mathcal{C}}[-1] & G_{\mathcal{B}} \,\overline{\otimes}_{\mathcal{B}} C_N \end{pmatrix}$$

respectively, and the image of f *is given by the morphism with components*

$$F_{\mathcal{B}} \otimes_{\mathcal{B}} \overline{\mathcal{B}} \otimes_{\mathcal{B}} N \otimes_{\mathcal{C}} M^{\mathcal{C}}[-1] \xrightarrow{f_{\mathcal{B}} \otimes \mathrm{id}} G_{\mathcal{B}} \otimes_{\mathcal{B}} \overline{\mathcal{B}} \otimes_{\mathcal{B}} N \otimes_{\mathcal{C}} M^{\mathcal{C}}[-1] \tag{3.20}$$

and

$$F_{\mathcal{B}} \otimes_{\mathcal{B}} \overline{\mathcal{B}} \otimes_{\mathcal{B}} C_N \xrightarrow{f_{\mathcal{B}} \otimes \mathrm{id}} G_{\mathcal{B}} \otimes_{\mathcal{B}} \overline{\mathcal{B}} \otimes_{\mathcal{B}} C_N. \tag{3.21}$$

Proof. This lemma follows from Proposition 3.3.2 and Remark 3.3.3. □

Remark 3.3.6. Notice that the images of (3.20) and (3.21) via the functor Υ of Theorem 2.4.19 are given by

$$F_{\mathcal{B}} \,\overline{\otimes}_{\mathcal{B}} N \otimes_{\mathcal{C}} M^{\mathcal{C}}[-1] \xrightarrow{\Upsilon(f_{\mathcal{B}}) \,\overline{\otimes}\, \mathrm{id}} G_{\mathcal{B}} \,\overline{\otimes}_{\mathcal{B}} N \otimes_{\mathcal{C}} M^{\mathcal{C}}[-1]$$

and

$$F_{\mathcal{B}} \,\overline{\otimes}_{\mathcal{B}} C_N \xrightarrow{\Upsilon(f_{\mathcal{B}}) \,\overline{\otimes}\, \mathrm{id}} G_{\mathcal{B}} \,\overline{\otimes}_{\mathcal{B}} C_N.$$

respectively. We show the claim for (3.20).

By definition $\Upsilon(f_{\mathcal{B}}) \overline{\otimes}$ id is given by the morphism

$$\begin{aligned} F_{\mathcal{B}} \overline{\otimes}_{\mathcal{B}} N \otimes_{\mathcal{C}} M^{\mathcal{C}}[-1] \otimes_{\mathcal{B}} \overline{\mathcal{B}} &= F_{\mathcal{B}} \otimes_{\mathcal{B}} \overline{\mathcal{B}} \otimes_{\mathcal{B}} N \otimes_{\mathcal{C}} M^{\mathcal{C}}[-1] \otimes_{\mathcal{B}} \overline{\mathcal{B}} \\ &\xrightarrow{\mathrm{id} \otimes \Delta \otimes \Delta \otimes \mathrm{id}} F_{\mathcal{B}} \otimes_{\mathcal{B}} \overline{\mathcal{B}} \otimes_{\mathcal{B}} \overline{\mathcal{B}} \otimes_{\mathcal{B}} \overline{\mathcal{B}} \otimes_{\mathcal{B}} N \otimes_{\mathcal{C}} M^{\mathcal{C}}[-1] \otimes_{\mathcal{B}} \overline{\mathcal{B}} \\ &\xrightarrow{\Upsilon(f_{\mathcal{B}}) \otimes \mathrm{id} \otimes \tau \otimes \mathrm{id} \otimes \tau} G_{\mathcal{B}} \otimes_{\mathcal{B}} \overline{\mathcal{B}} \otimes_{\mathcal{B}} N \otimes_{\mathcal{C}} M^{\mathcal{C}}[-1] = G_{\mathcal{B}} \overline{\otimes}_{\mathcal{B}} N \otimes_{\mathcal{C}} M^{\mathcal{C}}[-1] \end{aligned} \tag{3.22}$$

As $\Delta \colon \overline{\mathcal{B}} \to \overline{\mathcal{B}} \otimes_{\mathcal{B}} \overline{\mathcal{B}}$ and $\tau \colon \overline{\mathcal{B}} \to \mathcal{B}$ endow $\overline{\mathcal{B}}$ with the structure of a coalgebra, (3.22) is equal to

$$\begin{aligned} F_{\mathcal{B}} \otimes_{\mathcal{B}} \overline{\mathcal{B}} \otimes_{\mathcal{B}} N \otimes_{\mathcal{C}} M^{\mathcal{C}}[-1] \otimes_{\mathcal{B}} \overline{\mathcal{B}} &\xrightarrow{\mathrm{id} \otimes \Delta \otimes \mathrm{id}} F_{\mathcal{B}} \otimes_{\mathcal{B}} \overline{\mathcal{B}} \otimes_{\mathcal{B}} \overline{\mathcal{B}} \otimes_{\mathcal{B}} N \otimes_{\mathcal{C}} M^{\mathcal{C}}[-1] \otimes_{\mathcal{B}} \overline{\mathcal{B}} \\ &\xrightarrow{\Upsilon(f_{\mathcal{B}}) \otimes \mathrm{id} \otimes \tau} G_{\mathcal{B}} \otimes_{\mathcal{B}} \overline{\mathcal{B}} \otimes_{\mathcal{B}} N \otimes_{\mathcal{C}} M^{\mathcal{C}}[-1] \end{aligned} \tag{3.23}$$

To conclude, recall that $\Upsilon(f_{\mathcal{B}}) = f_{\mathcal{B}} \otimes \tau$, and therefore (3.23) is equal to

$$F_{\mathcal{B}} \otimes_{\mathcal{B}} \overline{\mathcal{B}} \otimes_{\mathcal{B}} N \otimes_{\mathcal{C}} M^{\mathcal{C}}[-1] \otimes_{\mathcal{B}} \overline{\mathcal{B}} \xrightarrow{f_{\mathcal{B}} \otimes \mathrm{id} \otimes \tau} G_{\mathcal{B}} \otimes_{\mathcal{B}} \overline{\mathcal{B}} \otimes_{\mathcal{B}} N \otimes_{\mathcal{C}} M^{\mathcal{C}}[-1] \tag{3.24}$$

which, by definition, is equal to $\Upsilon((3.20))$, as we wanted to show.

Lemma 3.3.7. *Let* $G = G_{\mathcal{A}} \overline{\otimes}_{\mathcal{A}} \mathcal{R} = \begin{pmatrix} G_{\mathcal{A}} \overline{\otimes}_{\mathcal{A}} \mathcal{A} & G_{\mathcal{A}} \overline{\otimes}_{\mathcal{A}} M \otimes_{\mathcal{C}} N^{\mathcal{C}} \end{pmatrix}$, $F = \begin{pmatrix} 0 & F_{\mathcal{B}} \end{pmatrix}$ *be two* $\mathcal{R}$*-dg-modules and* $f \colon F \to G$ *be a morphism in* **Mod**-$\mathcal{R}$ *given by the morphism* $f_{\mathcal{B}} \colon F_{\mathcal{B}} \to G_{\mathcal{A}} \overline{\otimes}_{\mathcal{A}} M \otimes_{\mathcal{C}} N^{\mathcal{C}}$ *in* **Mod**-$\mathcal{B}$*.*

Then, under the quasi-isomorphism of Lemma 3.3.4, the image of f via the functor (3.17) is quasi-isomorphic to the morphism

$$\begin{pmatrix} F_{\mathcal{B}} \overline{\otimes}_{\mathcal{B}} N \otimes_{\mathcal{C}} M^{\mathcal{C}}[-1] & F_{\mathcal{B}} \overline{\otimes}_{\mathcal{B}} C_N \end{pmatrix} \to \begin{pmatrix} G_{\mathcal{A}} \overline{\otimes}_{\mathcal{A}} C_M & 0 \end{pmatrix}$$

in **Mod**-$\mathcal{R}$ *given by* $(g_{\mathcal{A}}, 0)$*, where* $g_{\mathcal{A}}$ *is the morphism*

$$F_{\mathcal{B}} \overline{\otimes}_{\mathcal{B}} N \otimes_{\mathcal{C}} M^{\mathcal{C}}[-1] \xrightarrow{f_{\mathcal{B}} \otimes \mathrm{id}} G_{\mathcal{A}} \overline{\otimes}_{\mathcal{A}} M \otimes_{\mathcal{C}} N^{\mathcal{C}} \overline{\otimes}_{\mathcal{B}} N \otimes_{\mathcal{C}} M^{\mathcal{C}}[-1] \xrightarrow{\mathrm{id} \otimes (\sigma_M \circ (\mathrm{id} \otimes \mathrm{tr} \otimes \mathrm{id}))} G_{\mathcal{A}} \overline{\otimes}_{\mathcal{A}} C_M.$$

Proof. As F only has $\mathcal{B}$-component, the only remaining part of the square (3.18) is the bottom left. The morphism f is then sent via (3.18) to the morphism $g \colon F \otimes_{\mathcal{R}} \widetilde{\mathcal{R}} \otimes_{\mathcal{R}} C_P \to G \otimes_{\mathcal{R}} \widetilde{\mathcal{R}} \otimes_{\mathcal{R}} C_P$ whose only non-zero component is its $\mathcal{A}$-component $g_{\mathcal{A}}$. The morphism $g_{\mathcal{A}}$ is given by

$$F_{\mathcal{B}} \overline{\otimes}_{\mathcal{B}} N \otimes_{\mathcal{C}} M^{\mathcal{C}}[-1] \xrightarrow{(0, f_{\mathcal{B}} \otimes \mathrm{id})} G_{\mathcal{A}} \overline{\otimes}_{\mathcal{A}} \mathcal{A} \overline{\otimes}_{\mathcal{A}} C_M \oplus G_{\mathcal{A}} \overline{\otimes}_{\mathcal{A}} M \otimes_{\mathcal{C}} N^{\mathcal{C}} \overline{\otimes}_{\mathcal{B}} N \otimes_{\mathcal{C}} M^{\mathcal{C}}[-1]$$

Postcomposing $g_{\mathcal{A}}$ with the quasi-isomorphism of Lemma 3.3.4, $g_{\mathcal{A}}$ becomes the morphism

$$F_{\mathcal{B}} \overline{\otimes}_{\mathcal{B}} N \otimes_{\mathcal{C}} M^{\mathcal{C}}[-1] \xrightarrow{f_{\mathcal{B}} \otimes \mathrm{id}} G_{\mathcal{A}} \overline{\otimes}_{\mathcal{A}} M \otimes_{\mathcal{C}} N^{\mathcal{C}} \overline{\otimes}_{\mathcal{B}} N \otimes_{\mathcal{C}} M^{\mathcal{C}}[-1] \xrightarrow{\mathrm{id} \otimes (\sigma_M \circ (\mathrm{id} \otimes \mathrm{tr} \otimes \mathrm{id}))} G_{\mathcal{A}} \overline{\otimes}_{\mathcal{A}} C_M$$

which proves the claim of the lemma. □

The above series of lemmas tell us that the following diagrams commute[4]

$$
\begin{array}{ccc}
\mathrm{D}(\mathcal{A}) & \xrightarrow{C_{\alpha_M}} & \mathrm{D}(\mathcal{A}) \\
{\scriptstyle L\mathrm{Ind}_{i_\mathcal{A}}}\downarrow & & \downarrow{\scriptstyle \mathrm{Res}^R_{i_\mathcal{A}}} \\
\mathrm{D}(\mathcal{R}) & \xrightarrow{C_{\alpha_P}} & \mathrm{D}(\mathcal{R})
\end{array}
\qquad
\begin{array}{ccc}
\mathrm{D}(\mathcal{B}) & \xrightarrow{C_{\alpha_N}} & \mathrm{D}(\mathcal{B}) \\
{\scriptstyle L\mathrm{Ind}_{i_\mathcal{B}}}\downarrow & & \uparrow{\scriptstyle \mathrm{Res}_{i_\mathcal{B}}} \\
\mathrm{D}(\mathcal{R}) & \xrightarrow{C_{\alpha_P}} & \mathrm{D}(\mathcal{R})
\end{array}
\qquad
\begin{array}{ccc}
\mathrm{D}(\mathcal{B}) & \xrightarrow{-\overset{L}{\otimes}_\mathcal{B} N \overset{L}{\otimes}_\mathcal{C} M^{\tilde{\mathcal{C}}}[-1]} & \mathrm{D}(\mathcal{A}) \\
{\scriptstyle L\mathrm{Ind}_{i_\mathcal{B}}}\downarrow & & \uparrow{\scriptstyle \mathrm{Res}_{i_\mathcal{A}}} \\
\mathrm{D}(\mathcal{R}) & \xrightarrow{\qquad C_{\alpha_P} \qquad} & \mathrm{D}(\mathcal{R})
\end{array}
\tag{3.25}
$$

Remark 3.3.8. Combining the top left square in (3.25), Proposition 2.4.28, and the fact that C_{α_M} is an autoequivalence, we obtain the SOD

$$\mathrm{D}(\mathcal{R}) = \langle C_{\alpha_P}(L\mathrm{Ind}_{i_\mathcal{A}}(\mathrm{D}(\mathcal{A}))), L\mathrm{Ind}_{i_\mathcal{B}}(\mathrm{D}(\mathcal{B}))\rangle.$$

Therefore, for the spherical functor induced by the bimodule P the hypotheses of [HLS16, Theorem 4.14] are satisfied. However, let us remark that we cannot use the theorem from *ibidem* to prove Theorem 3.1.4 because in *ibidem* the authors assume that the functor is spherical to deduce that the twist around it factorises, whereas we use the description of the twist to prove that the functor is spherical.

We can now complete the proof of Theorem 3.1.4 by proving the following

Proposition 3.3.9. *The cotwist around the functor* (3.9) *is an autoequivalence of* $\mathrm{D}(\mathcal{R})$.

Proof. Let us first show that C_{α_P} is an autoequivalence if it is fully faithful. By Proposition 2.4.28, we have the SOD

$$\mathrm{D}(\mathcal{R}) = \langle \mathrm{Res}^R_{i_\mathcal{A}}(\mathrm{D}(\mathcal{A})), L\mathrm{Ind}_{i_\mathcal{B}}(\mathrm{D}(\mathcal{B}))\rangle.$$

Therefore, if C_{α_P} is fully faithful, to show that it is essentially surjective it is enough to show that its essential image $\mathrm{im}(C_{\alpha_P})$ contains the subcategories $\mathrm{Res}^R_{i_\mathcal{A}}(\mathrm{D}(\mathcal{A}))$ and $L\mathrm{Ind}_{i_\mathcal{B}}(\mathrm{D}(\mathcal{B}))$.

By (3.25) and the fact that C_{α_M} is an autoequivalence, we get $\mathrm{Res}^R_{i_\mathcal{A}}(\mathrm{D}(\mathcal{A})) \subset \mathrm{im}(C_{\alpha_P})$. Now take $G_\mathcal{B} \in \mathrm{D}(\mathcal{B})$. By (3.27) below and the fact that C_{α_N} is an autoequivalence, we know that there exists $F \in \mathrm{im}(C_{\alpha_P})$ such that $F_\mathcal{B} \simeq G_\mathcal{B}$ in $\mathrm{D}(\mathcal{B})$. Decomposing F with respect to the SOD of Proposition 2.4.28 and using the description of the projection functors given in § 2.4.9, we get

$$L\mathrm{Ind}_{i_\mathcal{B}}(G_\mathcal{B}) \simeq \mathrm{cone}(F \to \mathrm{Res}^R_{i_\mathcal{A}}(F_\mathcal{A}))[-1].$$

[4]Recall that M is $\mathcal{C}$-h-projective and $\mathcal{C}$-perfect, thus $\mathrm{RHom}_\mathcal{C}(M, -) \simeq - \overset{L}{\otimes}_\mathcal{C} M^{\tilde{\mathcal{C}}} \simeq - \otimes_\mathcal{C} M^\mathcal{C}$, and that we showed that $\mathrm{Res}_{i_\mathcal{A}}$ has a right adjoint in Lemma 2.4.27.

As $F, \mathrm{Res}^R_{i_\mathcal{A}}(F_\mathcal{A}) \in \mathrm{im}(C_{\alpha_P})$ and we are assuming that C_{α_P} is fully faithful, we get $L\mathrm{Ind}_{i_\mathcal{B}}(G_\mathcal{B}) \in \mathrm{im}(C_{\alpha_P})$.

Therefore, if C_{α_P} is fully faithful it is essentially surjective, and to prove that C_{α_P} is an equivalence it is enough to prove that it is fully faithful.

As by Proposition 2.4.25 we have the SOD $\mathrm{D}(\mathcal{R}) = \langle L\mathrm{Ind}_{i_\mathcal{B}}(\mathrm{D}(\mathcal{B})), L\mathrm{Ind}_{i_\mathcal{A}}(\mathrm{D}(\mathcal{A}))\rangle$, to show fully faithfulness of C_{α_P} it is enough to show that C_{α_P} is fully faithful on the objects of $L\mathrm{Ind}_{i_\mathcal{B}}(\mathrm{D}(\mathcal{B}))$ and $L\mathrm{Ind}_{i_\mathcal{A}}(\mathrm{D}(\mathcal{A}))$.

Let us fix $G_\mathcal{A} \in \mathrm{D}(\mathcal{A})$ and $F_\mathcal{B} \in \mathrm{D}(\mathcal{B})$. We now describe the action of C_{α_P} on the modules of the form

$$F = L\mathrm{Ind}_{i_\mathcal{B}}(F_\mathcal{B}) \overset{(2.26)}{\simeq} \begin{pmatrix} 0 & F_\mathcal{B} \overline{\otimes}_\mathcal{B} \mathcal{B} \end{pmatrix} \simeq \begin{pmatrix} 0 & F_\mathcal{B} \end{pmatrix}$$

and

$$G = L\mathrm{Ind}_{i_\mathcal{A}}(G_\mathcal{A}) \overset{(2.26)}{\simeq} \begin{pmatrix} G_\mathcal{A} \overline{\otimes}_\mathcal{A} \mathcal{A} & G_\mathcal{A} \overline{\otimes}_\mathcal{A} M \otimes_\mathcal{C} N^\mathcal{C} \end{pmatrix}$$

where the structure morphism of the latter is given by the identity, and the isomorphisms are in $\mathrm{D}(\mathcal{R})$.

By Lemma 3.3.4, we know that

$$C_{\alpha_P}(G) \simeq \begin{pmatrix} G_\mathcal{A} \overline{\otimes}_\mathcal{A} C_M & 0 \end{pmatrix} \in \mathrm{D}(\mathcal{R}). \tag{3.26}$$

Similarly, by Lemma 3.3.5 we know that

$$C_{\alpha_P}(F) \simeq \begin{pmatrix} F_\mathcal{B} \overline{\otimes}_\mathcal{B} N \otimes_\mathcal{C} M^\mathcal{C}[-1] & F_\mathcal{B} \overline{\otimes}_\mathcal{B} C_N \end{pmatrix} \in \mathrm{D}(\mathcal{R}) \tag{3.27}$$

with structure morphism

$$F_\mathcal{B} \otimes_\mathcal{B} \overline{\mathcal{B}} \otimes_\mathcal{B} N \otimes_\mathcal{C} M^\mathcal{C}[-1] \otimes_\mathcal{A} M \otimes_\mathcal{C} N^\mathcal{C} \xrightarrow{(\mathrm{id}^{\otimes 2} \otimes \sigma_N) \circ (\mathrm{id}^{\otimes 3} \otimes \mathrm{tr} \otimes \mathrm{id})} F_\mathcal{B} \otimes_\mathcal{B} \overline{\mathcal{B}} \otimes_\mathcal{B} C_N. \tag{3.28}$$

For the rest of the proof, when we talk about the modules F and G we refer to the ones above.

We are now ready to prove that C_{α_P} is fully faithful. We will deal with four different cases, and at the beginning of each new case we put an header in bold text specifying which case we are dealing with.

Case 1: $L\mathrm{Ind}_{i_\mathcal{A}}(\mathrm{D}(\mathcal{A})) \to L\mathrm{Ind}_{i_\mathcal{A}}(\mathrm{D}(\mathcal{A}))$

That C_{α_P} is fully faithful on morphisms from objects of $L\mathrm{Ind}_{i_\mathcal{A}}(\mathrm{D}(\mathcal{A}))$ to objects of $L\mathrm{Ind}_{i_\mathcal{A}}(\mathrm{D}(\mathcal{A}))$ follows from the left square in (3.25) because C_{α_M} is an equivalence and the vertical functors are fully faithful by Proposition 2.4.25 and Proposition 2.4.28, respectively.

Case 2: $L\mathrm{Ind}_{i_\mathcal{B}}(\mathrm{D}(\mathcal{B})) \to L\mathrm{Ind}_{i_\mathcal{A}}(\mathrm{D}(\mathcal{A}))$

We now consider morphisms from objects of $L\mathrm{Ind}_{i_{\mathcal{B}}}(\mathrm{D}(\mathcal{B}))$ to objects of $L\mathrm{Ind}_{i_{\mathcal{A}}}(\mathrm{D}(\mathcal{A}))$. What we want to prove is that the morphism

$$C_{\alpha_P}\colon \operatorname{Hom}_{\mathrm{D}(\mathcal{R})}(F, G) \to \operatorname{Hom}_{\mathrm{D}(\mathcal{R})}(C_{\alpha_P}(F), C_{\alpha_P}(G)) \tag{3.29}$$

is an isomorphism.

Let us fix a dg-module lifting the isomorphism class of $G_{\mathcal{A}}$ in $\mathrm{D}(\mathcal{A})$ and an h-projective dg-module lifting the isomorphism class of $F_{\mathcal{B}}$ in $\mathrm{D}(\mathcal{B})$, respectively. We will abuse notation and still write $G_{\mathcal{A}}$ and $F_{\mathcal{B}}$ for them.

Then, $G = \begin{pmatrix} G_{\mathcal{A}} \overline{\otimes}_{\mathcal{A}} \mathcal{A} & G_{\mathcal{A}} \overline{\otimes}_{\mathcal{A}} M \otimes_{\mathcal{C}} N^{\mathcal{C}} \end{pmatrix}$ is an h-projective $\mathcal{R}$-dg-module because it is of the form $G = G_{\mathcal{A}} \overline{\otimes}_{\mathcal{A}} \mathcal{R}$, and $F = \begin{pmatrix} 0 & F_{\mathcal{B}} \end{pmatrix}$ is an h-projective $\mathcal{R}$-dg-module because

$$\operatorname{Hom}_{\mathcal{R}}(F, S) = \operatorname{Hom}_{\mathcal{B}}(F_{\mathcal{B}}, S_{\mathcal{B}})$$

for any $\mathcal{R}$-dg-module S.[5]

Hence

$$\operatorname{Hom}_{\mathrm{D}(\mathcal{R})}(F, G) = \mathrm{H}^0(\operatorname{Hom}_{\mathcal{R}}(F, G)) = \mathrm{H}^0(\operatorname{Hom}_{\mathcal{B}}(F_{\mathcal{B}}, G_{\mathcal{A}} \overline{\otimes}_{\mathcal{A}} M \otimes_{\mathcal{C}} N^{\mathcal{C}}))$$

and thus, to prove that (3.29) is an isomorphism, it is enough to prove that

$$C_{\alpha_P}\colon \mathrm{H}^0(\operatorname{Hom}_{\mathcal{B}}(F_{\mathcal{B}}, G_{\mathcal{A}} \overline{\otimes}_{\mathcal{A}} M \otimes_{\mathcal{C}} N^{\mathcal{C}})) \to \operatorname{Hom}_{\mathrm{D}(\mathcal{R})}(C_{\alpha_P}(F), C_{\alpha_P}(G)) \tag{3.30}$$

is one.

Let us consider the dg-lift of C_{α_P} given by (3.17). Then, by Lemma 3.3.4, Lemma 3.3.5 and Proposition 2.4.14, we have the isomorphisms

$$\begin{aligned} \operatorname{Hom}_{\mathrm{D}(\mathcal{R})}(C_{\alpha_P}(F), C_{\alpha_P}(G)) &\simeq \operatorname{Hom}_{\mathrm{D}(\mathcal{R})}(C_{\alpha_P}(F), \operatorname{Res}^{\mathcal{R}}_{\mathcal{A}}(G_{\mathcal{A}} \overline{\otimes}_{\mathcal{A}} C_M)) \\ &\simeq \operatorname{Hom}_{\mathrm{D}(\mathcal{A})}(F_{\mathcal{B}} \overline{\otimes}_{\mathcal{B}} N \otimes_{\mathcal{C}} M^{\mathcal{C}}[-1], G_{\mathcal{A}} \overline{\otimes}_{\mathcal{A}} C_M), \end{aligned}$$

and therefore the morphism (3.30) takes the form

$$\mathrm{H}^0(\operatorname{Hom}_{\mathcal{B}}(F_{\mathcal{B}}, G_{\mathcal{A}} \overline{\otimes}_{\mathcal{A}} M \otimes_{\mathcal{C}} N^{\mathcal{C}})) \to \operatorname{Hom}_{\mathrm{D}(\mathcal{A})}(F_{\mathcal{B}} \overline{\otimes}_{\mathcal{B}} N \otimes_{\mathcal{C}} M^{\mathcal{C}}[-1], G_{\mathcal{A}} \overline{\otimes}_{\mathcal{A}} C_M) \tag{3.31}$$

Thus, we are left to prove that (3.31) is an isomorphism.

We have good control on (3.31) because by Lemma 3.3.7 we know that it sends

[5] We are implicitly using that acyclicity is defined fibrewise, and therefore $S \in \mathbf{Mod}\text{-}\mathcal{R}$ is acyclic if and only if its components are.

$f_{\mathcal{B}} \colon F_{\mathcal{B}} \to G_{\mathcal{A}} \overline{\otimes}_{\mathcal{A}} M \otimes_{\mathcal{C}} N^{\mathcal{C}}$ to

$$F_{\mathcal{B}} \overline{\otimes}_{\mathcal{B}} N \otimes_{\mathcal{C}} M^{\mathcal{C}}[-1] \xrightarrow{g_{\mathcal{B}} \otimes \mathrm{id}} G_{\mathcal{A}} \overline{\otimes}_{\mathcal{A}} M \otimes_{\mathcal{C}} N^{\mathcal{C}} \overline{\otimes}_{\mathcal{B}} N \otimes_{\mathcal{C}} M^{\mathcal{C}}[-1] \xrightarrow{\mathrm{id} \otimes (\sigma_M \circ (\mathrm{id} \otimes \mathrm{tr} \otimes \mathrm{id}))} G_{\mathcal{A}} \overline{\otimes}_{\mathcal{A}} C_M.$$

However, to see why this mapping is an isomorphism, let us rewrite it in the language of derived categories.

The module $F_{\mathcal{B}}$ is h-projective, hence we have an isomorphism

$$\mathrm{H}^0(\mathrm{Hom}_{\mathcal{B}}(F_{\mathcal{B}}, G_{\mathcal{A}} \overline{\otimes}_{\mathcal{A}} M \otimes_{\mathcal{C}} N^{\mathcal{C}})) \simeq \mathrm{Hom}_{\mathrm{D}(\mathcal{B})}(F_{\mathcal{B}}, G_{\mathcal{A}} \overline{\otimes}_{\mathcal{A}} M \otimes_{\mathcal{C}} N^{\mathcal{C}}).$$

Moreover, in every tensor product appearing in the above displayed equations at least one of the two bimodules being tensored is h-projective, hence every tensor product is derived. This means that we can rewrite[6] (3.31) as the morphism

$$\mathrm{Hom}_{\mathrm{D}(\mathcal{B})}(F_{\mathcal{B}}, G_{\mathcal{A}} \overset{L}{\otimes}_{\mathcal{A}} M \overset{L}{\otimes}_{\mathcal{C}} N^{\widetilde{\mathcal{C}}}) \to \mathrm{Hom}_{\mathrm{D}(\mathcal{A})}(F_{\mathcal{B}} \overset{L}{\otimes}_{\mathcal{B}} N \overset{L}{\otimes}_{\mathcal{C}} M^{\widetilde{\mathcal{C}}}, G_{\mathcal{A}} \overset{L}{\otimes}_{\mathcal{A}} C_M) \tag{3.32}$$

that sends $f_{\mathcal{B}} \colon F_{\mathcal{B}} \to G_{\mathcal{A}} \overset{L}{\otimes}_{\mathcal{A}} M \overset{L}{\otimes}_{\mathcal{C}} N^{\widetilde{\mathcal{C}}}$ to

$$g_{\mathcal{A}} \colon F_{\mathcal{B}} \overset{L}{\otimes}_{\mathcal{B}} N \overset{L}{\otimes}_{\mathcal{C}} M^{\widetilde{\mathcal{C}}}[-1] \xrightarrow{f_{\mathcal{B}} \otimes \mathrm{id}} G_{\mathcal{A}} \overset{L}{\otimes}_{\mathcal{A}} M \overset{L}{\otimes}_{\mathcal{C}} N^{\widetilde{\mathcal{C}}} \overset{L}{\otimes}_{\mathcal{B}} N \overset{L}{\otimes}_{\mathcal{C}} M^{\widetilde{\mathcal{C}}}[-1] \to \xrightarrow{\mathrm{id} \otimes (\sigma_M \circ (\mathrm{id} \otimes \mathrm{tr} \otimes \mathrm{id}))} G_{\mathcal{A}} \overset{L}{\otimes}_{\mathcal{A}} C_M. \tag{3.33}$$

Then, using Remark 2.5.5 we see that postcomposing (3.32) with the functor[7] $- \overset{L}{\otimes}_{\mathcal{A}} (C_M)^{\widetilde{\mathcal{A}}}$ we obtain the adjunction isomorphism[8]

$$\mathrm{Hom}_{\mathrm{D}(\mathcal{B})}(F_{\mathcal{B}}, G_{\mathcal{A}} \overset{L}{\otimes}_{\mathcal{A}} M \overset{L}{\otimes}_{\mathcal{C}} N^{\widetilde{\mathcal{C}}}) \simeq \mathrm{Hom}_{\mathrm{D}(\mathcal{A})}(F_{\mathcal{B}} \overset{L}{\otimes}_{\mathcal{B}} N \overset{L}{\otimes}_{\mathcal{C}} M^{\widetilde{\mathcal{A}}}, G_{\mathcal{A}}).$$

Thus, (3.32) is an isomorphism, as we wanted. Hence, C_{α_P} is fully faithful on morphisms from objects of $L\mathrm{Ind}_{i_{\mathcal{B}}}(\mathrm{D}(\mathcal{B}))$ to objects of $L\mathrm{Ind}_{i_{\mathcal{A}}}(\mathrm{D}(\mathcal{A}))$.

Case 3: $L\mathrm{Ind}_{i_{\mathcal{A}}}(\mathrm{D}(\mathcal{A})) \to L\mathrm{Ind}_{i_{\mathcal{B}}}(\mathrm{D}(\mathcal{B}))$

We now consider morphisms from objects of $L\mathrm{Ind}_{i_{\mathcal{A}}}(\mathrm{D}(\mathcal{A}))$ to objects of $L\mathrm{Ind}_{i_{\mathcal{B}}}(\mathrm{D}(\mathcal{B}))$. By Proposition 2.4.25 we know that there are no such morphisms in $\mathrm{D}(\mathcal{R})$. Thus, we have to prove that any morphism $C_{\alpha_P}(G) \to C_{\alpha_P}(F)$ in $\mathrm{D}(\mathcal{R})$ is isomorphic to zero. By[9] Theorem 2.4.19 and (3.26) and (3.27), it is enough to prove that any closed, degree zero

[6] Recall that both M and N are h-projective bimodules, hences their duals are their derived duals.
[7] The dual is taken as a right $\mathcal{A}$-module.
[8] Here we use that as C_{α_M} is an autoequivalence, $(C_M)^{\widetilde{\mathcal{A}}}$, which induces the right adjoint functor $C^R_{\alpha_M}$, induces the inverse functor $C^{-1}_{\alpha_M}$
[9] We put a reference to Theorem 2.4.19 only in this part of the proof, but we will use it implicitly throughout.

morphism

$$g\colon \left(G_{\mathcal{A}} \,\overline{\otimes}_{\mathcal{A}}\, C_M \quad 0 \right) \to \left(F_{\mathcal{B}} \,\overline{\otimes}_{\mathcal{B}}\, N \otimes_{\mathcal{C}} M^{\mathcal{C}}[-1] \quad F_{\mathcal{B}} \,\overline{\otimes}_{\mathcal{B}}\, C_N \right)$$

in $\overline{\mathbf{Mod}}\text{-}\mathcal{R}$ is homotopic to zero, and this is what we show.

Such a morphism g is given by a triple $(g_{\mathcal{A}}, 0, g_{\mathcal{AB}})$ where $g_{\mathcal{A}}\colon G_{\mathcal{A}} \,\overline{\otimes}_{\mathcal{A}}\, C_M \to F_{\mathcal{B}} \,\overline{\otimes}_{\mathcal{B}}\, N \otimes_{\mathcal{C}} M^{\mathcal{C}}[-1]$ and $g_{\mathcal{AB}}\colon G_{\mathcal{A}} \,\overline{\otimes}_{\mathcal{A}}\, C_M \,\overline{\otimes}_{\mathcal{A}}\, M \otimes_{\mathcal{C}} N^{\mathcal{C}} \to F_{\mathcal{B}} \,\overline{\otimes}_{\mathcal{B}}\, C_N$ are two morphisms in $\overline{\mathbf{Mod}}\text{-}\mathcal{A}$ and $\overline{\mathbf{Mod}}\text{-}\mathcal{B}$, respectively, such that (see (2.23) for the definition of the differential in $\overline{\mathbf{Mod}}\text{-}\mathcal{R}$) $d(g_{\mathcal{A}}) = 0$ and the diagram

$$\begin{array}{ccc} G_{\mathcal{A}} \,\overline{\otimes}_{\mathcal{A}}\, C_M \,\overline{\otimes}_{\mathcal{A}}\, M \otimes_{\mathcal{C}} N^{\mathcal{C}} & \longrightarrow & 0 \\ \big\downarrow{\scriptstyle g_{\mathcal{A}} \overline{\otimes}\, \mathrm{id}} & & \big\downarrow \\ F_{\mathcal{B}} \,\overline{\otimes}_{\mathcal{B}}\, N \otimes_{\mathcal{C}} M^{\mathcal{C}}[-1] \,\overline{\otimes}_{\mathcal{A}}\, M \otimes_{\mathcal{C}} N^{\mathcal{C}}[-1] & \xrightarrow{(3.28)} & F_{\mathcal{B}} \,\overline{\otimes}_{\mathcal{B}}\, C_N \end{array} \tag{3.34}$$

commutes in $\overline{\mathbf{Mod}}\text{-}\mathcal{B}$ up to the homotopy $g_{\mathcal{AB}}$, that is $d(g_{\mathcal{AB}}) + (3.28) \circ (g_{\mathcal{A}} \,\overline{\otimes}\, \mathrm{id}) = 0$.

Notice that the diagram (3.34) commutes on the nose in $\mathrm{D}(\mathcal{B})$. If we consider (3.34) as a diagram in $\mathrm{D}(\mathcal{B})$ and we tensor it with[10] $(C_{\alpha_N})^{\widetilde{\mathcal{B}}}$, then using that N is spherical, Remark 2.5.5, and the adjunctions $- \overset{L}{\otimes}_{\mathcal{A}} M \dashv - \overset{L}{\otimes}_{\mathcal{C}} M^{\widetilde{\mathcal{C}}}$ and $- \overset{L}{\otimes}_{\mathcal{C}} N^{\widetilde{\mathcal{B}}} \dashv - \overset{L}{\otimes}_{\mathcal{B}} N$, we obtain that the following diagram[11] commutes in $\mathrm{D}(\mathcal{A})$

$$\begin{array}{ccc} G_{\mathcal{A}} \overset{L}{\otimes}_{\mathcal{A}} C_M & \longrightarrow & 0 \\ \big\downarrow{\scriptstyle g_{\mathcal{A}}} & & \big\downarrow \\ F_{\mathcal{B}} \overset{L}{\otimes}_{\mathcal{B}} N \overset{L}{\otimes}_{\mathcal{C}} M^{\mathcal{C}}[-1] & \xrightarrow{\mathrm{id}} & F_{\mathcal{B}} \overset{L}{\otimes}_{\mathcal{B}} N \overset{L}{\otimes}_{\mathcal{C}} M^{\mathcal{C}}[-1] \end{array}$$

Thus, we see that (3.34) implies that $g_{\mathcal{A}} = 0$ in $\mathrm{D}(\mathcal{A})$, and therefore $g_{\mathcal{A}} = d(h_{\mathcal{A}})$ for some $h_{\mathcal{A}}\colon G_{\mathcal{A}} \,\overline{\otimes}_{\mathcal{A}}\, C_M \to F_{\mathcal{B}} \,\overline{\otimes}_{\mathcal{B}}\, N \otimes_{\mathcal{C}} M^{\mathcal{C}}[-1]$ in $\overline{\mathbf{Mod}}\text{-}\mathcal{A}$.

At this point, it is enough to prove that the morphism given by the triple

$$(g_{\mathcal{A}}, 0, g_{\mathcal{AB}}) - d((h_{\mathcal{A}}, 0, , 0)) = (0, 0, g_{\mathcal{AB}} + (3.28) \circ (h_{\mathcal{A}} \,\overline{\otimes}\, \mathrm{id}))$$

is homotopic to zero. It is enough to prove that any given triple $(0, 0, r_{\mathcal{AB}})$, where $r_{\mathcal{AB}}\colon : G_{\mathcal{A}} \,\overline{\otimes}_{\mathcal{A}}\, C_M \,\overline{\otimes}_{\mathcal{A}}\, M \otimes_{\mathcal{C}} N^{\mathcal{C}} \to F_{\mathcal{B}} \,\overline{\otimes}_{\mathcal{B}}\, C_N$ is a closed, degree -1 morphism in $\overline{\mathbf{Mod}}\text{-}\mathcal{B}$, is homotopic to zero in $\overline{\mathbf{Mod}}\text{-}\mathcal{R}$, and this is what we show.

[10] The dual is taken as a right $\mathcal{B}$-module.

[11] The reason why all the tensor products become derived when we pass to the derived category is that we are either considering bar tensor products, or tensor product where one of the two sides is given by M or N, which are h-projective.

As $r_{\mathcal{AB}}$ is a closed morphism, it can be interpreted as a morphism

$$r_{\mathcal{AB}}\colon G_{\mathcal{A}} \overset{L}{\otimes}_{\mathcal{A}} C_M \overset{L}{\otimes}_{\mathcal{A}} M \overset{L}{\otimes}_{\mathcal{C}} N^{\widetilde{\mathcal{C}}} \to F_{\mathcal{B}} \overset{L}{\otimes}_{\mathcal{B}} C_N[-1]$$

in $\mathrm{D}(\mathcal{B})$. Tensoring this morphism with $(C_{\alpha_N})^{\widetilde{\mathcal{B}}}$ and using that N is spherical, from $r_{\mathcal{AB}}$ we obtain a morphism

$$G_{\mathcal{A}} \overset{L}{\otimes}_{\mathcal{A}} C_M \overset{L}{\otimes}_{\mathcal{A}} M \overset{L}{\otimes}_{\mathcal{C}} N^{\widetilde{\mathcal{B}}}[1] \to F_{\mathcal{B}}[-1].$$

Then, using the adjunctions $- \overset{L}{\otimes}_{\mathcal{A}} M \dashv - \overset{L}{\otimes}_{\mathcal{C}} M^{\widetilde{\mathcal{C}}}$ and $- \overset{L}{\otimes}_{\mathcal{C}} N^{\widetilde{\mathcal{B}}} \dashv - \overset{L}{\otimes}_{\mathcal{B}} N$, we get a morphism

$$G_{\mathcal{A}} \overset{L}{\otimes}_{\mathcal{A}} C_M[1] \to F_{\mathcal{B}} \overset{L}{\otimes}_{\mathcal{B}} N \overset{L}{\otimes}_{\mathcal{C}} M^{\widetilde{\mathcal{C}}}[-1] \tag{3.35}$$

that can be lifted to a closed, degree zero morphism $s_{\mathcal{A}}\colon : G_{\mathcal{A}} \overline{\otimes}_{\mathcal{A}} C_M[1] \to F_{\mathcal{B}} \overline{\otimes}_{\mathcal{B}} N \otimes_{\mathcal{C}} M^{\mathcal{C}}[-1]$ in $\overline{\mathbf{Mod}}$-$\mathcal{A}$.

The relation between $s_{\mathcal{A}}$ and $r_{\mathcal{AB}}$ is easily understood thanks to Remark 2.5.5. Namely, as to pass from $r_{\mathcal{AB}}$ to $s_{\mathcal{A}}$ we used adjunction, it means that $r_{\mathcal{AB}} = (3.28) \circ (s_{\mathcal{A}} \overline{\otimes}\ \mathrm{id})$ in $\mathrm{D}(\mathcal{B})$. Hence, we know that there exists a morphism $s_{\mathcal{AB}}\colon : G_{\mathcal{A}} \overline{\otimes}_{\mathcal{A}} C_M \overline{\otimes}_{\mathcal{A}} M \otimes_{\mathcal{C}} N^{\mathcal{C}} \to F_{\mathcal{B}} \overline{\otimes}_{\mathcal{B}} C_N$ in $\overline{\mathbf{Mod}}$-$\mathcal{B}$ of degree -2 such that $d(s_{\mathcal{AB}}) = r_{\mathcal{AB}} - (3.28) \circ (s_{\mathcal{A}} \overline{\otimes}\ \mathrm{id})$.

Summing up, we proved that

$$(0, 0, r_{\mathcal{AB}}) = (0, 0, (3.28) \circ (s_{\mathcal{A}} \overline{\otimes}\ \mathrm{id}) + d(s_{\mathcal{AB}})) = d((-s_{\mathcal{A}}, 0, s_{\mathcal{AB}})),$$

and therefore $(0, 0, r_{\mathcal{AB}})$ is homotopic to zero in $\overline{\mathbf{Mod}}$-$\mathcal{R}$, as we wanted to show. Hence, C_{α_P} is fully faithful on morphisms from objects of $L\mathrm{Ind}_{i_{\mathcal{A}}}(\mathrm{D}(\mathcal{A}))$ to objects of $L\mathrm{Ind}_{i_{\mathcal{B}}}(\mathrm{D}(\mathcal{B}))$.

Case 4: $L\mathrm{Ind}_{i_{\mathcal{B}}}(\mathrm{D}(\mathcal{B})) \to L\mathrm{Ind}_{i_{\mathcal{B}}}(\mathrm{D}(\mathcal{B}))$

To conclude the proof of the proposition, we now consider morphism from objects of $L\mathrm{Ind}_{i_{\mathcal{B}}}(\mathrm{D}(\mathcal{B}))$ to objects of $L\mathrm{Ind}_{i_{\mathcal{B}}}(\mathrm{D}(\mathcal{B}))$. Let us take $F'_{\mathcal{B}} \in \mathrm{D}(\mathcal{B})$ and consider $F' = L\mathrm{Ind}_{i_{\mathcal{B}}}(F'_{\mathcal{B}})$.

We fix two lifts of $F_{\mathcal{B}}$ and $F'_{\mathcal{B}}$ to h-projective $\mathcal{B}$-dg-modules, and we keep denoting them by $F_{\mathcal{B}}$ and $F'_{\mathcal{B}}$. Then, a morphism $f\colon F \to F'$ in $\mathrm{D}(\mathcal{B})$ is given by a triple $(0, f_{\mathcal{B}}, 0)$, where $f_{\mathcal{B}}\colon F_{\mathcal{B}} \to F'_{\mathcal{B}}$ is a closed, degree 0 morphism in $\mathbf{Mod}$-$\mathcal{B}$. By Lemma 3.3.5, we know that $f_{\mathcal{B}}$ is sent to the morphism

$$\begin{pmatrix} F_{\mathcal{B}} \overline{\otimes}_{\mathcal{B}} N \otimes_{\mathcal{C}} M^{\mathcal{C}}[-1] & F_{\mathcal{B}} \overline{\otimes}_{\mathcal{B}} C_N \end{pmatrix} \to \\ \xrightarrow{(f_{\mathcal{B}} \otimes \mathrm{id}, f_{\mathcal{B}} \otimes \mathrm{id}, 0)} \begin{pmatrix} F'_{\mathcal{B}} \overline{\otimes}_{\mathcal{B}} N \otimes_{\mathcal{C}} M^{\mathcal{C}}[-1] & F'_{\mathcal{B}} \overline{\otimes}_{\mathcal{B}} C_N \end{pmatrix}. \tag{3.36}$$

It is clear that the morphism $f \mapsto$ (3.36) is injective when considered as a morphism

$$C_{\alpha_P} \colon \operatorname{Hom}_{\mathrm{D}(\mathcal{B})}(F_{\mathcal{B}}, F'_{\mathcal{B}}) \simeq \operatorname{Hom}_{\mathrm{D}(\mathcal{R})}(F, F') \to \operatorname{Hom}_{\mathrm{D}(\mathcal{R})}(C_{\alpha_P}(F), C_{\alpha_P}(F')).$$

Indeed, if $f_{\mathcal{B}}, g_{\mathcal{B}} \colon F_{\mathcal{B}} \to F'_{\mathcal{B}}$ are two morphisms in $\textbf{Mod-}\mathcal{B}$ with the same image, then $f_{\mathcal{B}} \otimes \mathrm{id}$ and $g_{\mathcal{B}} \otimes \mathrm{id}$ are equal in $\textbf{Mod-}\mathcal{B}$ as morphisms from $F_{\mathcal{B}} \overline{\otimes}_{\mathcal{B}} C_N$ to $F'_{\mathcal{B}} \overline{\otimes}_{\mathcal{B}} C_N$. Hence, they are equal in $\mathrm{D}(\mathcal{B})$. However, C_N induces an autoequivalence of $\mathrm{D}(\mathcal{B})$, and therefore $f_{\mathcal{B}}$ and $g_{\mathcal{B}}$ are equal as morphisms in $\mathrm{D}(\mathcal{B})$.

Thus, to conclude that C_{α_P} is fully faithful on morphisms from objects of $L\mathrm{Ind}_{i_{\mathcal{B}}}(\mathrm{D}(\mathcal{B}))$ to objects of $L\mathrm{Ind}_{i_{\mathcal{B}}}(\mathrm{D}(\mathcal{B}))$ we only have to prove that the morphism $f \mapsto$ (3.36) is surjective up to quasi-isomorphism. Instead of proving this statement, we will prove the equivalent statement that $f \mapsto \Upsilon($(3.36)$)$ is surjective up to homotopy, where Υ is the functor of Theorem 2.4.19.

Recall that by Remark 3.3.6 the morphism $\Upsilon($(3.36)$)$ is equal to

$$\begin{pmatrix} F_{\mathcal{B}} \overline{\otimes}_{\mathcal{B}} N \otimes_{\mathcal{C}} M^{\mathcal{C}}[-1] & F_{\mathcal{B}} \overline{\otimes}_{\mathcal{B}} C_N \end{pmatrix} \to \\ \xrightarrow{(f_{\mathcal{B}} \overline{\otimes}\ \mathrm{id}, f_{\mathcal{B}} \overline{\otimes}\ \mathrm{id}, 0)} \begin{pmatrix} F'_{\mathcal{B}} \overline{\otimes}_{\mathcal{B}} N \otimes_{\mathcal{C}} M^{\mathcal{C}}[-1] & F'_{\mathcal{B}} \overline{\otimes}_{\mathcal{B}} C_N \end{pmatrix} \tag{3.37}$$

Let us take a closed, degree 0 morphism $g \colon C_{\alpha_P}(F) \to C_{\alpha_P}(F')$. Using (3.27), we see that such g is given by a triple of morphisms $(g_{\mathcal{A}}, g_{\mathcal{B}}, g_{\mathcal{AB}})$ such that $d(g_{\mathcal{A}}) = d(g_{\mathcal{B}}) = 0$ and

$$d(g_{\mathcal{AB}}) = g_{\mathcal{B}} \circ (3.28) - (3.28) \circ (g_{\mathcal{A}} \overline{\otimes}\ \mathrm{id})$$

in $\overline{\textbf{Mod}}\textbf{-}\mathcal{B}$. These relations imply that the following diagram commutes in $\mathrm{D}(\mathcal{B})$

$$\begin{array}{ccc} F_{\mathcal{B}} \overset{L}{\otimes}_{\mathcal{B}} N \overset{L}{\otimes}_{\mathcal{C}} M^{\widetilde{\mathcal{C}}}[-1] \overset{L}{\otimes}_{\mathcal{A}} M \overset{L}{\otimes}_{\mathcal{C}} N^{\widetilde{\mathcal{C}}} & \xrightarrow{(3.28)} & F_{\mathcal{B}} \overset{L}{\otimes}_{\mathcal{B}} C_N \\ \downarrow{\scriptstyle g_{\mathcal{A}} \otimes \mathrm{id}} & & \downarrow{\scriptstyle g_{\mathcal{B}}} \\ F'_{\mathcal{B}} \overset{L}{\otimes}_{\mathcal{B}} N \overset{L}{\otimes}_{\mathcal{C}} M^{\widetilde{\mathcal{C}}}[-1] \overset{L}{\otimes}_{\mathcal{A}} M \overset{L}{\otimes}_{\mathcal{C}} N^{\widetilde{\mathcal{C}}} & \xrightarrow{(3.28)} & F_{\mathcal{B}} \overset{L}{\otimes}_{\mathcal{B}} C_N \end{array} \tag{3.38}$$

Let us write $f_{\mathcal{B}} \colon F_{\mathcal{B}} \to F'_{\mathcal{B}}$ for a lift to $\overline{\textbf{Mod}}\textbf{-}\mathcal{B}$ of the morphism $C_{\alpha_N}^{-1}(g_{\mathcal{B}})$ in $\mathrm{D}(\mathcal{B})$. Then, tensoring the diagram (3.38) with $(C_{\alpha_N})^{\widetilde{\mathcal{B}}}$, using that N is spherical, Remark 2.5.5, and the adjuctions $- \overset{L}{\otimes}_{\mathcal{A}} M \dashv - \overset{L}{\otimes}_{\mathcal{C}} M^{\widetilde{\mathcal{C}}}$ and $- \overset{L}{\otimes}_{\mathcal{C}} N^{\widetilde{\mathcal{B}}} \dashv - \overset{L}{\otimes}_{\mathcal{B}} N$, we get that the following diagram commutes in $\mathrm{D}(\mathcal{B})$

$$\begin{array}{ccc} F_{\mathcal{B}} \overset{L}{\otimes}_{\mathcal{B}} N \overset{L}{\otimes}_{\mathcal{C}} M^{\widetilde{\mathcal{C}}}[-1] & \xrightarrow{\mathrm{id}} & F_{\mathcal{B}} \overset{L}{\otimes}_{\mathcal{B}} N \overset{L}{\otimes}_{\mathcal{C}} M^{\widetilde{\mathcal{C}}}[-1] \\ \downarrow{\scriptstyle g_{\mathcal{A}}} & & \downarrow{\scriptstyle f_{\mathcal{B}}\otimes \mathrm{id}} \\ F'_{\mathcal{B}} \overset{L}{\otimes}_{\mathcal{B}} N \overset{L}{\otimes}_{\mathcal{C}} M^{\widetilde{\mathcal{C}}}[-1] & \xrightarrow{\mathrm{id}} & F'_{\mathcal{B}} \overset{L}{\otimes}_{\mathcal{B}} N \overset{L}{\otimes}_{\mathcal{C}} M^{\widetilde{\mathcal{C}}}[-1] \end{array} \tag{3.39}$$

The diagram (3.39) implies that $g_{\mathcal{A}} = f_{\mathcal{B}} \otimes \mathrm{id}$ in $\mathrm{D}(\mathcal{B})$, and therefore that

$$g_{\mathcal{A}} - f_{\mathcal{B}} \overline{\otimes}\ \mathrm{id} = d(h_{\mathcal{A}})$$

for some morphism $h_{\mathcal{A}} \colon F_{\mathcal{B}} \overline{\otimes}_{\mathcal{B}} N \otimes_{\mathcal{C}} M^{\mathcal{C}}[-1] \to F'_{\mathcal{B}} \overline{\otimes}_{\mathcal{B}} N \otimes_{\mathcal{C}} M^{\mathcal{C}}[-1]$ in $\overline{\mathbf{Mod}}\text{-}\mathcal{A}$.

As by the choice of $f_{\mathcal{B}}$ we know that $g_{\mathcal{B}} - f_{\mathcal{B}} \overline{\otimes}\ \mathrm{id} = d(h_{\mathcal{B}})$ for some morphism $h_{\mathcal{B}} \colon F_{\mathcal{B}} \overline{\otimes}\, C_N \to F'_{\mathcal{B}} \overline{\otimes}\, C_N$ in $\overline{\mathbf{Mod}}\text{-}\mathcal{B}$, we get

$$(g_{\mathcal{A}}, g_{\mathcal{B}}, g_{\mathcal{AB}}) - d((h_{\mathcal{A}}, h_{\mathcal{B}}, 0)) = (f_{\mathcal{B}} \overline{\otimes}\ \mathrm{id}, f_{\mathcal{B}} \overline{\otimes}\ \mathrm{id}, g_{\mathcal{AB}} - (h_{\mathcal{B}} \circ (3.28) - (3.28) \circ (h_{\mathcal{A}} \overline{\otimes}\ \mathrm{id})))$$

and to prove that g is homotopic to an element of the form (3.37) it is enough to prove that any triple $(0, 0, r_{\mathcal{AB}})$, where $r_{\mathcal{AB}} \colon F_{\mathcal{B}} \overline{\otimes}_{\mathcal{B}} N \otimes_{\mathcal{C}} M^{\mathcal{C}}[-1] \overline{\otimes}_{\mathcal{A}} M \otimes_{\mathcal{C}} N^{\mathcal{C}} \to F'_{\mathcal{B}} \overline{\otimes}_{\mathcal{B}} C_N$ is a closed, degree -1 morphism in $\overline{\mathbf{Mod}}\text{-}\mathcal{B}$, is homotopic to zero in $\overline{\mathbf{Mod}}\text{-}\mathcal{R}$.

This last claim is proved with a strategy similar to the one we employed in the previous step of the proof. Namely, as $r_{\mathcal{AB}}$ is a closed morphism, we can interpet it as a morphism

$$r_{\mathcal{AB}} \colon F_{\mathcal{B}} \overset{L}{\otimes}_{\mathcal{B}} N \overset{L}{\otimes}_{\mathcal{C}} M^{\widetilde{\mathcal{C}}}[-1] \overset{L}{\otimes}_{\mathcal{A}} M \overset{L}{\otimes}_{\mathcal{C}} N^{\widetilde{\mathcal{C}}} \to F'_{\mathcal{B}} \overset{L}{\otimes}_{\mathcal{B}} C_N[-1]$$

in $\mathrm{D}(\mathcal{B})$. Then, using that C_N is spherical, Remark 2.5.5, and the adjunctions $- \overset{L}{\otimes}_{\mathcal{C}} N^{\widetilde{\mathcal{B}}} \dashv - \overset{L}{\otimes}_{\mathcal{B}} N$ and $- \overset{L}{\otimes}_{\mathcal{A}} M \dashv - \overset{L}{\otimes}_{\mathcal{C}} M^{\widetilde{\mathcal{C}}}$, we get a morphism

$$F_{\mathcal{B}} \overset{L}{\otimes}_{\mathcal{B}} N \overset{L}{\otimes}_{\mathcal{C}} M^{\widetilde{\mathcal{C}}} \to F'_{\mathcal{B}} \overset{L}{\otimes}_{\mathcal{B}} N \overset{L}{\otimes}_{\mathcal{C}} M^{\widetilde{\mathcal{C}}}[-1]$$

that we can lift to a closed, degree -1 morphism $s_{\mathcal{A}} \colon F_{\mathcal{B}} \overline{\otimes}_{\mathcal{B}} N \otimes_{\mathcal{C}} M^{\mathcal{C}} \to F'_{\mathcal{B}} \overline{\otimes}_{\mathcal{B}} N \otimes_{\mathcal{C}} M^{\mathcal{C}}$ in $\overline{\mathbf{Mod}}\text{-}\mathcal{A}$ with the property that $r_{\mathcal{AB}} - (3.28) \circ s_{\mathcal{A}} = d(s_{\mathcal{AB}})$ for some $s_{\mathcal{AB}} \colon F_{\mathcal{B}} \overline{\otimes}_{\mathcal{B}} N \otimes_{\mathcal{C}} M^{\mathcal{C}}[-1] \overline{\otimes}_{\mathcal{A}} M \otimes_{\mathcal{C}} N^{\mathcal{C}} \to F'_{\mathcal{B}} \overline{\otimes}_{\mathcal{B}} C_N$.

Therefore, we get

$$(0, 0, r_{\mathcal{AB}}) = (0, 0, d(s_{\mathcal{AB}}) + (3.28) \circ s_{\mathcal{A}}) = d((-s_{\mathcal{A}}, 0, s_{\mathcal{AB}})),$$

that is $(0, 0, r_{\mathcal{AB}})$ is homotopic to zero in $\overline{\mathbf{Mod}}\text{-}\mathcal{R}$, as we wanted. Hence, we proved that C_{α_P} is fully faithful on morphisms from objects of $L\mathrm{Ind}_{i_{\mathcal{B}}}(\mathrm{D}(\mathcal{B}))$ to objects of

$L\mathrm{Ind}_{i_{\mathcal{B}}}(\mathrm{D}(\mathcal{B}))$, and the proof of the proposition is complete.

□

3.4 Spherical objects

Now that we have completed the proof of Theorem 3.1.4, we survey some examples of glued spherical functors, and we show that these objects arise naturally in geometric situations. In this subsection, we consider the example of spherical objects.

Recall that spherical objects were defined by Seidel and Thomas in [ST01] as objects whose endomorphism algebra is isomorphic, as a graded algebra, to the cohomology algebra of a sphere. In the treatment we gave in § 1 we restricted ourselves to the case of Calabi–Yau varieties, thus hiding one more condition that is needed in the definition of a spherical object. Namely, spherical objects need to be invariant under the Serre functor.

Let $\mathcal{C}$ be a small, proper dg-category, *i.e.*, the complex $\mathrm{Hom}_{\mathcal{C}}(c_1, c_2)$ is cohomologically bounded and has finite dimensional cohomology for any $c_1, c_2 \in \mathcal{C}$. Notice that every module $E \in \mathrm{D}(\mathcal{C})$ can be considered as a $\star_k$-$\mathcal{C}$-bimodule where $\star_k$ is the dg-category with a single object such that $\mathrm{Hom}_{\star_k}(\star_k, \star_k) = k$; here k sits in degree 0. For such a bimodule being $\star_k$-perfect means that, for every $c \in \mathcal{C}$, the complex E_c is cohomologically bounded and has finite dimensional cohomology.

For the rest of this section we will assume that the category $\mathrm{D}(\mathcal{C})^{\mathrm{c}}$ has a Serre functor $\mathbb{S}$ which is given by tensor product with a bimodule. We fix $S_{\mathcal{C}} \in \mathrm{D}(\mathcal{C}\text{-}\mathcal{C})$ such that $\mathbb{S}(-) = - \overset{L}{\otimes}_{\mathcal{C}} S_{\mathcal{C}}$.

Definition 3.4.1. Let $E \in \mathrm{D}(\mathcal{C})$. We say that E is a *d-spherical object* if the following three conditions are satisfied:

1. E is both $\star_k$- and $\mathcal{C}$-perfect
2. $\mathrm{Hom}^{\bullet}_{\mathrm{D}(\mathcal{C})}(E, E) \simeq k \oplus k[-d]$ as graded vector spaces
3. $E \overset{L}{\otimes}_{\mathcal{C}} S_{\mathcal{C}} \simeq E[d]$ in $\mathrm{D}(\mathcal{C})$

Remark 3.4.2. Notice that this notion is slightly more general than the one in [ST01] because it allows d to be negative and zero. This generalisation has also been considered in [HKP16].

Notice that when $d > 0$ there exists a unique structure of graded algebra on the graded vector space $k \oplus k[-d]$, and thus we obtain that for a d-spherical object $E \in \mathrm{D}(\mathcal{C})$ we have

$$\mathrm{Hom}^{\bullet}_{\mathrm{D}(\mathcal{C})}(E, E) \simeq \mathrm{H}^{\bullet}(S^d, k).$$

Remark 3.4.3. The reader might be wondering how does the notion of a spherical object tie up with the notion of a spherical functor. The answer is that giving a spherical object in $\mathrm{D}(\mathcal{C})$ is equivalent to give a conservative spherical functor $\Psi\colon \mathrm{D}(\star_k) \to \mathrm{D}(\mathcal{C})$, *i.e.*, a spherical functor such that $\ker \Psi = 0$.

It is clear that giving an object in $\mathrm{D}(\mathcal{C})$ is the same thing as giving a functor $\Psi\colon \mathrm{D}(\star_k) \to \mathrm{D}(\mathcal{C})$. However, it is not obvious that Ψ is spherical if and only if $\Psi(k)$ is a spherical object in the sense of Definition 3.4.1. We now show how to prove the equivalence of the two statements.

One direction of the equivalence is proved by Theorem 3.4.6. The converse, namely that any conservative spherical functor $\Psi\colon \mathrm{D}(\star_k) \to \mathrm{D}(\mathcal{C})$ corresponds to a unique spherical object in $\mathrm{D}(\mathcal{C})$, can be easily proved as follows.

Let $E = \Psi(k) \in \mathrm{D}(\mathcal{C})$ be the image of k considered as a $\star_k$-right-module, which is non-zero because Ψ is conservative. Then, as $\mathrm{Aut}(\mathrm{D}(k)) = \mathbb{Z}$ is generated by the shift [1], we must have $C_\Psi = [-d-1]$, and therefore (2) holds. Indeed, $\mathrm{Hom}^\bullet_{\mathrm{D}(\mathcal{C})}(E,E)$ sits in the distinguished triangle

$$k \to \mathrm{Hom}^\bullet_{\mathrm{D}(\mathcal{C})}(E,E) \to C_\Psi(k)[1] \simeq k[-d],$$

and thus $\mathrm{Hom}^\bullet_{\mathrm{D}(\mathcal{C})}(E,E) \simeq k \oplus k[-d]$ as graded vector spaces.

Now, as $\mathbb{S}\Psi = \Psi^{RR} = \Psi C_\Psi^{-1}[-1]$, we get $\mathbb{S}\Psi(k) = \Psi^{RR}(k) = \Psi(k)[d]$, proving that $E = \Psi(k)$ is invariant under the Serre functor, and the proof is complete.

Remark 3.4.4. The reason why we require $\mathrm{D}(\mathcal{C})^{\mathrm{c}}$ to have a Serre functor is to simplify Definition 3.4.1. If we did not have a Serre functor, instead of (3) above we would have to require the existence of an isomorphism $E^{\widetilde{\mathcal{C}}} \simeq E^{\widetilde{\star_k}}[-d]$ in $\mathrm{D}(\mathcal{C})$. While we could get by with this for the proof of Theorem 3.4.6, we would run into functoriality issues in the proof of Theorem 3.4.11.

Remark 3.4.5. If $\mathrm{D}(\mathcal{C}) \simeq \mathrm{D}_{\mathrm{qc}}(X)$ for some smooth, projective variety X of dimension d, then an object $E \in \mathrm{D}(\mathcal{C})$ is d-spherical if and only if its image in $\mathrm{D}_{\mathrm{qc}}(X)$ is spherical according to the standard definition given in [ST01].

The following is well known.

Theorem 3.4.6 ([ST01])**.** *For a d-spherical object E the functor $- \overset{L}{\otimes}_{\star_k} E\colon \mathrm{D}(\star_k) \to \mathrm{D}(\mathcal{C})$ is spherical.*

Let us now consider d-spherical objects $E_1, \dots, E_n$ and assume we replaced them with h-projective resolutions. To ease the notation, we write T_{E_i} for the spherical twist around the spherical functor

$$- \overset{L}{\otimes}_{\star_k} E_i\colon \mathrm{D}(\star_k) \to \mathrm{D}(\mathcal{C})$$

of Theorem 3.4.6. Notice that this is not ambiguous by Remark 3.4.3.

Applying Theorem 3.1.4 inductively we obtain that the autoequivalence $T_{E_n} \dots T_{E_2} T_{E_1}$ can be realised as the twist around the functor

$$\mathrm{D}(\mathcal{R}) \xrightarrow{-\overset{L}{\otimes}_{\mathcal{R}}(E_n \oplus \cdots \oplus E_1)} \mathrm{D}(\mathcal{C}), \tag{3.40}$$

where the dg-category $\mathcal{R}$ is the dg-category with objects $\{1, \dots, n\}$ and morphisms[12]

$$\mathrm{Hom}_{\mathcal{R}}(i,j) = \begin{cases} 0 & i < j \\ k & i = j \\ \mathrm{Hom}_{\mathcal{C}}(E_i, E_j) & i > j \end{cases}.$$

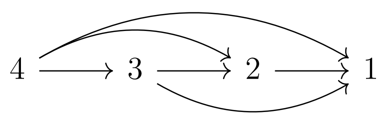
Therefore, we see that $\mathrm{D}(\mathcal{R}) = \mathrm{D}(\star_R)$ where

$$R = \bigoplus_{i=1}^{n} k \cdot \mathrm{id}_{E_i} \oplus \bigoplus_{i>j} \mathrm{Hom}_{\mathcal{C}}(E_i, E_j) \tag{3.41}$$

(considered as a sub-dg-algebra of $\mathrm{Hom}_{\mathcal{C}}(\oplus_{i=1}^n E_i, \oplus_{i=1}^n E_j)$), and (3.40) can be rewritten as

$$\mathrm{D}(\star_R) \xrightarrow{-\overset{L}{\otimes}_{\star_R}(E_n \oplus \cdots \oplus E_1)} \mathrm{D}(\mathcal{C}). \tag{3.42}$$

Remark 3.4.7. The description (3.41) gives us an interpretation of the category $\mathrm{D}(R)$ as that of the derived category of modules over the path algebra of a quiver with relations. Indeed, one can think of a quiver with n-vertices and arrows from i to j labelled by $\mathrm{Hom}_{\mathcal{C}}(E_i, E_j)$ whenever $i > j$, 0 if $i < j$, and by k if $i = j$. We draw the example $n = 4$



Example 3.4.8. Let us give a first geometric example of the above construction; we thank Timothy Logvinenko for explaining it to us. Let X be a smooth, projective variety, and consider two spherical objects $E, F \in \mathrm{D^b}(X)$ such that

$$\mathrm{Hom}^\bullet_{\mathrm{D^b}(X)}(E, F) = \mathrm{Hom}_{\mathrm{D^b}(X)}(E, F[1])[-1] \simeq \mathbb{C}^2[-1].$$

Consider $U \in \mathrm{D^b}(\mathbb{P}(\mathrm{Hom}_{\mathrm{D^b}(X)}(E, F[1])) \times X)$ the universal family that parametrises non-zero extensions of E by F up to the action of $\mathbb{C}^\times$. This object has the property that its fibre over any $p \in \mathbb{P}(\mathrm{Hom}_{\mathrm{D^b}(X)}(E, F[1]))$ gives the corresponding extension of E by F. Considering $U[1]$ as a Fourier–Mukai kernel, we get a functor $\Psi\colon \mathrm{D^b}(\mathbb{P}(\mathrm{Hom}_{\mathrm{D^b}(X)}(E, F[1]))) \to$

[12] We think of the object i as the one corresponding to the i-th copy of the category $\star_k$.

$\mathrm{D^b}(X)$ which is spherical, and whose twist is the composition $T_E T_F$. This can be seen as an example of the above construction considering the spherical objects E and $F[1]$. Indeed, in this case the algebra (3.41) is $k \oplus k^2 \oplus k$, which is the endomorphism algebra of the object $\mathcal{O}_{\mathbb{P}^1}(-2) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$ in $\mathrm{D^b}(\mathbb{P}(\mathrm{Hom}_{\mathrm{D^b}(X)}(E, F[1])))$, and therefore we have an equivalence

$$\mathrm{D}(k \oplus k^2 \oplus k) \simeq \mathrm{D_{qc}}(\mathbb{P}(\mathrm{Hom}_{\mathrm{D^b}(X)}(E, F[1])))$$

under which the spherical functor of Theorem 3.1.4 gets identified with Ψ.

Example 3.4.9. In § 4.4.1, we will construct another geometric example of a glued spherical functor whose spherical twist is the composition of spherical twists around (families of) spherical objects. This example arises as the flop-flop autoequivalence for standard flops.

Example 3.4.10. We thank Tobias Dyckerhoff for explaining to us the following symplectic interpretation of the construction (3.41).

Consider $f\colon E \to \mathbb{D}$ a Lefschetz fibration with base the disk $\mathbb{D}$ with n marked points $p_1, \dots, p_n$ corresponding to the critical points of f. Assume for simplicity that $p_i \neq 1$ for any i, and write $X = f^{-1}(1)$ for the smooth fibre of f. To X we can associate the Fukaya–Seidel category $\mathrm{Fuk}(X)$, which in this case is generated by the vanishing cycles S_i's associated to the p_i's. The fundamental group $\pi_1(\mathbb{D} \setminus \{p_1, \dots, p_n\}, 1)$ acts on $\mathrm{Fuk}(X)$ via a braid group action whose generators are given by the Dehn twists around the spherical objects $S_1, \dots, S_n$.

We can also define the directed Fukaya–Seidel category $\mathrm{Fuk}^{\to}(f)$ of f, see [Sei01, § 6]. This category is generated by the vanishing thimbles associated to the p_i's, *i.e.*, the vanishing cycle together with the choice of a vanishing path. We then get a functor $\partial\colon \mathrm{Fuk}^{\to}(f) \to \mathrm{Fuk}(X)$ given by sending each vanishing thimble to its boundary (which is the corresponding vanishing cycle).

The functor ∂ is spherical, and the spherical twist around it is the *total monodromy action*. More precisely, T_∂ is the composition of the Dehn twists around the S_i's. The connection with Theorem 3.4.6 is that $\mathrm{Fuk}^{\to}(f)$ is the category $\mathrm{D}(R)^{\mathrm{c}}$ for R as defined in (3.41) with respect to the vanishing cycles S_i's, see *ibidem*.

The dg-algebra R defined in (3.41) is smooth (being the gluing of smooth dg-algebras along perfect bimodules) and proper. Therefore, the category $\mathrm{D}(R)^{\mathrm{c}}$ has a Serre duality functor given by tensor product with $R^* := \mathrm{RHom}_k(R, k)$, see [Shk07]. We now describe the cotwist around (3.42) in terms of Serre duality for the category $\mathrm{D}(R)^{\mathrm{c}}$. We have

Theorem 3.4.11. *If $d \neq 0$ the cotwist around (3.42) is given by tensor product with $R^*[-1-d]$.*

Remark 3.4.12. The reason why we need $d \neq 0$ is that in the proof below we need Serre duality to identify id_{E_i} with the non-trivial extension $E_i \to E_i[d]$, which is not necessarily the case if $d = 0$.

Remark 3.4.13. The above theorem appeared in the author's published work [Bar22, Theorem 4.1.9]. Notice however that the statement in *ibidem* lacks the requirement $d \neq 0$.

Proof. We give the proof for the case $n = 2$, the general case being similar.

By Proposition 3.3.1, we know that the cotwist is described by the matrix[13]

$$\begin{pmatrix} \mathrm{Hom}_{\mathrm{D}(\mathcal{C})}(E_1, E_1[d])[-1-d] & 0 \\ \mathrm{Hom}^\bullet_{\mathrm{D}(\mathcal{C})}(E_1, E_2)[-1] & \mathrm{Hom}_{\mathrm{D}(\mathcal{C})}(E_2, E_2[d])[-1-d] \end{pmatrix}$$

with structure morphisms

$$\mathrm{Hom}^\bullet_{\mathrm{D}(\mathcal{C})}(E_1, E_2)[-1] \otimes_k \mathrm{Hom}^\bullet_{\mathrm{D}(\mathcal{C})}(E_2, E_1) \xrightarrow{\mathrm{pr}_{1+d} \circ \mathrm{cmp}} \mathrm{Hom}_{\mathrm{D}(\mathcal{C})}(E_2, E_2[d])[-1-d]$$
$$\mathrm{Hom}^\bullet_{\mathrm{D}(\mathcal{C})}(E_2, E_1) \otimes_k \mathrm{Hom}^\bullet_{\mathrm{D}(\mathcal{C})}(E_1, E_2)[-1] \xrightarrow{\mathrm{pr}_{1+d} \circ \mathrm{cmp}} \mathrm{Hom}_{\mathrm{D}(\mathcal{C})}(E_1, E_1[d])[-1-d],$$

where pr_{1+d} is the projection to the degree $1 + d$ part and cmp is the composition of morphisms.

The bimodule $R^*[-1-d]$ is given by the matrix

$$\begin{pmatrix} \mathrm{Hom}_{\mathrm{D}(\mathcal{C})}(E_1, E_1)^*[-1-d] & 0 \\ \mathrm{Hom}^\bullet_{\mathrm{D}(\mathcal{C})}(E_2, E_1)^*[-1-d] & \mathrm{Hom}_{\mathrm{D}(\mathcal{C})}(E_2, E_2)^*[-1-d] \end{pmatrix}$$

with structure morphisms

$$\mathrm{Hom}^\bullet_{\mathrm{D}(\mathcal{C})}(E_2, E_1)^*[-1-d] \otimes_k \mathrm{Hom}^\bullet_{\mathrm{D}(\mathcal{C})}(E_2, E_1) \to \xrightarrow{\psi \otimes f \mapsto (g \mapsto \psi(f \circ g))} \mathrm{Hom}^\bullet_{\mathrm{D}(\mathcal{C})}(E_2, E_2)^*[-1-d] \to \xrightarrow{\mathrm{pr}_{1+d}} \mathrm{Hom}_{\mathrm{D}(\mathcal{C})}(E_2, E_2)^*[-1-d]$$

and

$$\mathrm{Hom}^\bullet_{\mathrm{D}(\mathcal{C})}(E_2, E_1) \otimes_k \mathrm{Hom}^\bullet_{\mathrm{D}(\mathcal{C})}(E_2, E_1)^*[-1-d] \to \xrightarrow{f \otimes \psi \mapsto (g \mapsto \psi(g \circ f))} \mathrm{Hom}^\bullet_{\mathrm{D}(\mathcal{C})}(E_1, E_1)^*[-1-d] \to \xrightarrow{\mathrm{pr}_{1+d}} \mathrm{Hom}_{\mathrm{D}(\mathcal{C})}(E_1, E_1)^*[-1-d]$$

To conclude we now consider the morphism of bimodules induced by the matrix of morphisms $\begin{pmatrix} \alpha_{11} & 0 \\ \alpha_{12} & \alpha_{22} \end{pmatrix}$ where $\alpha_{ij} : \mathrm{Hom}^\bullet_{\mathrm{D}(\mathcal{C})}(E_i, E_j)[-1] \to \mathrm{Hom}^\bullet_{\mathrm{D}(\mathcal{C})}(E_j, E_i)^*[-1-d]$ is the isomorphism given by Serre duality. This matrix of isomorphisms can be lifted[14] to

[13] We can pass from $\mathrm{RHom}_{\mathcal{C}}(E_1, E_2)$ to the underlying graded vector space because the category of k-k-bimodules is semisimple.

[14] To lift the matrix to a morphism of bimodules we use that the Serre duality isomorphism intertwines

a morphism in the derived category of bimodules which is itself an isomorphism using an argument similar to [AL19, Lemma 7.3] for the case in which the top right components of the bimodules are zero. Thus, the proof is complete. □

Remark 3.4.14. The above theorem together with [Shk07, Theorem 4.2, 4.3] prove that the cotwist around the spherical functor (3.42) is isomorphic to the Serre duality functor for $\mathrm{D}(R)^{\mathrm{c}}$ shifted by $[-1-d]$.

An application of this result to compute the categorical entropy of the composition of two spherical twists around spherical objects can be found in [BK21].

3.5 $\mathbb{P}$-objects

In this subsection we consider the case of $\mathbb{P}$-objects.

$\mathbb{P}$-objects were introduced by Huybrechts and Thomas in [HT06], and their introduction was motivated, as for spherical objects, by Homological Mirror Symmetry. More precisely, the idea is that in the Fukaya category we can twist not only around Lagrangian spheres, but also around Lagrangian $\mathbb{P}^n$'s.

Drawing inspiration from this idea, Huybrechts and Thomas defined the notion of a $\mathbb{P}$-object, and showed that to any such object one can associate an autoequivalence, which they called the $\mathbb{P}$-twist around the $\mathbb{P}$-object.

The definition of a $\mathbb{P}$-object has been later generalised to that of a split $\mathbb{P}$-functor in [Add16], [Cau12b], and further to general $\mathbb{P}$-functors in [AL19].

Let us now give the formal definition of a $\mathbb{P}$-object. Let $\mathcal{C}$ be a small, proper dg-category over a field k such that $\mathrm{D}(\mathcal{C})^{\mathrm{c}}$ has a Serre functor $\mathbb{S}$ which is given by tensor product with a bimodule: $\mathbb{S}(-) = - \overset{L}{\otimes}_{\mathcal{C}} S_{\mathcal{C}}$.

Definition 3.5.1. An object $P \in \mathrm{D}(\mathcal{C})$ is said to be a $\mathbb{P}^n$-object if the following conditions are satisfied:

1. P is both $\star_k$- and $\mathcal{C}$-perfect
2. $\mathrm{Hom}^{\bullet}_{\mathrm{D}(\mathcal{C})}(P, P) \simeq k[t]/t^{n+1}$, $\deg(t) = 2$, as graded algebras
3. We have an isomorphism $P \overset{L}{\otimes}_{\mathcal{C}} S_{\mathcal{C}} \simeq P[2n]$ in $\mathrm{D}(\mathcal{C})$

Remark 3.5.2. If X is a smooth projective variety of dimension $2n$ such that $\mathrm{D}(\mathcal{C}) \simeq \mathrm{D}_{\mathrm{qc}}(X)$, then an object $P \in \mathrm{D}(\mathcal{C})$ is a $\mathbb{P}^n$-object if and only if the corresponding object in $\mathrm{D}_{\mathrm{qc}}(X)$ is a $\mathbb{P}^n$-object in the sense of [HT06].

the structure morphisms we described. This is were we used that $d \neq 0$ and Remark 3.4.12.

Once again, the reader might wonder what is the relationship between $\mathbb{P}$-objects and spherical functors. This relationship will be clarified momentarily, and we devote further attention to this question in Remark 3.5.5.

In [Seg18] Segal describes two ways to realise the $\mathbb{P}$-twist around a $\mathbb{P}$-object as a spherical twist. The first one considers P as a dg-module over the dg-algebra $k[t]$, $\deg(t) = 2$, and defines the functor

$$- \overset{L}{\otimes}_{\star_{k[t]}} P \colon \mathrm{D}(\star_{k[t]}) \to \mathrm{D}(\mathcal{C}). \tag{3.43}$$

Here to define the action of $k[t]$ on P we assume that we replaced P with an h-projective resolution, that we fixed a representative $\tilde{t} \in \mathrm{Hom}_{\mathcal{C}}(P, P)$ for the generator of degree 2, and we make t act as $\tilde{t}$.

Remark 3.5.3. Notice that $\star_k$-perfectness of P implies $\star_{k[t]}$-perfectness because $k[t]$ is smooth, see e.g. [Shk07, pag. 7].

Then, under a technical assumption (which was subsequently shown to be always satisfied [HK19]), [Seg18, Proposition 4.2] proves that (3.43) is spherical, that its twist is the $\mathbb{P}$-twist around P, and that its cotwist is $[-2n-2]$.

The second construction of [Seg18] uses Koszul duality to rewrite (3.43) in a different way. More precisely, the object $k \in \mathrm{D}(\star_{k[t]})$ is compact and we have $\mathrm{RHom}_{k[t]}(k, k) \simeq k[\varepsilon]/\varepsilon^2$, with $\deg(\varepsilon) = -1$. The $\star_{k[t]}$-module $k[t, e]$, $\deg(e) = 1$, $d(e) = t$, is h-projective and gives a resolution of k as a right $k[t]$-module. Moreover, $k[t, e]$ carries a left action of $k[\varepsilon]/\varepsilon^2$ via the degree -1 map of $k[t]$-modules $k[t, e] \to k[t, e]$ that sends $p(t) + eq(t)$ to $q(t)$. Hence, we get a functor

$$- \overset{L}{\otimes}_{\star_{k[\varepsilon]/\varepsilon^2}} P' \colon \mathrm{D}(\star_{k[\varepsilon]/\varepsilon^2}) \to \mathrm{D}(\mathcal{C}) \tag{3.44}$$

where

$$P' := k[t, e] \otimes_{\star_{k[t]}} P = \left\{ P[-2] \xrightarrow{\tilde{t}} P \right\}.$$

The twist around (3.44) is the $\mathbb{P}$-twist around P, and its cotwists is given by $[-2n-2]$.

Remark 3.5.4. Construction (3.44) was generalised to the case of split $\mathbb{P}^n$-functors in [AL19, Theorem 5.1].

Remark 3.5.5. Let us spend a few more words on the relationship between $\mathbb{P}$-objects and spherical functors.

Above, we recalled the construction by Segal that shows how to attach to any $\mathbb{P}$-object two spherical functors, one with source category $\mathrm{D}(\star_{k[t]})$, $\deg(t) = 2$, and one with source category $\mathrm{D}(\star_{k[\varepsilon]/\varepsilon^2})$, $\deg(\varepsilon) = -1$.

In fact it is possible to prove that any conservative spherical functor $\Psi \colon \mathrm{D}(\star_{k[\varepsilon]/\varepsilon^2}) \to \mathrm{D}(\mathcal{C})$ such that $C_\Psi \simeq [-1-m]$ comes from a $\mathbb{P}^n$-object such that $2n + 1 = m$.[15]

[15] This is a special case of a result the author proved jointly with Pieter Belmans, Alessio Bottini,

3.5.1 The $k[t]$-model

Take $\mathbb{P}^n$-objects $P_1, \dots, P_m$. Assume we replaced them with h-projective resolutions and that for each $i = 1, \dots, m$ we fixed $\tilde{t}_i$ a lift of the generator of degree 2 of $\mathrm{Hom}^\bullet_{\mathrm{D}(\mathcal{C})}(P_i, P_i)$. Then, we can apply Theorem 3.1.4 to (3.43) to obtain a spherical functor

$$\mathrm{D}(\star_R) \xrightarrow{-\overset{L}{\otimes}_{\star_R}(P_m \oplus \cdots \oplus P_1)} \mathrm{D}(\mathcal{C})$$

whose twist is given by the composition of the $\mathbb{P}$-twists around $P_m, \dots, P_1$. Here (beware that e_i is just a placeholder to distinguish between the different copies of t)

$$R = \bigoplus_{i=1}^{m} k[t] \cdot e_i \oplus \bigoplus_{i>j} \mathrm{Hom}_{\mathcal{C}}(P_i, P_j)$$

and the composition is defined as follows. Elements of $\mathrm{Hom}_{\mathcal{C}}(P_i, P_j)$ and $\mathrm{Hom}_{\mathcal{C}}(P_k, P_l)$ compose according to the composition rule in **Mod**-$\mathcal{C}$, $p(t)e_i$ and $q(t)e_j$ compose as

$$p(t)e_i \cdot q(t)e_j = \delta_{ij} p(t)q(t)e_i$$

and finally $p(t)e_i$ composes with $f \in \mathrm{Hom}_{\mathcal{C}}(P_i, P_j)$ and $g \in \mathrm{Hom}_{\mathcal{C}}(P_k, P_i)$ as

$$f \cdot p(t)e_i \cdot g = f \circ p(\tilde{t}_i) \circ g$$

where $\circ$ is the composition law in **Mod**-$\mathcal{C}$.

The cotwist cannot be described as in Theorem 3.4.11 because $k[t]^*$ is not a shift of $k[t]$.

Example 3.5.6. For a geometric example of this construction, see Remark 4.4.19.

3.5.2 The $k[\varepsilon]/\varepsilon^2$-model

Take $\mathbb{P}^n$-objects $P_1, \dots, P_m$. Assume we replaced them with h-projective resolutions and that for each $i = 1, \dots, m$ we fixed $\tilde{t}_i$ a lift of the generator of degree 2 of $\mathrm{Hom}^\bullet_{\mathrm{D}(\mathcal{C})}(P_i, P_i)$. We write $P_i' := k[t, e] \otimes_{\star_{k[t]}} P_i = \left\{ P_i[-2] \xrightarrow{\tilde{t}_i} P_i \right\}$. Applying Theorem 3.1.4 to (3.44) we obtain a spherical functor

$$\mathrm{D}(\star_R) \xrightarrow{-\overset{L}{\otimes}_{\star_R}(P_m' \oplus \cdots \oplus P_1')} \mathrm{D}(\mathcal{C}) \tag{3.45}$$

Emma Lepri and Johannes Krah, at a 2021 summer school at the Hausdorff Centre for Mathematics, following a suggestion of Alexander Kuznetsov.

whose twist is given by the composition of the $\mathbb{P}$-twists around $P_m, \dots, P_1$. Here

$$R = \bigoplus_{i=1}^{m} k[\varepsilon]/\varepsilon^2 \cdot \mathrm{id}_{P_i'} \oplus \bigoplus_{i>j} \mathrm{Hom}_{\mathcal{C}}(P_i', P_j')$$

considered as a sub-dg-algebra of $\mathrm{Hom}_{\mathcal{C}}(\oplus_{i=1}^m P_i', \oplus_{i=1}^m P_i')$.

Example 3.5.7. In § 4.4.2, we will provide a geometric example of this construction by looking at the flop-flop autoequivalence for Mukai flops.

Remark 3.5.8. Our hope was to prove an isomorphism $C_{(3.45)}(-) \simeq - \overset{L}{\otimes}_{\star R} R^*[-2n-1]$ in analogy with Theorem 3.4.11. Unfortunately, we stumble upon technical issues we do not know how to fix. More precisely, the components of the cotwist outside the diagonal are of the form $\mathrm{Hom}_{\mathcal{C}}(P_j', P_i')[-1]$, $j < i$, and we would like to use Serre duality to relate them to $\mathrm{Hom}_{\mathcal{C}}(P_i', P_j')^*[-2n-1]$. The problem is that Serre duality provides us with an isomorphism $\mathrm{Hom}^\bullet_{\mathrm{D}(\mathcal{C})}(P_j', P_i')[-1] \simeq \mathrm{Hom}^\bullet_{\mathrm{D}(\mathcal{C})}(P_i', P_j')^*[-2n-1]$ of $k[\varepsilon]/\varepsilon^2$-$k[\varepsilon]/\varepsilon^2$-bimodules, but it is not clear whether one can lift this isomorphism to a quasi-isomorphism $\mathrm{Hom}_{\mathcal{C}}(P_j', P_i')[-1] \to \mathrm{Hom}_{\mathcal{C}}(P_i', P_j')^*[-2n-1]$ of $k[\varepsilon]/\varepsilon^2$-$k[\varepsilon]/\varepsilon^2$-bimodules.

Chapter 4

Flop-flop autoequivalences

In § 1, we explained the *raison d'être* that connects the various projects whose results are presented in this thesis. Namely, the results of § 3 were motivated by the will to understand how spherical twists behave when composed with each other, whereas the present chapter was born from the will to show that glued spherical functors appear naturally in geometric situations. Along the way, the search for geometric examples of glued spherical functors resulted in the discovery of various interesting results, which we present in this chapter.

Here, we just want to spend a few words in motivating our use of cocomplete triangulated categories. Since the very beginning of this thesis, we have been working with cocomplete triangulated categories, and the reader might be wondering why, as most of the questions geometry poses are about the bounded derived category of coherent sheaves. The reason is that the use of cocomplete triangulated categories allows us to leverage powerful theorems such as Brown representability [Nee96]. This turns out to be extremely useful, especially when one deals with Verdier quotients because we can (in good cases) realise the quotient as a subcategory of the parent category.

It is often the case that requiring the category to be cocomplete is not restrictive, see also Remark 2.3.9. However, there are cases in which there is a tangible difference between cocomplete and non-cocomplete triangulated categories, and in our treatment we see this when we consider singular algebraic varieties, see Remark 4.2.4 and Remark 4.2.5.

4.1 General case

In this subsection we introduce the notion of a *flop-flop diagram*. The intuition behind this notion, as we explain in Remark 4.1.2 below, is that they arise from correspondences of schemes, and therefore appear naturally in the geometric context.

In the following, when we speak of triangulated categories we always mean enhanced triangulated categories. The question of which enhancement one uses does not make a difference for the purposes of this chapter, and thus we do not fix any particular enhancement. For more on the question of the wealth of available enhancements, the reader is directed to § 2.1.

For the rest of section, all triangulated categories are assumed to be cocomplete and compactly generated (see Definition 2.3.27). For simplicity, we assume that our triangulated categories are generated by a single compact object that we call a *compact generator*, but all the results apply in the general case.

We are now ready to introduce the central notion of this chapter. Let

$$\mathcal{B}_- \xleftarrow{\alpha_-} \mathcal{A} \xrightarrow{\alpha_+} \mathcal{B}_+ \tag{4.1}$$

be a diagram of cocomplete triangulated categories and cocontinuous functors.

Definition 4.1.1. We call a diagram as (4.1) a *flop-flop diagram* if

(i) α_- and α_+ have fully faithful left adjoints $\alpha_-^L\colon \mathcal{B}_- \hookrightarrow \mathcal{A}$ and $\alpha_+^L\colon \mathcal{B}_+ \hookrightarrow \mathcal{A}$, respectively

(ii) the functors

$$\Phi_+ = \alpha_+\alpha_-^L\colon \mathcal{B}_- \to \mathcal{B}_+ \quad \text{and} \quad \Phi_- = \alpha_-\alpha_+^L\colon \mathcal{B}_+ \to \mathcal{B}_-$$

are equivalences.

Given a flop-flop diagram $\mathcal{B}_- \xleftarrow{\alpha_-} \mathcal{A} \xrightarrow{\alpha_+} \mathcal{B}_+$, we call the autoequivalences

$$\Phi_+\Phi_-\colon \mathcal{B}_+ \to \mathcal{B}_+ \quad \text{and} \quad \Phi_-\Phi_+\colon \mathcal{B}_- \to \mathcal{B}_-$$

the *flop-flop autoequivalences*.

Remark 4.1.2. We chose the name *flop-flop diagram* because the typical example of a diagram as (4.1) arises from a birational contraction $f_-\colon X_- \to Y$ together with its flop $f_+\colon X_+ \to Y$. From f_- and f_+ we can construct the diagram of schemes

$$X_- \xleftarrow{p_-} X_- \times_Y X_+ \xrightarrow{p_+} X_+,$$

which gives rise to the following diagram of categories

$$\mathrm{D}_{\mathrm{qc}}(X_-) \xleftarrow{(p_-)_*} \mathrm{D}_{\mathrm{qc}}(X_- \times_Y X_+) \xrightarrow{(p_+)_*} \mathrm{D}_{\mathrm{qc}}(X_+)$$

By work of Bridgeland [Bri02] and Chen [Che02], we know that this construction always gives an example of a flop-flop diagram when X_- and X_+ are projective Calabi–Yau threefolds, Y has rational singularities, and we are flopping a single curve. We will explore flop-flop diagrams arising from diagrams of schemes in § 4.2.

As we explained in § 1, our aim is to express the flop-flop autoequivalences arising from a flop-flop diagram as inverses of spherical twists around spherical functors.

We define the subcategories

$$\mathcal{K} := \ker \alpha_- \cap \ker \alpha_+ \quad \text{and} \quad \mathcal{S}_\pm = {}^\perp\mathcal{K} \cap (\alpha_\mp^L \mathcal{B}_\mp)^\perp = {}^\perp\mathcal{K} \cap \ker \alpha_\mp$$

and write $i_{\mathcal{S}_\pm} \colon \mathcal{S}_\pm \hookrightarrow \mathcal{A}$ for the inclusions. The following is the main theorem of this section.

Theorem 4.1.3. *Let $\mathcal{B}_- \xleftarrow{\alpha_-} \mathcal{A} \xrightarrow{\alpha_+} \mathcal{B}_+$ be a flop-flop diagram. Then, $\mathcal{K}$ is left admissible in $\mathcal{A}$ and we have four periodic SODs*

$$ {}^\perp\mathcal{K} = \langle \mathcal{S}_+, \alpha_-^L \mathcal{B}_- \rangle = \langle \alpha_-^L \mathcal{B}_-, \mathcal{S}_- \rangle = \langle \mathcal{S}_-, \alpha_+^L \mathcal{B}_+ \rangle = \langle \alpha_+^L \mathcal{B}_+, \mathcal{S}_+ \rangle \tag{4.2}$$

$$ ({}^\perp\mathcal{K})^c = \langle \mathcal{S}_+^c, \alpha_-^L \mathcal{B}_-^c \rangle = \langle \alpha_-^L \mathcal{B}_-^c, \mathcal{S}_-^c \rangle = \langle \mathcal{S}_-^c, \alpha_+^L \mathcal{B}_+^c \rangle = \langle \alpha_+^L \mathcal{B}_+^c, \mathcal{S}_+^c \rangle \tag{4.3}$$

Furthermore, the functors

$$\Psi_+ := \alpha_+ i_{\mathcal{S}_+} \colon \mathcal{S}_+ \to \mathcal{B}_+ \quad \text{and} \quad \Psi_- := \alpha_- i_{\mathcal{S}_-} \colon \mathcal{S}_- \to \mathcal{B}_-$$

are conservative spherical functors such that $T_{\Psi_\pm}^{-1} = \Phi_\pm \Phi_\mp \in \operatorname{Aut}(\mathcal{B}_\pm)$, and their restrictions $\Psi_+|_{\mathcal{S}_+^c}$ and $\Psi_-|_{\mathcal{S}_-^c}$ are conservative spherical functors such that $T_{\Psi_\pm|_{\mathcal{S}_\pm^c}}^{-1} = \Phi_\pm \Phi_\mp|_{\mathcal{B}_\pm^c}$.

Remark 4.1.4. Recall that a functor is called *conservative* if it has no kernel.

Proof. First, we prove that $\mathcal{K}$ is left admissible, see Definition 2.3.23. We write $\mathcal{C} = \langle \alpha_\pm^L \mathcal{B}_\pm \rangle^\oplus$ for the smallest cocomplete subcategory generated by the essential images of α_+^L and α_-^L. By definition, $\mathcal{C}$ is closed under arbitrary small direct sums in $\mathcal{A}$, and therefore it is localising. Moreover, as α_+^L and α_-^L are fully faithful functors, if we fix compact generators $B_+ \in \mathcal{B}_+^c$ and $B_- \in \mathcal{B}_-^c$, then $\mathcal{C} = \langle \alpha_+^L(B_+) \oplus \alpha_-^L(B_-) \rangle^\oplus$.

As α_+ and α_- are cocontinuous functors, the functors α_+^L and α_-^L preserve compactness. Hence, $\alpha_+^L(B_+) \oplus \alpha_-^L(B_-) \in \mathcal{A}^c$, and by Lemma 2.3.31 we have the SOD $\mathcal{A} = \langle \mathcal{C}^\perp, \mathcal{C} \rangle$. However, it is clear that $\mathcal{C}^\perp = \mathcal{K}$, and thus $\mathcal{K}$ is left admissible by Lemma 2.3.24.

Notice that the SOD $\mathcal{A} = \langle \mathcal{K}, \mathcal{C} \rangle$ implies $\mathcal{C} = {}^\perp\mathcal{K}$. From now on, we will write ${}^\perp\mathcal{K}$ in place of $\mathcal{C}$.

Notice that the subcategories $\mathcal{S}_+, \mathcal{S}_-, \alpha_+^L \mathcal{B}_+$, and $\alpha_-^L \mathcal{B}_-$ are localising because the functors $\alpha_+, \alpha_-, \alpha_+^L$, and α_-^L are cocontinuous, see Remark 2.3.11. Therefore, if we prove the existence of the SODs (4.2), we get the SODs (4.3) from Lemma 2.3.22.

We now prove the existence of the SODs (4.2). As the functors α_+^L and α_-^L are fully faithful, the subcategories $\alpha_+^L\mathcal{B}_+, \alpha_-^L\mathcal{B}_- \subset {}^\perp\mathcal{K}$ are right admissible. Therefore, by Lemma 2.3.24 we have

$$ {}^\perp\mathcal{K} = \langle {}^\perp\mathcal{K} \cap (\alpha_\pm^L\mathcal{B}_\pm)^\perp, \alpha_\pm^L\mathcal{B}_\pm\rangle = \langle \mathcal{S}_\mp, \alpha_\pm^L\mathcal{B}_\pm\rangle, $$

where in the second equality we used the definition of $\mathcal{S}_+$ and $\mathcal{S}_-$. Thus, we established the existence of the first and the third SOD in (4.2).

To prove the existence of the remaining SODs in (4.2), we have to do some more work. To simplify the notation, we will focus on $\mathcal{S}_+$ and prove the existence of the fourth SOD. The existence of the second SOD in (4.2) can be proven similarly.

The first step to prove that we have the SOD ${}^\perp\mathcal{K} = \langle \alpha_+^L\mathcal{B}_+, \mathcal{S}_+\rangle$ is to show that $\mathcal{S}_+ \subset {}^\perp\mathcal{K}$ is a right admissible subcategory. By Lemma 2.3.31, it is enough to show that $\mathcal{S}_+$ is compactly generated. Take $B_+ \in \mathcal{B}_+^c$ a compact generator. Then, the object

$$ S_+ := \mathrm{cone}(\alpha_-^L\alpha_-\alpha_+^L(B_+) \to \alpha_+^L(B_+)) $$

belongs to $\mathcal{S}_+$. Indeed, S_+ clearly belongs to ${}^\perp\mathcal{K}$. Moreover, $\alpha_-(S_+) \simeq 0$ because α_-^L is fully faithful, and therefore $S_+ \in {}^\perp\mathcal{K} \cap \ker\alpha_- = \mathcal{S}_+$.

Now notice that $S_+ \in \mathcal{A}^c$ because $\alpha_+\alpha_-^L$ is a cocontinuous equivalence and α_-^L and α_+^L preserve compactness (see above). Therefore, $S_+ \in \mathcal{S}_+^c$ because $\mathcal{S}_+$ is a localising subcategory of $\mathcal{A}$. We claim that S_+ is a compact generator of $\mathcal{S}_+$, *i.e.*, $\mathcal{S}_+ = \langle S_+\rangle^\oplus$.

As S_+ is a compact object, proving that $\mathcal{S}_+ = \langle S_+\rangle^\oplus$ is equivalent to prove $S_+^\perp \cap \mathcal{S}_+ = 0$. Take $T_+ \in S_+^\perp \cap \mathcal{S}_+$, then

$$ 0 \simeq \mathrm{Hom}^\bullet_{\mathcal{S}_+}(S_+, T_+) \simeq \mathrm{Hom}^\bullet_{\mathcal{B}_+}(B_+, \alpha_+(T_+)). $$

As B_+ is a compact generator, the above vanishing implies $\alpha_+(T_+) \simeq 0$. Hence, we have

$$ T_+ \in \mathcal{S}_+ \cap \ker\alpha_+ = {}^\perp\mathcal{K} \cap \ker\alpha_- \cap \ker\alpha_+ = {}^\perp\mathcal{K} \cap \mathcal{K} = 0. $$

Thus, we proved that $\mathcal{S}_+$ is compactly generated by S_+, and therefore by Lemma 2.3.31 we have the SOD

$$ {}^\perp\mathcal{K} = \langle \mathcal{S}_+^\perp, \mathcal{S}_+\rangle. \tag{4.4} $$

The next step in proving the existence of the fourth SOD in (4.2) is to show that $\alpha_+^L\mathcal{B}_+ \subset \mathcal{S}_+^\perp$. By the definition of $B_+ \in \mathcal{B}_+^c$ and S_+, since Φ_- is an equivalence and α_+^L is fully faithful, we have

$$ \mathrm{Hom}^\bullet_{\mathcal{A}}(S_+, \alpha_+^L(B_+)) \simeq \mathrm{cone}\left(\mathrm{Hom}^\bullet_{\mathcal{B}_+}(B_+, B_+) \xrightarrow{\simeq} \mathrm{Hom}^\bullet_{\mathcal{B}_-}(\Phi_-(B_+), \Phi_-(B_+))\right)[-1] \simeq 0. $$

Hence, as S_+ and B_+ are compact generators of $\mathcal{S}_+$ and $\alpha_+^L\mathcal{B}_+$, respectively, we have

$$\alpha_+^L\mathcal{B}_+ = \langle\alpha_+^L(B_+)\rangle^\oplus \subset (\langle S_+\rangle^\oplus)^\perp = \mathcal{S}_+^\perp.$$

Applying Lemma 2.3.24, from (4.4) we get

$$^\perp\mathcal{K} = \langle{}^\perp\mathcal{K}\cap(\alpha_+^L\mathcal{B}_+)^\perp\cap\mathcal{S}_+^\perp, \alpha_+^L\mathcal{B}_+, \mathcal{S}_+\rangle. \tag{4.5}$$

To conclude, we now prove ${}^\perp\mathcal{K}\cap(\alpha_+^L\mathcal{B}_+)^\perp\cap\mathcal{S}_+^\perp = 0$. Indeed, since Φ_- is an equivalence, the distinguished triangle

$$\alpha_-^L\Phi_- = \alpha_-^L\alpha_-\alpha_+^L \to \alpha_+^L \to i_{\mathcal{S}_+}i_{\mathcal{S}_+}^L\alpha_+^L,$$

which comes from the SOD $\langle\mathcal{S}_+, \alpha_-^L\mathcal{B}_-\rangle = {}^\perp\mathcal{K}$, shows that

$$^\perp\mathcal{K} = \langle\mathcal{S}_+, \alpha_-^L\mathcal{B}_-\rangle = \langle\mathcal{S}_+, \alpha_-^L\Phi_-\mathcal{B}_+\rangle = \langle\alpha_+^L\mathcal{B}_+, \mathcal{S}_+\rangle^\oplus.$$

Therefore

$$^\perp\mathcal{K}\cap(\alpha_+^L\mathcal{B}_+)^\perp\cap\mathcal{S}_+^\perp = {}^\perp\mathcal{K}\cap\big(\langle\alpha_+^L\mathcal{B}_+, \mathcal{S}_+\rangle^\oplus\big)^\perp = {}^\perp\mathcal{K}\cap({}^\perp\mathcal{K})^\perp = 0$$

and plugging this equality in (4.5) the existence of the fourth SOD in (4.2) follows.

Given the existence of the SODs (4.2), the statements about the spherical functors follow from Lemma 2.5.7. □

Remark 4.1.5. [BB15] was the first paper in the literature to consider a particular class of flop-flop diagrams (without calling them so) as per Definition 4.1.1. In *ibidem*, Bodzenta and Bondal consider $f_-\colon X_- \to Y$ a morphism satisfying certain assumptions (among which there is $(f_-)_*\mathcal{O}_{X_-} \simeq \mathcal{O}_Y$ and that the fibres of f_- must have at most dimension 1), take $f_+\colon X_+ \to Y$ a flop of f_-, and set $\mathcal{B}_\pm = \mathrm{D^b}(X_\pm)$, $\mathcal{A} = \mathrm{D^b}(X_- \times_Y X_+)$, and $\alpha_\pm = (f_\pm)_*$. Among other things, Bodzenta and Bondal also prove the statement analogous to Corollary 4.1.7 (see below) for their setup.

Let us now reformulate the above theorem using a language similar to [BB15, § 5.2]. In § 4.2, this will allow us to prove a statement about bounded derived categories of coherent sheaves.

As $\mathcal{K}$ is left admissible, we have $\mathcal{A} = \langle\mathcal{K}, {}^\perp\mathcal{K}\rangle$. This implies that the quotient functor[1] $\pi\colon \mathcal{A} \to \mathcal{A}/\mathcal{K}$ induces an equivalence ${}^\perp\mathcal{K} \simeq \mathcal{A}/\mathcal{K}$. Indeed, π is clearly essentially surjective. The following lemma shows that it is also fully faithful.

[1]Let us spend a couple of words on the choice of the letter π as the notation for the quotient functor. The reader might think that this is an inconvenient choice because it clashes with the notation for projection functors of SODs. However, this clash of notation is intentional, and it is not a clash of

Lemma 4.1.6 ([BB15, Lemma 5.7]). *Let $\mathcal{A}$ be a triangulated category and $\mathcal{S} \subset \mathcal{A}$ be a thick subcategory. Then, for any $E \in {}^{\perp}\mathcal{S}$ and any $F \in \mathcal{A}$, the quotient functor $\pi \colon \mathcal{A} \to \mathcal{A}/\mathcal{S}$ induces an isomorphism*

$$\operatorname{Hom}_{\mathcal{A}}(E, F) \simeq \operatorname{Hom}_{\mathcal{A}/\mathcal{S}}(\pi(E), \pi(F)).$$

Hence, if we write $\overline{\alpha}_{\pm}$ for the functors induced by $\alpha_{\pm}$ on $\mathcal{A}/\mathcal{K}$, and $\overline{\alpha}_{\pm}^{L} = \pi\alpha_{\pm}^{L}$ (which one can easily check, using Lemma 4.1.6, are the left adjoints to $\overline{\alpha}_{\pm}$, see also page 105 for a similar argument), then Theorem 4.1.3 can be reformulated as follows

Corollary 4.1.7. *Let $\mathcal{B}_{-} \xleftarrow{\alpha_{-}} \mathcal{A} \xrightarrow{\alpha_{+}} \mathcal{B}_{+}$ be a flop-flop diagram. Then, we have four periodic SODs*

$$\mathcal{A}/\mathcal{K} = \langle \ker \overline{\alpha}_{-}, \overline{\alpha}_{-}^{L}\mathcal{B}_{-} \rangle = \langle \overline{\alpha}_{-}^{L}\mathcal{B}_{-}, \ker \overline{\alpha}_{+} \rangle = \langle \ker \overline{\alpha}_{+}, \overline{\alpha}_{+}^{L}\mathcal{B}_{+} \rangle = \langle \overline{\alpha}_{+}^{L}\mathcal{B}_{+}, \ker \overline{\alpha}_{-} \rangle.$$

$$\begin{aligned}(\mathcal{A}/\mathcal{K})^{c} &= \langle (\ker \overline{\alpha}_{-})^{c}, \overline{\alpha}_{-}^{L}\mathcal{B}_{-}^{c} \rangle = \langle \overline{\alpha}_{-}^{L}\mathcal{B}_{-}^{c}, (\ker \overline{\alpha}_{+})^{c} \rangle \\ &= \langle (\ker \overline{\alpha}_{+})^{c}, \overline{\alpha}_{+}^{L}\mathcal{B}_{+}^{c} \rangle = \langle \overline{\alpha}_{+}^{L}\mathcal{B}_{+}^{c}, (\ker \overline{\alpha}_{-})^{c} \rangle.\end{aligned}$$

Furthermore, the functors

$$\overline{\Psi}_{-} := \overline{\alpha}_{-} i_{\ker \overline{\alpha}_{+}} \colon \ker \overline{\alpha}_{+} \to \mathcal{B}_{-} \quad \text{and} \quad \overline{\Psi}_{+} := \overline{\alpha}_{+} i_{\ker \overline{\alpha}_{-}} \colon \ker \overline{\alpha}_{-} \to \mathcal{B}_{+}$$

are conservative spherical functors such that $T_{\overline{\Psi}_{\pm}}^{-1} = \Phi_{\pm}\Phi_{\mp} \in \operatorname{Aut}(\mathcal{B}_{\pm})$, and their restrictions $\overline{\Psi}_{\pm}|_{(\ker \overline{\alpha}_{\mp})^{c}}$ are conservative spherical functors such that $T^{-1}_{\overline{\Psi}_{\pm}|_{(\ker \overline{\alpha}_{\mp})^{c}}} = \Phi_{\pm}\Phi_{\mp}|_{\mathcal{B}_{\pm}^{c}}$.

Proof. All the claims follow from Theorem 4.1.3 once we notice that the quotient functor induces equivalences $\mathcal{S}_{\mp} \simeq \ker \overline{\alpha}_{\pm}$ and $\mathcal{S}_{\mp}^{c} \simeq (\ker \overline{\alpha}_{\pm})^{c}$. □

4.2 Bounded derived categories

Let us continue the discussion we started in Remark 4.1.2. Take X_{-}, X_{+}, and $\widehat{X}$ three separated, finite type schemes of finite Krull dimension together with finite type maps $X_{-} \xleftarrow{p_{-}} \widehat{X} \xrightarrow{p_{+}} X_{+}$. Assume that

$$(p_{-})_{*}\mathcal{O}_{\widehat{X}} \simeq \mathcal{O}_{X_{-}} \qquad (p_{+})_{*}\mathcal{O}_{\widehat{X}} \simeq \mathcal{O}_{X_{+}} \tag{4.6}$$

and that

$$\Phi_{+} = (p_{+})_{*}p_{-}^{*} \colon \mathrm{D}_{\mathrm{qc}}(X_{-}) \xrightarrow{\simeq} \mathrm{D}_{\mathrm{qc}}(X_{+}) \qquad \Phi_{-} = (p_{-})_{*}p_{+}^{*} \colon \mathrm{D}_{\mathrm{qc}}(X_{+}) \xrightarrow{\simeq} \mathrm{D}_{\mathrm{qc}}(X_{-}). \tag{4.7}$$

notation at all. Indeed, as $\mathcal{K}$ sits in an SOD $\mathcal{A} = \langle \mathcal{K}, {}^{\perp}\mathcal{K} \rangle$, the quotient functor π has a fully faithful left adjoint π^{L}, and if we identify $\mathcal{A}/\mathcal{K}$ with the essential image of π^{L} (which is ${}^{\perp}\mathcal{K}$) the quotient functor π becomes the projection functor to ${}^{\perp}\mathcal{K}$ in the previous SOD.

Then,

$$\mathrm{D}_{\mathrm{qc}}(X_-) \xleftarrow{(p_-)_*} \mathrm{D}_{\mathrm{qc}}(\widehat{X}) \xrightarrow{(p_+)_*} \mathrm{D}_{\mathrm{qc}}(X_+) \tag{4.8}$$

is a flop-flop diagram, and we say that $X_- \xleftarrow{p_-} \widehat{X} \xrightarrow{p_+} X_+$ induces a flop-flop diagram.

Applying Corollary 4.1.7 to the flop-flop diagram (4.8), we obtain the SODs

$$\begin{aligned}\mathrm{D}_{\mathrm{qc}}(\widehat{X})/\mathcal{K} &= \langle \ker(\overline{p}_-)_*, \overline{p}_-^*\mathrm{D}_{\mathrm{qc}}(X_-)\rangle = \langle \overline{p}_-^*\mathrm{D}_{\mathrm{qc}}(X_-), \ker(\overline{p}_+)_*\rangle \\ &= \langle \ker(\overline{p}_+)_*, \overline{p}_+^*\mathrm{D}_{\mathrm{qc}}(X_+)\rangle = \langle \overline{p}_+^*\mathrm{D}_{\mathrm{qc}}(X_+), \ker(\overline{p}_-)_*\rangle.\end{aligned} \tag{4.9}$$

We explained in Remark 4.1.2 that, in fact, these diagrams were the examples from which this research started. We now want to investigate when we can pass from the categories of complexes with quasi-coherent cohomology to the categories of cohomologically bounded complexes with coherent cohomology, *i.e.*, $\mathrm{D}^{\mathrm{b}}(-)$. For this reason, in this subsection we make the following

Assumption. p_- and p_+ are proper and of finite Tor dimension.

Under this assumption, we have the functors

$$p_\pm^*\colon \mathrm{D}^{\mathrm{b}}(X_\pm) \to \mathrm{D}^{\mathrm{b}}(\widehat{X}) \qquad (p_\pm)_*\colon \mathrm{D}^{\mathrm{b}}(\widehat{X}) \to \mathrm{D}^{\mathrm{b}}(X_\pm) \qquad p_\pm^\times\colon \mathrm{D}^{\mathrm{b}}(X_\pm) \to \mathrm{D}^{\mathrm{b}}(\widehat{X}).$$

The functors[2] $p_\pm^\times$ are the right adjoints to $(p_\pm)_*$, and the fact that they preserve $\mathrm{D}^{\mathrm{b}}(-)$ is proved in [Nee18b, Lemma 3.12]. Moreover, [Nee18a, Remark 6.1.1] together with (4.6) imply that $p_\pm^\times$ are fully faithful.

From now on, all the functors we write are assumed to be between bounded derived categories of coherent sheaves unless otherwise stated.

Let us consider $\mathcal{K}^{\mathrm{b}} = \mathcal{K} \cap \mathrm{D}^{\mathrm{b}}(\widehat{X})$, and take the quotient $\mathrm{D}^{\mathrm{b}}(\widehat{X})/\mathcal{K}^{\mathrm{b}}$ with quotient functor $\pi\colon \mathrm{D}^{\mathrm{b}}(\widehat{X}) \to \mathrm{D}^{\mathrm{b}}(\widehat{X})/\mathcal{K}^{\mathrm{b}}$. We consider the functors induced on the quotient by $(p_\pm)_*$, *i.e.*,

$$(\overline{p}_-)_*\colon \mathrm{D}^{\mathrm{b}}(\widehat{X})/\mathcal{K}^{\mathrm{b}} \to \mathrm{D}^{\mathrm{b}}(X_-) \quad \text{and} \quad (\overline{p}_+)_*\colon \mathrm{D}^{\mathrm{b}}(\widehat{X})/\mathcal{K}^{\mathrm{b}} \to \mathrm{D}^{\mathrm{b}}(X_+)$$

and the functors

$$\overline{p}_\pm^*\colon \mathrm{D}^{\mathrm{b}}(X_\pm) \xrightarrow{\pi p_\pm^*} \mathrm{D}^{\mathrm{b}}(\widehat{X})/\mathcal{K}^{\mathrm{b}} \quad \text{and} \quad \overline{p}_\pm^\times\colon \mathrm{D}^{\mathrm{b}}(X_\pm) \xrightarrow{\pi p_\pm^\times} \mathrm{D}^{\mathrm{b}}(\widehat{X})/\mathcal{K}^{\mathrm{b}}.$$

By Lemma 4.1.6, we have for any $E, F \in \mathrm{D}^{\mathrm{b}}(X_\pm)$

$$\mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}(\widehat{X})}(p_\pm^*(E), F) \simeq \mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}(\widehat{X})/\mathcal{K}^{\mathrm{b}}}(\overline{p}_\pm^*(E), \pi(F)).$$

[2] The functors $p_\pm^\times$ coincide with $p_\pm^!$ because $p_\pm$ are proper, but to be consistent with the notation of [Nee18b], we denote them by $p_\pm^\times$.

Therefore, fully faithfulness of $p_\pm^*$ implies fully faithfulness of $\overline{p}_\pm^*$, and the adjunction $p_\pm^* \dashv (p_\pm)_*$ induces an adjunction $\overline{p}_\pm^* \dashv (\overline{p}_\pm)_*$. The following lemma shows that the same holds true for $\overline{p}_\pm^\times$, *i.e.*, they are fully faithful, and we have an adjunction $(\overline{p}_\pm)_* \dashv \overline{p}_\pm^\times$.

Lemma 4.2.1. *Let $\mathcal{A}$ be a triangulated category and $\mathcal{S} \subset \mathcal{A}$ be a thick subcategory. Then, for any $E \in \mathcal{S}^\perp$ and any $F \in \mathcal{A}$, the quotient functor $\pi\colon \mathcal{A} \to \mathcal{A}/\mathcal{S}$ induces an isomorphism*

$$\mathrm{Hom}_{\mathcal{A}}(F, E) \simeq \mathrm{Hom}_{\mathcal{A}/\mathcal{S}}(\pi(F), \pi(E))$$

Proof. This can be proved similarly to Lemma 4.1.6. □

Therefore, the subcategories $\overline{p}_-^*\mathrm{D^b}(X_-)$ and $\overline{p}_+^*\mathrm{D^b}(X_+)$ are left admissible in the quotient category $\mathrm{D^b}(\widehat{X})/\mathcal{K}^{\mathrm{b}}$, while the subcategories $\overline{p}_-^\times\mathrm{D^b}(X_-)$ and $\overline{p}_+^\times\mathrm{D^b}(X_+)$ are right admissible in $\mathrm{D^b}(\widehat{X})/\mathcal{K}^{\mathrm{b}}$. Hence, by Lemma 2.3.24 we have the following SODs

$$\begin{aligned} \mathrm{D^b}(\widehat{X})/\mathcal{K}^{\mathrm{b}} &= \langle \ker(\overline{p}_-)_*, \overline{p}_-^*\mathrm{D^b}(X_-)\rangle = \langle \overline{p}_+^\times\mathrm{D^b}(X_+), \ker(\overline{p}_+)_*\rangle \\ &= \langle \ker(\overline{p}_+)_*, \overline{p}_+^*\mathrm{D^b}(X_+)\rangle = \langle \overline{p}_-^\times\mathrm{D^b}(X_-), \ker(\overline{p}_-)_*\rangle. \end{aligned} \tag{4.10}$$

Theorem 4.2.2. *Assume that $X_- \xleftarrow{p_-} \widehat{X} \xrightarrow{p_+} X_+$ induces a flop-flop diagram and that p_- and p_+ are proper and of finite Tor dimension. Then, we have a four periodic SOD*

$$\begin{aligned} \mathrm{D^b}(\widehat{X})/\mathcal{K}^{\mathrm{b}} &= \langle \ker(\overline{p}_-)_*, \overline{p}_-^*\mathrm{D^b}(X_-)\rangle = \langle \overline{p}_-^*\mathrm{D^b}(X_-), \ker(\overline{p}_+)_*\rangle \\ &= \langle \ker(\overline{p}_+)_*, \overline{p}_+^*\mathrm{D^b}(X_+)\rangle = \langle \overline{p}_+^*\mathrm{D^b}(X_+), \ker(\overline{p}_-)_*\rangle \end{aligned}$$

and the functors

$$\Psi_-^{\mathrm{b}} = (\overline{p}_-)_*\colon \ker(\overline{p}_+)_* \to \mathrm{D^b}(X_-) \quad \text{and} \quad \Psi_+^{\mathrm{b}} = (\overline{p}_+)_*\colon \ker(\overline{p}_-)_* \to \mathrm{D^b}(X_+)$$

are conservative spherical functors such that $T_{\Psi_\pm^{\mathrm{b}}}^{-1} = \Phi_\pm\Phi_\mp \in \mathrm{Aut}(\mathrm{D^b}(X_\pm))$.

Proof. We will prove that $\overline{p}_-^*\mathrm{D^b}(X_-) = \overline{p}_+^\times\mathrm{D^b}(X_+)$ and that $\overline{p}_+^*\mathrm{D^b}(X_+) = \overline{p}_-^\times\mathrm{D^b}(X_-)$ as subcategories of $\mathrm{D^b}(\widehat{X})/\mathcal{K}^{\mathrm{b}}$. Then, substituting in (4.10) we get the four periodic SOD in the statement of the theorem. Once we have the four periodic SOD, the statement about the spherical functors follows from Lemma 2.5.7.

We prove $\overline{p}_-^*\mathrm{D^b}(X_-) = \overline{p}_+^\times\mathrm{D^b}(X_+)$, the other equality being analogous. By (4.9) and the adjunction $(\overline{p}_+)_* \dashv \overline{p}_+^\times$, we have the SODs

$$\mathrm{D_{qc}}(\widehat{X})/\mathcal{K} = \langle \overline{p}_-^*\mathrm{D_{qc}}(X_-), \ker(\overline{p}_+)_*\rangle = \langle \overline{p}_+^\times\mathrm{D_{qc}}(X_+), \ker(\overline{p}_+)_*\rangle,$$

where the functors are considered between unbounded derived categories of quasi-coherent sheaves. Therefore, we have $\overline{p}_-^*\mathrm{D_{qc}}(X_-) = \overline{p}_+^\times\mathrm{D_{qc}}(X_+)$ as subcategories of $\mathrm{D_{qc}}(\widehat{X})/\mathcal{K}$.

Hence, for every $E_- \in \mathrm{D^b}(X_-)$ there exists $E_+ \in \mathrm{D_{qc}}(X_+)$ and an isomorphism $\overline{p}_-^*(E_-) \simeq \overline{p}_+^\times(E_+)$ in $\mathrm{D_{qc}}(\widehat{X})/\mathcal{K}$.

As $\overline{p}_+^\times$ is fully faithful, applying $(\overline{p}_+)_*$ to this isomorphism we get $E_+ \simeq (\overline{p}_+)_*\overline{p}_-^*(E_-) \in \mathrm{D^b}(X_+)$. Hence, for every $E_- \in \mathrm{D^b}(X_-)$ there exists $E_+ \in \mathrm{D^b}(X_+)$ such that $\overline{p}_-^*(E_-) \simeq \overline{p}_+^\times(E_+)$ in $\mathrm{D_{qc}}(\widehat{X})/\mathcal{K}$.

By Lemma 4.1.6, the isomorphism $\overline{p}_-^*(E_-) \simeq \overline{p}_+^\times(E_+)$ is the image via $\pi\colon \mathrm{D_{qc}}(\widehat{X}) \to \mathrm{D_{qc}}(\widehat{X})/\mathcal{K}$ of a morphism $f\colon p_-^*(E_-) \to p_+^\times(E_+)$ such that $\mathrm{cone}(f) \in \mathcal{K}$. However, $p_-^*(E_-)$ and $p_+^\times(E_+)$ are cohomologically bounded complexes with coherent cohomology, thus $\mathrm{cone}(f)$ is too. This means that $\pi(f)$ is an isomorphism in $\mathrm{D^b}(\widehat{X})/\mathcal{K}^\mathrm{b}$. Hence, we have $\overline{p}_-^*\mathrm{D^b}(X_-) \subset \overline{p}_+^\times\mathrm{D^b}(X_+)$, and similarly one proves the other containment. □

Remark 4.2.3. The biggest drawback of the above theorem is that we can prove it only at the level of quotient categories. It is not clear whether in general the subcategory $\mathcal{K}^\mathrm{b}$ is left admissible in $\mathrm{D^b}(\widehat{X})$, and therefore whether the SODs of Theorem 4.2.2 come from SODs of $\mathrm{D^b}(\widehat{X})$ (as it happens in Theorem 4.1.3).

Remark 4.2.4. It is worth spending some time explaining what is the relationship between the spherical functors of Theorem 4.2.2 and the ones obtained by applying Corollary 4.1.7 to the flop-flop diagram induced by $X_- \xleftarrow{p_-} \widehat{X} \xrightarrow{p_+} X_+$.

Let us focus on the autoequivalence $\Phi_+\Phi_- = (p_+)_*p_-^*(p_-)_*p_+^*$. Corollary 4.1.7 and Theorem 4.2.2 construct for us three spherical functors all of which have twist equal to $\Phi_+\Phi_-$. However, these functors have different source and target categories, and in this remark we want to explain what is the relationship among them.

As both $\mathrm{D_{qc}}(\widehat{X})/\mathcal{K}$ and $\mathrm{D^b}(\widehat{X})/\mathcal{K}^\mathrm{b}$ will play a role in what follows, we introduce some auxiliary notation: we write $\ker(\overline{p}_-)_*$ and $\ker(\overline{p}_-)_*^\mathrm{b}$ for the kernel of the functor induced by $(p_-)_*$ on $\mathrm{D_{qc}}(\widehat{X})/\mathcal{K}$ and $\mathrm{D^b}(\widehat{X})/\mathcal{K}^\mathrm{b}$, respectively. The category $\ker(\overline{p}_-)_*^\mathrm{c}$ is the subcategory of compact objects of $\ker(\overline{p}_-)_*$.

Applying Corollary 4.1.7 to the flop-flop diagram (4.8), we obtain the spherical functors

$$\overline{\Psi}_+\colon \ker(\overline{p}_-)_* \to \mathrm{D_{qc}}(X_+) \quad \text{and} \quad \overline{\Psi}_+\colon \ker(\overline{p}_-)_*^\mathrm{c} \to \mathrm{D_{qc}}(X_+)^\mathrm{c}.$$

On the other hand, Theorem 4.2.2 tells us that we have a spherical functor

$$\Psi_+^\mathrm{b}\colon \ker(\overline{p}_-)_*^\mathrm{b} \to \mathrm{D^b}(X_+).$$

In general, these spherical functors only fit into a commutative diagram

$$\begin{array}{ccccc} \ker(\overline{p}_-)_*^\mathrm{c} & \longrightarrow & \ker(\overline{p}_-)_*^\mathrm{b} & \longrightarrow & \ker(\overline{p}_-)_* \\ \downarrow{\scriptstyle \overline{\Psi}_+} & & \downarrow{\scriptstyle \Psi_+^\mathrm{b}} & & \downarrow{\scriptstyle \overline{\Psi}_+} \\ \mathrm{D_{qc}}(X_+)^\mathrm{c} & \hookrightarrow & \mathrm{D^b}(X_+) & \hookrightarrow & \mathrm{D_{qc}}(X_+). \end{array} \tag{4.11}$$

As a guiding example, the reader can think of Mukai flops in dimension 2. In § 4.4.2 we prove that the top row of (4.11) in this case takes the form

$$\mathrm{D}(k[\varepsilon]/\varepsilon^2)^{\mathrm{c}} \hookrightarrow \mathrm{D}^{\mathrm{b}}(k[\varepsilon]/\varepsilon^2) \hookrightarrow \mathrm{D}(k[\varepsilon]/\varepsilon^2) \tag{4.12}$$

where $\deg(\varepsilon) = -1$ and $\mathrm{D}^{\mathrm{b}}(k[\varepsilon]/\varepsilon^2)$ is the triangulated subcategory of $\mathrm{D}(k[\varepsilon]/\varepsilon^2)$ generated by k seen as a trivial $k[\varepsilon]/\varepsilon^2$-dg-module.

The functors in the top row of the diagram (4.11) are induced by the inclusions $\mathrm{D}_{\mathrm{qc}}(\widehat{X})^{\mathrm{c}} \hookrightarrow \mathrm{D}^{\mathrm{b}}(\widehat{X}) \hookrightarrow \mathrm{D}_{\mathrm{qc}}(\widehat{X})$. However, notice that while it is obvious that $\ker(\overline{p}_-)_*^{\mathrm{b}}$ maps to $\ker(\overline{p}_-)_*$, the fact that $\ker(\overline{p}_-)_*^{\mathrm{c}}$ maps to $\ker(\overline{p}_-)_*^{\mathrm{b}}$ follows from the equivalence $\ker(\overline{p}_-)_*^{\mathrm{c}} \simeq \mathcal{S}_+^{\mathrm{c}}$ and the fact, proved in the proof of Theorem 4.1.3, that $\mathcal{S}_+$ is generated by an object which is compact in $\mathrm{D}_{\mathrm{qc}}(\widehat{X})$, and thus, $\mathcal{S}_+^{\mathrm{c}} = \mathcal{S}_+ \cap \mathrm{D}_{\mathrm{qc}}(\widehat{X})^{\mathrm{c}} \subset \mathrm{D}^{\mathrm{b}}(\widehat{X})$.

In the general case, we cannot say much about the functor $\ker(\overline{p}_-)_*^{\mathrm{b}} \to \ker(\overline{p}_-)_*$. However, we will see in § 4.3 that if the fibres of p_- and p_+ have dimension at most one, then this functor is fully faithful.

On the other hand, the functor $\ker(\overline{p}_-)_*^{\mathrm{c}} \to \ker(\overline{p}_-)_*^{\mathrm{b}}$ is always fully faithful. To see this, notice that this functor is constructed as

$$\ker(\overline{p}_-)_*^{\mathrm{c}} \xrightarrow{\pi^{-1}} \mathcal{S}_+^{\mathrm{c}} \hookrightarrow \mathrm{D}^{\mathrm{b}}(\widehat{X}) \xrightarrow{\pi} \mathrm{D}^{\mathrm{b}}(\widehat{X})/\mathcal{K}^{\mathrm{b}}$$

and that the functor $\pi\colon \mathcal{S}_+^{\mathrm{c}} \to \mathrm{D}^{\mathrm{b}}(\widehat{X})/\mathcal{K}^{\mathrm{b}}$ (which lands in $\ker(\overline{p}_-)_*^{\mathrm{b}}$) is fully faithful by the definition of $\mathcal{S}_+$ and Lemma 4.1.6.

The example of Mukai flops (4.12) shows that we cannot expect $\ker(\overline{p}_-)_*^{\mathrm{c}} \hookrightarrow \ker(\overline{p}_-)_*^{\mathrm{b}}$ to be essentially surjective in general. Indeed, $\mathrm{D}^{\mathrm{b}}(k[\varepsilon]/\varepsilon^2)$ strictly contains $\mathrm{D}(k[\varepsilon]/\varepsilon^2)^{\mathrm{c}}$ because $k \in \mathrm{D}(k[\varepsilon]/\varepsilon^2)$ is not a perfect $k[\varepsilon]/\varepsilon^2$-dg-module. To see this, notice that $\mathrm{RHom}_{k[\varepsilon]/\varepsilon^2}(k,k) \simeq k[q]$, $\deg(q) = 2$, is infinite dimensional, while $k[\varepsilon]/\varepsilon^2$ is proper.

Remark 4.2.5. In the previous remark, we explained the relationship between the various spherical functors that one can obtain by applying Corollary 4.1.7 and Theorem 4.2.2 to the flop-flop diagram (4.8).

In this remark we want to stress that, if X_- and X_+ are smooth, then $\mathrm{D}^{\mathrm{b}}(X_\pm) = \mathrm{D}_{\mathrm{qc}}(X_\pm)^{\mathrm{c}}$, and therefore, regardless of putting further assumptions on p_- and p_+, we can always realise the flop-flop autoequivalence of $\mathrm{D}^{\mathrm{b}}(X_\pm)$ associated to the equivalences (4.7) by restricting the spherical functors of Theorem 4.1.3 to compact objects.

This approach, when pursuable, is better suited for computations for two reasons. First, because it is easier to compute morphisms in $\mathrm{D}_{\mathrm{qc}}(\widehat{X})$ rather than in $\mathrm{D}^{\mathrm{b}}(\widehat{X})/\mathcal{K}^{\mathrm{b}}$. Second, because we have generators for $\mathcal{S}_-$ and $\mathcal{S}_+$, whereas we do not have them in general for $\ker(\overline{p}_-)_*, \ker(\overline{p}_+)_* \subset \mathrm{D}^{\mathrm{b}}(\widehat{X})/\mathcal{K}^{\mathrm{b}}$.

4.3 Fibres of dimension at most one

In this subsection, we keep employing the notation we introduced in § 4.2.

Our aim is to compare our work to [BB15]. For this reason, in this subsection we make the following

Assumption. p_- and p_+ are proper, of finite Tor dimension, and have fibres of dimension at most one.

The following lemma is well known.

Lemma 4.3.1 ([Bri02, Lemma 3.1]). *An object $K \in \mathrm{D}_{\mathrm{qc}}(\widehat{X})$ is in $\mathcal{K}$ if and only if its cohomology sheaves are.*

Proof. Our assumptions on $p_\pm$ imply, by [Sta18, Tag 08D5], that for any $E \in \mathrm{D}_{\mathrm{qc}}(\widehat{X})$ to compute $\mathcal{H}^i((p_\pm)_*E)$ we can assume $E \in \mathrm{D}^+_{\mathrm{qc}}(\widehat{X})$. Then, we have a convergent, second page spectral sequence $\mathcal{H}^i((p_\pm)_*\mathcal{H}^j(E)) \implies \mathcal{H}^{i+j}((p_\pm)_*E)$ that degenerates at page 2 and the statement follows. □

In [BB15, Lemma 5.5] Bodzenta and Bondal use the above lemma to prove that $\mathrm{D}^{\mathrm{b}}(\widehat{X})/\mathcal{K}^{\mathrm{b}}$ is a full subcategory of $\mathrm{D}^-(\widehat{X})/\mathcal{K}^-$, $\mathcal{K}^- = \mathcal{K} \cap \mathrm{D}^-(\widehat{X})$. We claim that the same argument proves that $\mathrm{D}^-(\widehat{X})/\mathcal{K}^-$ is a full subcategory of $\mathrm{D}_{\mathrm{qc}}(\widehat{X})/\mathcal{K}$. Indeed, the proof of [BB15, Lemma 5.5] carries on *verbatim* replacing the functors of truncation above with those of truncation below, and choosing $l \in \mathbb{Z}$ such that all the objects appearing in the relevant diagrams belong to $\mathrm{D}^{\leq l}_{\mathrm{qc}}$ rather than to $\mathrm{D}^{\geq l}_{\mathrm{qc}}$. Thus, it follows that $\mathrm{D}^{\mathrm{b}}(\widehat{X})/\mathcal{K}^{\mathrm{b}}$ is a full subcategory of $\mathrm{D}_{\mathrm{qc}}(\widehat{X})/\mathcal{K}$.

For the convenience of the reader, we recall that applying Corollary 4.1.7 to the flop-flop diagram (4.8) we obtain the SODs

$$\begin{aligned} \mathrm{D}_{\mathrm{qc}}(\widehat{X})/\mathcal{K} &= \langle \ker(\overline{p}_-)_*, \overline{p}_-^*\mathrm{D}_{\mathrm{qc}}(X_-)\rangle = \langle \overline{p}_-^*\mathrm{D}_{\mathrm{qc}}(X_-), \ker(\overline{p}_+)_*\rangle \\ &= \langle \ker(\overline{p}_+)_*, \overline{p}_+^*\mathrm{D}_{\mathrm{qc}}(X_+)\rangle = \langle \overline{p}_+^*\mathrm{D}_{\mathrm{qc}}(X_+), \ker(\overline{p}_-)_*\rangle. \end{aligned} \tag{4.13}$$

Given that $\mathrm{D}^{\mathrm{b}}(\widehat{X})/\mathcal{K}^{\mathrm{b}} \subset \mathrm{D}_{\mathrm{qc}}(\widehat{X})/\mathcal{K}$ is a full subcategory, it makes sense to ask whether the SODs (4.13) induce the SODs of Theorem 4.2.2 (see Definition 2.3.15 for the definition of an induced SOD). The answer is yes, as the following theorem shows.

Theorem 4.3.2. *Assume that $X_- \xleftarrow{p_-} \widehat{X} \xrightarrow{p_+} X_+$ induces a flop-flop diagram and that p_- and p_+ are proper, of finite Tor dimension, and with fibres of dimension at most one. Then, the SODs (4.13) induce the SODs of Theorem 4.2.2.*

Proof. We prove that the SODs

$$\mathrm{D}_{\mathrm{qc}}(\widehat{X})/\mathcal{K} = \langle \ker(\overline{p}_-)_*, \overline{p}_-^*\mathrm{D}_{\mathrm{qc}}(X_-)\rangle = \langle \overline{p}_-^*\mathrm{D}_{\mathrm{qc}}(X_-), \ker(\overline{p}_+)_*\rangle \tag{4.14}$$

induce the SODs

$$\mathrm{D}^{\mathrm{b}}(\widehat{X})/\mathcal{K}^{\mathrm{b}} = \langle \ker(\overline{p}_-)_*, \overline{p}_-^*\mathrm{D}^{\mathrm{b}}(X_-)\rangle = \langle \overline{p}_-^*\mathrm{D}^{\mathrm{b}}(X_-), \ker(\overline{p}_+)_*\rangle.$$

The proof for the other two SODs in (4.13) is similar.

By Definition 2.3.15, we have to prove that the projection functors of the SODs (4.14) preserve the subcategory $\mathrm{D}^{\mathrm{b}}(\widehat{X})/\mathcal{K}^{\mathrm{b}}$. We begin with the SOD

$$\mathrm{D}_{\mathrm{qc}}(\widehat{X})/\mathcal{K} = \langle \ker(\overline{p}_-)_*, \overline{p}_-^*\mathrm{D}_{\mathrm{qc}}(X_-)\rangle. \tag{4.15}$$

Take $\overline{E} = \pi(E) \in \mathrm{D}^{\mathrm{b}}(\widehat{X})/\mathcal{K}^{\mathrm{b}}$. Then, Lemma 4.1.6 shows that the adjunction counit for the adjoint pair $\overline{p}_-^* \dashv (\overline{p}_-)_*$ is the image, via the quotient functor $\pi\colon \mathrm{D}_{\mathrm{qc}}(\widehat{X}) \to \mathrm{D}_{\mathrm{qc}}(\widehat{X})/\mathcal{K}$, of the adjunction counit for the adjoint pair $p_-^* \dashv (p_-)_*$. Therefore, we have an isomorphism between the following distinguished triangles

$$\overline{p}_-^*(\overline{p}_-)_*(\overline{E}) \to \overline{E} \to \overline{E}' \quad \simeq \quad \pi\left(p_-^*(p_-)_*(E) \to E \to E'\right).$$

As p_- is proper and of finite Tor dimension, the fact that $E \in \mathrm{D}^{\mathrm{b}}(\widehat{X})$ implies that $p_-^*(p_-)_*(E) \in \mathrm{D}^{\mathrm{b}}(\widehat{X})$, and thus that $E' \in \mathrm{D}^{\mathrm{b}}(\widehat{X})$. Therefore, we get

$$\overline{p}_-^*(\overline{p}_-)_*(\overline{E}) \simeq \pi(p_-^*(p_-)_*(E)) \in \mathrm{D}^{\mathrm{b}}(\widehat{X})/\mathcal{K}^{\mathrm{b}} \quad i_{\ker(\overline{p}_-)_*} i^L_{\ker(\overline{p}_-)_*}(\overline{E}') \simeq \pi(E') \in \mathrm{D}^{\mathrm{b}}(\widehat{X})/\mathcal{K}^{\mathrm{b}},$$

which means that the projection functors of the SOD (4.15) preserve the subcategory $\mathrm{D}^{\mathrm{b}}(\widehat{X})/\mathcal{K}^{\mathrm{b}} \subset \mathrm{D}_{\mathrm{qc}}(\widehat{X})/\mathcal{K}$.

Now we prove that the projection functors of the SOD

$$\mathrm{D}_{\mathrm{qc}}(\widehat{X})/\mathcal{K} = \langle \overline{p}_-^*\mathrm{D}_{\mathrm{qc}}(X_-), \ker(\overline{p}_+)_*\rangle \tag{4.16}$$

preserve $\mathrm{D}^{\mathrm{b}}(\widehat{X})/\mathcal{K}^{\mathrm{b}}$. By Corollary 4.1.7 we know that $\overline{p}_+^\times\mathrm{D}_{\mathrm{qc}}(X_+) = \overline{p}_-^*\mathrm{D}_{\mathrm{qc}}(X_-)$ as subcategories of $\mathrm{D}_{\mathrm{qc}}(\widehat{X})/\mathcal{K}$, and by the proof of Theorem 4.2.2 we know that $\overline{p}_+^\times\mathrm{D}^{\mathrm{b}}(X_+) = \overline{p}_-^*\mathrm{D}^{\mathrm{b}}(X_-)$ as subcategories of $\mathrm{D}^{\mathrm{b}}(\widehat{X})/\mathcal{K}^{\mathrm{b}}$. Hence, it is enough to prove that the SOD $\mathrm{D}_{\mathrm{qc}}(\widehat{X})/\mathcal{K} = \langle \overline{p}_+^\times\mathrm{D}_{\mathrm{qc}}(X_+), \ker(\overline{p}_+)_*\rangle$ induces an SOD of $\mathrm{D}^{\mathrm{b}}(\widehat{X})/\mathcal{K}^{\mathrm{b}}$. We prove this statement.

Take $\overline{E} = \pi(E) \in \mathrm{D}^{\mathrm{b}}(\widehat{X})/\mathcal{K}^{\mathrm{b}}$. Then, using Lemma 4.2.1 in place of Lemma 4.1.6, we see that the following distinguished triangles are isomorphic

$$\overline{E}' \to \overline{E} \to \overline{p}_+^\times(\overline{p}_+)_*(\overline{E}) \quad \simeq \quad \pi\left(E' \to E \to p_+^\times(p_+)_*(E)\right).$$

As p_+ is proper and of finite Tor dimension, $E \in \mathrm{D}^{\mathrm{b}}(X_+)$ implies that $p_+^\times(p_+)_*(E) \in$

$\mathrm{D^b}(\widehat{X})$, see [Nee96, Lemma 3.12], and thus also that $E' \in \mathrm{D^b}(\widehat{X})$. Therefore, we get

$$\overline{p}_+^*(\overline{p}_+)_*(\overline{E}) \simeq \pi(p_+^\times(p_+)_*(E)) \in \mathrm{D^b}(\widehat{X})/\mathcal{K}^\mathrm{b} \quad i_{\ker(\overline{p}_+)_*} i^R_{\ker(\overline{p}_+)_*}(\overline{E}') \simeq \pi(E') \in \mathrm{D^b}(\widehat{X})/\mathcal{K}^\mathrm{b},$$

which means that the projection functors of the SOD (4.16) preserve the subcategory $\mathrm{D^b}(\widehat{X})/\mathcal{K}^\mathrm{b} \subset \mathrm{D_{qc}}(\widehat{X})/\mathcal{K}$. □

Remark 4.3.3. We are finally in a position to tie up our results with some of the results in [BB15]. To do so, let us recall the setup of *ibidem* and the relevant results.

In their work Bodzenta and Bondal consider a cartesian diagram

$$\begin{array}{ccccc} & & \widehat{X} & & \\ & \overset{p_-}{\swarrow} & & \overset{p_+}{\searrow} & \\ X_- & & & & X_+ \\ & \underset{f_-}{\searrow} & & \underset{f_+}{\swarrow} & \\ & & Y & & \end{array} \tag{4.17}$$

where the schemes and f_- are subject to various assumptions, f_+ is the flop of f_-, and Y is affine. Among the assumptions that f_- must satisfy there are two that are of particular importance for us: f_- has fibres of dimension at most one and $(f_-)_*\mathcal{O}_{X_-} \simeq \mathcal{O}_Y$.

Under their assumptions, Bodzenta and Bondal prove that the functors $(p_\pm)_*p_\mp^*$ are equivalences [BB15, Corollary 4.23] and that $(p_\pm)_*\mathcal{O}_{\widehat{X}} \simeq \mathcal{O}_{X_\pm}$ [BB15, Remark 4.2], *i.e.*, in our terminology, the upper half of the diagram (4.17) induces a flop-flop diagram. Moreover, they give an explicit construction of the flop-flop autoequivalence as the inverse of the spherical twist around a spherical functor.

Namely, they show that, as f_- has fibres of dimension at most one, the null-category $\mathcal{A}_{f_-} = \{E \in \mathrm{Coh}(X_-) : (f_-)_*E = 0\}$ is abelian [BB15, Lemma 2.1], and it has a projective generator [BB15, Proposition 2.4]. Therefore, one can derive the inclusion $\mathcal{A}_{f_-} \hookrightarrow \mathrm{Coh}(X_-)$ to a functor $\iota\colon \mathrm{D^b}(\mathcal{A}_{f_-}) \to \mathrm{D^b}(X_-)$. In [BB15, Corollary 5.18] Bodzenta and Bondal prove that ι is spherical and that the inverse of the spherical twist around it is the flop-flop autoequivalence.

The connection between ι and $\overline{\Psi}_-$ is provided by [BB15, Proposition 5.11, Theorem 5.17], which show that $\mathrm{D^b}(\mathcal{A}_{f_-}) \simeq \ker(\overline{p}_+)_* \subset \mathrm{D^b}(\widehat{X})/\mathcal{K}^\mathrm{b}$, and by [BB15, Lemma 5.10], which proves that under this equivalence ι is identified with $\overline{\Psi}_-$. Therefore, up to equivalences, in this setup Theorem 4.2.2 and Bodzenta–Bondal's construction produce the same functor.

Let us conclude this remark by noticing that when $Y = \operatorname{Spec} R$ for R a complete local k-algebra, Bodzenta and Bondal prove that the endomorphism algebra of the projective generator of $\mathcal{A}_{f_-}$ is isomorphic to the contraction algebra A_{con} as defined in [DW13], see

[BB15, Theorem 6.2]. Therefore, in this situation we get equivalences

$$\mathrm{D}^{\mathrm{b}}_{\mathrm{f.g.}}(A_{\mathrm{con}}) \simeq \mathrm{D}^{\mathrm{b}}(\mathcal{A}_{f_-}) \simeq \ker(\overline{p}_+)_* \quad \text{and} \quad \mathrm{D}(A_{\mathrm{con}}) \simeq \ker(\overline{p}_+)_*$$

where f.g. means *finitely generated* modules and the first copy of $\ker(\overline{p}_+)_*$ is considered as a subcategory of $\mathrm{D}^{\mathrm{b}}(\widehat{X})/\mathcal{K}^{\mathrm{b}}$, while the second as a subcategory of $\mathrm{D}_{\mathrm{qc}}(\widehat{X})/\mathcal{K}$.

4.4 Examples

As we explained in § 1, the research project that brought to Theorem 4.1.3 started from the will of finding geometric examples of glued spherical functors as constructed in Theorem 3.1.4.

In [ADM19], Addington–Donovan–Meachan proved that the flop-flop autoequivalence for standard flops and Mukai flops has a factorisation in terms of inverses of spherical twists around spherical functors. For this reason, it seemed a good idea to study these examples in further detail.

The reader not well-acquainted with the geometry of standard flops and Mukai flops should not worry, we will recall the setup when the time comes. For now, let us outline the approach we will use to tackle both examples.

Theorem 4.1.3 provides us with a spherical functor[3] Ψ_+ whose twist has inverse isomorphic to the flop-flop autoequivalence, and it is natural to guess that, if a glued spherical functor is hiding behind the scene, Ψ_+ should be that functor.

To identify Ψ_+ as a glued spherical functor, we have to check that the source category $\mathcal{S}_+$ of Ψ_+ admits an SOD with gluing functors as the ones described by Theorem 3.1.4, and that Ψ_+ restricted to the components of this SOD restricts to the spherical functors we are supposed to be gluing.

This strategy is easy to explain, but rather difficult to carry out. Let us summarise what the problem is, as this shows once more that passing from $\mathrm{D}^{\mathrm{b}}(\widehat{X})$ to $\mathrm{D}_{\mathrm{qc}}(\widehat{X})$ did indeed make things easier.

The point is that in $\mathcal{S}_+$ it is "easy" to compute morphisms because it is a full subcategory of $\mathrm{D}_{\mathrm{qc}}(\widehat{X})$. However, the *geometric* source category of Ψ_+, the one that naturally arises from the geometry, is really $\ker(\overline{p}_-)_*$.

Thus, one should really look at the structure of $\ker(\overline{p}_-)_*$ rather than at the one of $\mathcal{S}_+$. Fortunately, the quotient functor $\pi\colon \mathrm{D}_{\mathrm{qc}}(\widehat{X}) \to \mathrm{D}_{\mathrm{qc}}(\widehat{X})/\mathcal{K}$ induces an equivalence $\mathcal{S}_+ \simeq \ker(\overline{p}_-)_*$, and we can leverage both strengths at once: the insight on the structure of $\ker(\overline{p}_-)_*$, and the ability to carry out computations in $\mathcal{S}_+$. However, while we can often

[3] The choice of Ψ_+ over Ψ_- is just a matter of convention, and it does not play any role.

easily guess what the SOD should look like for $\ker(\overline{p}_-)_*$, transporting it back to $\mathcal{S}_+$ to check all the necessary properties requires us to compute π^L, which is rather complicated.

On top of this problem, when we want to work with cohomologically bounded complexes an important difference between $\mathrm{D}^{\mathrm{b}}(\widehat{X})$ and $\mathrm{D}_{\mathrm{qc}}(\widehat{X})$ shows up. Namely, that we have no analogue of $\mathcal{S}_+$ in the former, and the only category we can work with is $\ker(\overline{p}_-)_* \subset \mathrm{D}^{\mathrm{b}}(\widehat{X})/\mathcal{K}^{\mathrm{b}}$, which is hard to deal with because we are not always able to compute morphisms in this quotient category.

The problem of computing morphisms in $\mathrm{D}^{\mathrm{b}}(\widehat{X})/\mathcal{K}^{\mathrm{b}}$ already appears in the example of Mukai flops, § 4.4.2, and we are only able to circumvent it because we can use the theory of base change for SODs as developed in [Kuz11], see also the proof of Theorem 4.4.13.

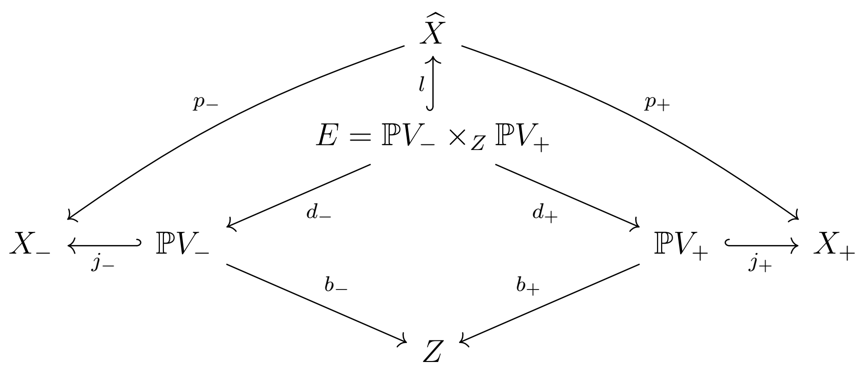
Regardless of these problems, in the examples of standard flops and Mukai flops we are able to match the picture of glued spherical functors with that of the flop-flop autoequivalence. The relevant theorems are Theorem 4.4.1 and Theorem 4.4.13, respectively.

After dealing with the examples of standard flops § 4.4.1 and Mukai flops § 4.4.2, we will briefly survey two more examples: Grassmannian flops and the Abuaf flop, § 4.4.3. In these last examples, we are not able to match the factorisation of the flop-flop autoequivalence with the construction of Theorem 3.1.4, but we will explain what our expectations are.

For the rest of this section, k denotes an algebraically closed field of characteristic zero.

4.4.1 Standard flops

Let Z be a smooth, projective variety and consider V_- and V_+ two locally free sheaves of rank $n+1$ over Z. Let X_- be a smooth, projective variety such that we have a closed embedding $j_- \colon \mathbb{P}V_- \hookrightarrow X_-$ with $N_{\mathbb{P}V_-/X_-} = \mathcal{O}_{\mathbb{P}V_-}(-1) \otimes b_-^* V_+$, where $b_- \colon \mathbb{P}V_- \to Z$ is the projection. Let us assume, as in [Huy06], that we can flop X_- along $j_-(\mathbb{P}V)$. Then, we obtain the following diagram



Here, $b_\pm$ and $d_\pm$ are the projections, i and $j_\pm$ are closed embeddings, the normal bundle of $\mathbb{P}V_-$ and $\mathbb{P}V_+$ in X_- and X_+ are given by

$$N_{\mathbb{P}V_-/X_-} = \mathcal{O}_{\mathbb{P}V_-}(-1) \otimes b_-^* V_+ \quad \text{and} \quad N_{\mathbb{P}V_+/X_+} = \mathcal{O}_{\mathbb{P}V_+}(-1) \otimes b_+^* V_-$$

respectively, and $\widehat{X} = \mathrm{Bl}_{\mathbb{P}V_\pm} X_\pm$.

Notice that $(p_\pm)_* \mathcal{O}_{\widehat{X}} \simeq \mathcal{O}_{X_\pm}$ because $p_\pm$ are blow-ups of smooth projective varieties in smooth subvarieties. Moreover, by [BO95] we have equivalences $(p_\mp)_* p_\pm^* \colon \mathrm{D^b}(X_\pm) \xrightarrow{\simeq} \mathrm{D^b}(X_\mp)$, and by [KL15, Lemma 2.12] we deduce equivalences $(p_\mp)_* p_\pm^* \colon \mathrm{D_{qc}}(X_\pm) \xrightarrow{\simeq} \mathrm{D_{qc}}(X_\mp)$. Therefore, the hypotheses of Theorem 4.1.3 are satisfied. Even more, $p_\pm$ are proper and of finite Tor dimension, hence also Theorem 4.2.2 applies.

We now want to describe the source categories for the spherical functors produced by these theorems. We concentrate on the spherical functor Ψ_+ and the subcategories[4] $\mathcal{S}_+ = {}^\perp\mathcal{K} \cap (p_-^* \mathrm{D_{qc}}(X_-))^\perp$ and $\mathcal{S}_+^{\mathrm{b}} = \mathcal{S}_+ \cap \mathrm{D^b}(\widehat{X})$, but clearly this is not restrictive by the symmetry of the situation.

For $i \in \mathbb{Z}$, consider the functors

$$\alpha_{-i}(-) = (j_+)_*(\mathcal{O}_{\mathbb{P}V_+}(-i) \otimes b_+^*(-)) \colon \mathrm{D_{qc}}(Z) \to \mathrm{D_{qc}}(X_+) \tag{4.18}$$

and write

$$A_{-i} = (b_+ \times j_+)_* \Delta_* \mathcal{O}_{\mathbb{P}V_+}(-i) \in \mathrm{D^b}(Z \times X_+) \tag{4.19}$$

for their Fourier–Mukai kernels. In [ADM19], Addington–Donovan–Meachan prove that the functors α_{-i} are spherical, and that we have an isomorphism

$$(p_+)_* p_-^* (p_-)_* p_+^* \simeq T_{\alpha_{-1}}^{-1} T_{\alpha_{-2}}^{-1} \dots T_{\alpha_{-n}}^{-1}.$$

It follows that we can apply the construction of Theorem 3.1.4 to produce a glued spherical functor whose twist has inverse isomorphic to this flop-flop autoequivalence. On the other hand, we can also use our geometric construction to produce a spherical functor for this autoequivalence. Our next theorem states that these two approaches produce exactly the same spherical functor.

Theorem 4.4.1. *Let us write $\mathcal{S}_+$ and Ψ_+ for the source category and the spherical functor, respectively, obtained by applying Theorem 4.1.3 to the setup of standard flops. Then, the category $\mathcal{S}_+$ has an SOD*

$$\mathcal{S}_+ = \langle \mathrm{D_{qc}}(Z), \mathrm{D_{qc}}(Z), \dots, \mathrm{D_{qc}}(Z) \rangle \tag{4.20}$$

[4] A priori the quotient functor $\pi \colon \mathrm{D_{qc}}(\widehat{X}) \to \mathrm{D_{qc}}(\widehat{X})/\mathcal{K}$ does not identify $\mathcal{S}_+^{\mathrm{b}}$ with the kernel of $(\overline{p}_-)_*$ in $\mathrm{D^b}(\widehat{X})/\mathcal{K}^{\mathrm{b}}$ because we do not know whether the SODs of Theorem 4.1.3 for the flop-flop diagram (4.8) induce SODs of $\mathrm{D^b}(\widehat{X})$. However, for standard flops this will be the case, as Theorem 4.4.1 shows.

where n copies of $\mathrm{D}_{\mathrm{qc}}(Z)$ appear. Moreover, the functor Ψ_+ restricted to the i-th copy of $\mathrm{D}_{\mathrm{qc}}(Z)$ (counting right to left) is identified with α_{-i}, and for any $1 \leq i < j \leq n$ the functor $\alpha^R_{-j}\alpha_{-i}[1]$ is the right gluing functor for the couple formed by the j-th and i-th copy of $\mathrm{D}_{\mathrm{qc}}(Z)$ (counting right to left).

Furthermore, the SODs obtained by applying Theorem 4.1.3 to the setup of standard flops induce SODs of ${}^{\perp}\mathcal{K} \cap \mathrm{D}^{\mathrm{b}}(\widehat{X})$, and thus the category $\mathcal{S}^{\mathrm{b}}_+ = \mathcal{S}_+ \cap \mathrm{D}^{\mathrm{b}}(\widehat{X})$ has an SOD

$$\mathcal{S}^{\mathrm{b}}_+ = \langle \mathrm{D}^{\mathrm{b}}(Z), \mathrm{D}^{\mathrm{b}}(Z), \ldots, \mathrm{D}^{\mathrm{b}}(Z)\rangle \tag{4.21}$$

and the quotient functor $\pi\colon \mathrm{D}_{\mathrm{qc}}(\widehat{X}) \to \mathrm{D}_{\mathrm{qc}}(\widehat{X})/\mathcal{K}$ induces an equivalence $\pi\colon \mathcal{S}^{\mathrm{b}}_+ \xrightarrow{\simeq} \ker(\overline{p}_-)_* \subset \mathrm{D}^{\mathrm{b}}(\widehat{X})/\mathcal{K}^{\mathrm{b}}$.

Remark 4.4.2. In [ADM19], at the end of § 2, Addington–Donovan–Meachan point out that the flop-flop functor should fit into the framework of [HLS16, Theorem 3.11]. The four periodic SOD of Theorem 4.1.3 implements this framework.

We now prepare the ground for the proof of the above theorem. We will proceed in steps. First, we prove that $\mathcal{S}_+$ and $\mathcal{S}^{\mathrm{b}}_+$ have the claimed SODs, Proposition 4.4.5, and then we prove the claim about the gluing functors, Proposition 4.4.6.

Let us define the fully faithful functors

$$\begin{aligned} \iota_{a,b}(-) &:= l_*\left(\mathcal{O}_E(a,b) \otimes d^*_{\pm} b^*_{\pm}(-)\right) : \mathrm{D}_{\mathrm{qc}}(Z) \to \mathrm{D}_{\mathrm{qc}}(\widehat{X}) \qquad a, b \in \mathbb{Z} \\ \upsilon_m(-) &:= l_*\left(\mathcal{O}_E(mE) \otimes d^*_-(-)\right) : \mathrm{D}_{\mathrm{qc}}(\mathbb{P}V_-) \to \mathrm{D}_{\mathrm{qc}}(\widehat{X}) \qquad m \in \mathbb{Z} \end{aligned} \tag{4.22}$$

and notice that by Grothendieck–Verdier duality $\iota_{a,b}$ has a left adjoint given by

$$\iota^L_{a,b}(-) = (b_-)_*(d_-)_*\left(\omega_{E/Z}[\dim E - \dim Z] \otimes (\mathcal{O}_E(-a,-b) \otimes l^*(-))\right)$$

where $\omega_{E/Z} = \omega_E \otimes (d_\pm b_\pm)^* \omega_Z^\vee$. Furthermore, let us recall the definition of the right mutation along $\mathrm{im}(\iota_{a,b})$ as an endofunctor of $\mathrm{D}_{\mathrm{qc}}(\widehat{X})$:

$$\mathbb{R}_{a,b} = \mathrm{cone}\left(\mathrm{id} \to \iota_{a,b}\iota^L_{a,b}\right)[-1] \colon \mathrm{D}_{\mathrm{qc}}(\widehat{X}) \to \mathrm{D}_{\mathrm{qc}}(\widehat{X}).$$

The blow-up formula together with [HL15, Lemma 3.20] tell us that we have an SOD

$$\mathrm{D}_{\mathrm{qc}}(\widehat{X}) = \langle \mathrm{im}(\upsilon_n), \ldots, \mathrm{im}(\upsilon_2), \mathrm{im}(\upsilon_1), p^*_-\mathrm{D}_{\mathrm{qc}}(X_-)\rangle.$$

Moreover, noticing that $\mathcal{O}_E(mE) \simeq \mathcal{O}_{\mathbb{P}V_- \times_Z \mathbb{P}V_+}(-m,-m)$, we can use Orlov's SOD of projective bundles to get that

$$\mathrm{im}(\upsilon_m) = \langle \mathrm{im}(\iota_{-m+a,-m}), \ldots, \mathrm{im}(\iota_{-m+n+a,-m})\rangle \quad a \in \mathbb{Z}.$$

Putting these two things together, we get the SOD

$$\mathrm{D_{qc}}(\widehat{X}) = \langle \mathrm{im}(\iota_{-n,-n}), \dots, \mathrm{im}(\iota_{0,-n}), \dots, \mathrm{im}(\iota_{-n,-1}), \dots, \mathrm{im}(\iota_{0,-1}), p_-^* \mathrm{D_{qc}}(X_-) \rangle \tag{4.23}$$

Set $\mathbb{R}_0 = \mathrm{id}$ and for $m = -1, \dots, -n+1$ define the functors

$$\mathbb{R}_m := \mathbb{R}_{-1,-1}\mathbb{R}_{-2,-1} \dots \mathbb{R}_{-n,-1} \dots \mathbb{R}_{-1,m}\mathbb{R}_{-2,m} \dots \mathbb{R}_{-n,m}.$$

Then, mutating (4.23) (see [BK89] for mutations of SODs), we get the SOD (read each block top to bottom, left to right)

$$\mathrm{D_{qc}}(\widehat{X}) = \langle \underbrace{\begin{matrix} \mathrm{im}(\iota_{-n,-n}), \dots, \mathrm{im}(\iota_{-1,-n}), \\ \mathrm{im}(\iota_{-n,-n+1}), \dots \mathrm{im}(\iota_{-1,-n+1}), \\ \dots \\ \mathrm{im}(\iota_{-n,-1}), \dots, \mathrm{im}(\iota_{-1,-1}), \end{matrix}}_{\mathcal{A}} \underbrace{\begin{matrix} \mathbb{R}_{-n+1}\,\mathrm{im}(\iota_{0,-n}), \\ \mathbb{R}_{-n+2}\,\mathrm{im}(\iota_{0,-n+1}), \\ \dots, \\ \mathbb{R}_{-1}\,\mathrm{im}(\iota_{0,-2}), \\ \mathrm{im}(\iota_{0,-1}), \end{matrix}}_{\mathcal{B}} p_-^*\mathrm{D_{qc}}(X_-) \rangle. \tag{4.24}$$

Remark 4.4.3. A similar SOD exists if we replace $\mathrm{D_{qc}}(-)$ with $\mathrm{D^b}(-)$ because all the varieties appearing are smooth, and both the blow-up formula and Orlov's SOD for projective bundles exist for $\mathrm{D^b}(-)$.

The following lemma is the first step in the proof of Theorem 4.1.3, and it shows that (4.24) is the SOD constructed in Theorem 4.1.3.

Lemma 4.4.4. *In the SOD* (4.24)*, we have* $\mathcal{K} = \mathcal{A}$ *and* $\mathcal{S}_+ = \mathcal{B}$*.*

Proof. It is enough to prove $\mathcal{K} = \mathcal{A}$, as then Theorem 4.1.3 implies

$$\mathcal{B} = {}^\perp\mathcal{K} \cap (p_-^*\mathrm{D_{qc}}(X_-))^\perp = {}^\perp\mathcal{K} \cap \ker(p_-)_* = \mathcal{S}_+.$$

It is clear that $\mathcal{A} \subset \mathcal{K}$, so we only have to show that $\mathcal{K} \subset \mathcal{A}$. Given an object $K \in \mathcal{K}$, its projection to $p_-^*\mathrm{D_{qc}}(X_-)$ is zero. Therefore, K can be decomposed in terms of the remaining subcategories in the SOD (4.24). In other words, there exists a distinguished triangle $B \to K \to A$ where $B \in \mathcal{B}$ and $A \in \mathcal{A}$.

Our aim is to show that $K \in \mathcal{A}$, which is equivalent to say that B is zero. Thus, we can assume we are in the following local situation: Z is affine and such that V_+ is trivial. Hence, $\mathbb{P}V_+ \simeq \mathbb{P}^n \times Z \subset X_+$.

As $B \in \mathcal{B}$, by the definition of an SOD Definition 2.3.1, we know that there exist objects $E_i \in \mathcal{B}$ and maps

$$0 = E_n \to E_{n-1} \to \dots \to E_0 = B \tag{4.25}$$

such that for every $i = 1, \dots, n$ there exists $Z_i \in \mathrm{D}_{\mathrm{qc}}(Z)$ such that

$$\operatorname{cone}(E_i \to E_{i-1}) = \mathbb{R}_{-n+i}\iota_{0,-n+i-1}(Z_i).$$

Now consider $B(1) := B \otimes p_+^* \mathcal{O}_{X_+}(1)$. As $(p_+)_*(B) = 0$, the projection formula tells us that $\Gamma(B(1)) = 0$. We will use this vanishing to prove $Z_1 = 0$.

The object $\Gamma(B(1))$ carries a filtration given by image via Γ of the tensor product of (4.25) with $p_+^*\mathcal{O}_{X_+}(1)$. The graded pieces of this filtration are given by

$$\Gamma(\mathbb{R}_{-n+i}\iota_{0,-n+i-1}(Z_i) \otimes p_+^*\mathcal{O}_{X_+}(1)) \simeq \Gamma(\mathcal{O}_{\mathbb{P}V_+}(-n+i) \otimes b_+^* Z_i) \tag{4.26}$$

for $i = 1, \dots, n$. As $i - n \in \{1-n, \dots, 0\}$, the right hand side of (4.26) vanishes for all i except $i = n$, in which case it is given by $\Gamma(Z_1)$. As all the graded pieces of the filtration of $\Gamma(B(1))$ are zero and $\Gamma(B(1)) = 0$, we get $\Gamma(Z_1) = 0$, and, as Z is affine, $Z_1 = 0$.

The vanishing of Z_1 implies that $E_1 \simeq E_0$, and therefore we can rewrite (4.25) ending with E_1. Then, we proceed inductively, *i.e.*, we tensor with $p_+^*\mathcal{O}_{X_+}(m)$, $m = 2, \dots, n$, and we deduce that $Z_i = 0$ for every i. Therefore, $B = 0$, and $K \in \mathcal{A}$. □

Let us define the subcategories

$$\mathcal{D}_{-j} = \mathbb{R}_{-j+1} \operatorname{im} \iota_{0,-j} \simeq \operatorname{im} \mathbb{R}_{-j+1}\iota_{0,-j} \simeq \mathrm{D}_{\mathrm{qc}}(Z) \quad \text{and} \quad \mathcal{D}^{\mathrm{b}}_{-j} = \mathcal{D}_{-j} \cap \mathrm{D}^{\mathrm{b}}(\widehat{X}) \simeq \mathrm{D}^{\mathrm{b}}(Z).$$

The following proposition bring us one step closer to proving Theorem 4.4.1: it shows that the categories $\mathcal{S}_+$ and $\mathcal{S}_+^{\mathrm{b}}$ have the desired SODs (4.20) and (4.21), respectively.

Proposition 4.4.5. *The subcategory $\mathcal{S}_+$ has an SOD*

$$\mathcal{S}_+ = \langle \mathcal{D}_{-n}, \dots, \mathcal{D}_{-1} \rangle$$

and the functor Ψ_+ restricted to $\mathcal{D}_{-i}$ is identified with α_{-i} for any $i = 1, \dots, n$.

Furthermore, the SODs obtained by applying Theorem 4.1.3 to the setup of standard flops induce SODs of ${}^\perp\mathcal{K} \cap \mathrm{D}^{\mathrm{b}}(\widehat{X})$, and thus the subcategory $\mathcal{S}_+^{\mathrm{b}}$ has an SOD

$$\mathcal{S}_+^{\mathrm{b}} = \langle \mathcal{D}^{\mathrm{b}}_{-n}, \dots, \mathcal{D}^{\mathrm{b}}_{-1} \rangle$$

and the quotient functor $\pi \colon \mathrm{D}_{\mathrm{qc}}(\widehat{X}) \to \mathrm{D}_{\mathrm{qc}}(\widehat{X})/\mathcal{K}$ induces an equivalence $\pi \colon \mathcal{S}_+^{\mathrm{b}} \xrightarrow{\simeq} \ker(\overline{p}_-)_* \subset \mathrm{D}^{\mathrm{b}}(\widehat{X})/\mathcal{K}^{\mathrm{b}}$.

Proof. The SOD of $\mathcal{S}_+$ follows from Lemma 4.4.4, while the statement about Ψ_+ follows from observing that $\mathbb{R}_{-i+1}\iota_{0,-i}$ is related to $\iota_{0,-i}$ by mutations through subcategories contained in $\mathcal{K}$, and therefore $\Psi_+\mathbb{R}_{-i+1}\iota_{0,-i} \simeq \Psi_+\iota_{0,-i} \simeq \alpha_{-i}$.

The statement that the SODs obtained by applying Theorem 4.1.3 to the setup of standard flops induce SODs of ${}^{\perp}\mathcal{K} \cap \mathrm{D^b}(\widehat{X})$ follows from Remark 4.4.3 and [Kuz11, Lemma 3.3].

Finally, once we have the SOD ${}^{\perp}\mathcal{K} \cap \mathrm{D^b}(\widehat{X}) = \langle \mathcal{S}^{\mathrm{b}}_+, p_-^* \mathrm{D^b}(X_-) \rangle$, the equivalence $\pi \colon \mathcal{S}^{\mathrm{b}}_+ \xrightarrow{\simeq} \ker(\overline{p}_-)_* \subset \mathrm{D^b}(\widehat{X})/\mathcal{K}^{\mathrm{b}}$ follows by e.g. Lemma 4.1.6. □

To complete the proof of Theorem 4.4.1, we are now only left to prove that in the SOD $\mathcal{S}_+ = \langle \mathcal{D}_{-n}, \dots, \mathcal{D}_{-1} \rangle$ the right gluing functor for the couple $(\mathcal{D}_{-j}, \mathcal{D}_{-i})$, $1 \leq i < j \leq n$, is given by $\alpha_{-j}^R \alpha_{-i}[1]$.

Let us fix $1 \leq i < j \leq n$. Then, Remark 2.3.6 tells us that the right gluing functor for the couple $(\mathcal{D}_{-j}, \mathcal{D}_{-i})$ is given by $(\mathbb{R}_{-j+1}\iota_{0,-j})^R \mathbb{R}_{-i+1}\iota_{0,-i}[1]$. If we had to identify this functor with $\alpha_{-j}^R \alpha_{-i}[1]$ by hand, it would be complicated. However, both $(\mathbb{R}_{-j+1}\iota_{0,-j})^R \mathbb{R}_{-i+1}\iota_{0,-i}$ and $\alpha_{-j}^R \alpha_{-i}$ are Fourier–Mukai transforms, thus it is enough to show that they have the same kernel.

From now on, we agree on the following: if $E \in \mathrm{D_{qc}}(X_1 \times X_2)$ and $F \in \mathrm{D_{qc}}(X_2 \times X_3)$ are the Fourier–Mukai kernels of the functors $\mathrm{FM}_E \colon \mathrm{D_{qc}}(X_1) \to \mathrm{D_{qc}}(X_2)$, $\mathrm{FM}_F \colon \mathrm{D_{qc}}(X_2) \to \mathrm{D_{qc}}(X_3)$, then we write $FE := (p_{13})_*(p_{12}^*(E) \otimes p_{23}^*(F))$, where $p_{ij} : X_1 \times X_2 \times X_3 \to X_i \times X_j$ are the projections, for the Fourier–Mukai kernel of the functor $\mathrm{FM}_F \mathrm{FM}_E \colon \mathrm{D_{qc}}(X_1) \to \mathrm{D_{qc}}(X_3)$.

We write $R_{a,b}$, $I_{a,b}$, and $I_{a,b}^L$ for the Fourier–Mukai kernels of $\mathbb{R}_{a,b}$, $\iota_{a,b}$, and $\iota_{a,b}^L$, respectively. Explicitly, they are given by[5]

$$I_{a,b} = (b_+ d_+ \times i)_* \Delta_* \mathcal{O}_{\mathbb{P}V_- \times_Z \mathbb{P}V_+}(a,b) \in \mathrm{D^b}(Z \times \widehat{X})$$
$$I_{a,b}^L = I_{a,b}^{\vee} \otimes p_{\widehat{X}}^* \omega_{\widehat{X}}[\dim \widehat{X}] \in \mathrm{D^b}(\widehat{X} \times Z)$$
$$R_{a,b} = \mathrm{cone}\left(\Delta_* \mathcal{O}_{\widehat{X}} \to I_{a,b} I_{a,b}^L\right)[-1] \in \mathrm{D^b}(\widehat{X} \times \widehat{X})$$

where Δ is the diagonal inclusion, and the map in the third line is given by adjunction.

Following the above convention, we write

$$R_{-i} := R_{-1,-1} R_{-2,-1} \dots R_{-n,-1} \dots R_{-1,-i} \dots R_{-n,-i} \in \mathrm{D^b}(\widehat{X} \times \widehat{X})$$

for the Fourier–Mukai kernel of $\mathbb{R}_{-i}$. Then, the functor $(\mathbb{R}_{-j+1}\iota_{0,-j})^R \mathbb{R}_{-i+1}\iota_{0,-i}$ has Fourier–Mukai kernel given by

$$(p_{Z \times Z})_*(p_{\widehat{X} \times Z}^* (R_{-j+1} I_{-j})^{\vee} \otimes p_Z^* \omega_Z \otimes p_{Z \times \widehat{X}}^* R_{-i+1} I_{-i})[\dim Z] \in \mathrm{D^b}(Z \times Z) \tag{4.27}$$

[5] Here we are suppressing the twist morphism $\widehat{X} \times Z \to Z \times \widehat{X}$ and we consider $I_{a,b}^{\vee}$ as a complex on $\widehat{X} \times Z$. We will always suppress the twist morphism in the following. Moreover, we are computing the Fourier–Mukai kernel of a left adjoint functor following [Huy06, Proposition 5.9].

where $p_{\widehat{X}\times Z}$, $p_{Z\times\widehat{X}}$, $p_{Z\times Z}$ and p_Z are the projections from $Z\times\widehat{X}\times Z$ to $\widehat{X}\times Z$, $Z\times\widehat{X}$, $Z\times Z$ and Z, respectively. Similarly, the Fourier–Mukai kernel of $\alpha^R_{-j}\alpha_{-i}$ is given by

$$(p_{Z\times Z})_*(p^*_{X_+\times Z}A^\vee_{-j}\otimes p^*_Z\omega_Z\otimes p^*_{Z\times X_+}A_{-i})[\dim Z]\in \mathrm{D^b}(Z\times Z) \tag{4.28}$$

with the projection maps defined from $Z\times X_+\times Z$ to $Z\times X_+$, $X_+\times Z$, $Z\times Z$ and Z, respectively.

To prove an isomorphism of functors $(\mathbb{R}_{-j+1}\iota_{0,-j})^R\mathbb{R}_{-i+1}\iota_{0,-i}\simeq\alpha^R_{-j}\alpha_{-i}$ it is enough to prove that the Fourier–Mukai kernels (4.27) and (4.28) are isomorphic. By the projection formula, it is enough to prove

$$(p_{Z\times Z})_*(p^*_{\widehat{X}\times Z}(R_{-j+1}I_{-j})^\vee\otimes p^*_{Z\times\widehat{X}}R_{-i+1}I_{-i})\simeq(p_{Z\times Z})_*(p^*_{X_+\times Z}A^\vee_{-j}\otimes p^*_{Z\times X_+}A_{-i}).$$

We prove such isomorphism in the following proposition, thus concluding the proof of Theorem 4.4.1.

Proposition 4.4.6. *For $1\le i<j\le n$ we have an isomorphism*

$$(p_{Z\times Z})_*(p^*_{\widehat{X}\times Z}(R_{-j+1}I_{-j})^\vee\otimes p^*_{Z\times\widehat{X}}R_{-i+1}I_{-i})\simeq(p_{Z\times Z})_*(p^*_{X_+\times Z}A^\vee_{-j}\otimes p^*_{Z\times X_+}A_{-i})$$

and therefore the right gluing functor for the couple $(\mathcal{D}_{-j},\mathcal{D}_{-i})$, $1\le i<j\le n$, in the SOD of Proposition 4.4.5 is given by $\alpha^R_{-j}\alpha_{-i}[1]$.

Proof. We first construct a map

$$(p_{Z\times Z})_*(p^*_{\widehat{X}\times Z}(R_{-j+1}I_{-j})^\vee\otimes p^*_{Z\times\widehat{X}}R_{-i+1}I_{-i})\to(p_{Z\times Z})_*(p^*_{X_+\times Z}A^\vee_{-j}\otimes p^*_{Z\times X_+}A_{-i}) \tag{4.29}$$

that we will then prove to be an isomorphism.

First of all, notice that all as the Fourier–Mukai kernels involved are perfect objects in the respective derived categories, we can rewrite the left hand side of (4.29) as

$$(p_{Z\times Z})_*(\mathrm{R}\mathcal{H}om(p^*_{\widehat{X}\times Z}R_{-j+1}I_{-j},p^*_{Z\times\widehat{X}}R_{-i+1}I_{-i}))$$

and the right hand side as

$$(p_{Z\times Z})_*(\mathrm{R}\mathcal{H}om(p^*_{X_+\times Z}A_{-j},p^*_{Z\times X_+}A_{-i})).$$

Then, using flat base change we get a map

$$\begin{aligned} &(\operatorname{id} \times p_+ \times \operatorname{id})_*(\mathrm{R}\mathcal{H}om(p^*_{\widehat{X}\times Z}R_{-j+1}I_{-j}, p^*_{Z\times\widehat{X}}R_{-i+1}I_{-i})) \\ \to\, &\mathrm{R}\mathcal{H}om((\operatorname{id} \times p_+ \times \operatorname{id})_*p^*_{\widehat{X}\times Z}R_{-j+1}I_{-j}, (\operatorname{id} \times p_+ \times \operatorname{id})_*p^*_{Z\times\widehat{X}}R_{-i+1}I_{-i})) \\ \simeq\, &\mathrm{R}\mathcal{H}om(p^*_{X_+\times Z}(p_+ \times \operatorname{id})_*R_{-j+1}I_{-j}, p^*_{Z\times X_+}(\operatorname{id} \times p_+)_*R_{-i+1}I_{-i})). \end{aligned} \tag{4.30}$$

To get from (4.30) to (4.29) we now prove that we have an isomorphism

$$(\operatorname{id} \times p_+)_*R_{-i+1}I_{0,-i} \simeq A_{-i}$$

for any $1 \leq i \leq n$. Indeed, as by definition $R_{a,b}$ maps to $\Delta_*\mathcal{O}_{\widehat{X}}$ for any a and b, we can construct a map $R_{-i+1}I_{0,-i} \to I_{0,-i}$. The cone of this map is an iterated extension of complexes of the form $I_{a,b}I^L_{a,b}$ with $a, b \in \{-n, \ldots, -1\}$, and therefore it is in the kernel of $(\operatorname{id} \times p_+)_*$, *i.e.*, $(\operatorname{id} \times p_-)_*R_{-i+1}I_{0,-i} \simeq A_{-i}$.

Summing up, we constructed a map

$$(\operatorname{id} \times p_+ \times \operatorname{id})_*(\mathrm{R}\mathcal{H}om(p^*_{\widehat{X}\times Z}R_{-j+1}I_{-j}, p^*_{Z\times\widehat{X}}R_{-i+1}I_{-i})) \to \mathrm{R}\mathcal{H}om(p^*_{X_+\times Z}A_{-j}, p^*_{Z\times X_+}A_{-i}) \tag{4.31}$$

and we set (4.29) to be $(p_{Z\times Z})_*$(4.31).

Our aim is now to prove that (4.29) is an isomorphism. To do so we will use the technique of the deformation to the normal bundle, see [Ful98, Chapter 5].

Let us briefly recall this piece of theory. Given any closed immersion $X \hookrightarrow Y$, we can construct a flat, proper family $f\colon \mathcal{X} \to \mathbb{A}^1$ together with a closed immersion $X \times \mathbb{A}^1 \hookrightarrow \mathcal{X}$ such that for any $t \in \mathbb{A}^1 \setminus \{0\}$ we have $\mathcal{X}_t = \mathcal{X} \times_{\mathbb{A}^1} \{t\} = Y$ with the given embedding, and $\mathcal{X}_0 = \operatorname{Tot}(N_{X/Y})$ with the embedding given by the zero section.

Let us write $\widehat{f}\colon \widehat{\mathcal{X}} \to \mathbb{A}^1$ for the deformation to the normal bundle of $\widehat{X}$, and similarly $f_\pm\colon \mathcal{X}_\pm \to \mathbb{A}^1$ for the deformation to the normal bundle of $X_\pm$. The maps $p_\pm$ lift to maps $p_\pm\colon \widehat{\mathcal{X}} \to \mathcal{X}_\pm$, which remain proper. Moreover, the functors α_{-i}, $\iota_{a,b}$ and $\mathbb{R}_{a,b}$ can be constructed flatly in the family and we write A^{fam}_{-i}, $I^{\mathrm{fam}}_{a,b}$, $R^{\mathrm{fam}}_{a,b}$ for their Fourier–Mukai kernels. Convolving the kernels $R^{\mathrm{fam}}_{a,b}$ we get a family version R^{fam}_{-i} of $\mathbb{R}_{-i}$. Then, it is easy to see that $(\operatorname{id} \times p_\pm)_*R^{\mathrm{fam}}_{-i+1}I^{\mathrm{fam}}_{0,-i} \simeq A^{\mathrm{fam}}_{-i}$. Therefore, we get a map

$$\begin{aligned} &(p_{Z\times\mathbb{A}^1\times Z})_*(\mathrm{R}\mathcal{H}om(p^*_{\widehat{\mathcal{X}}\times Z}R^{\mathrm{fam}}_{-j+1}I^{\mathrm{fam}}_{-j}, p^*_{Z\times\widehat{\mathcal{X}}}R^{\mathrm{fam}}_{-i+1}I_{-i})^{\mathrm{fam}}) \\ \to\, &(p_{Z\times\mathbb{A}^1\times Z})_*(\mathrm{R}\mathcal{H}om(p^*_{\mathcal{X}_+\times Z}A^{\mathrm{fam}}_{-j}, p^*_{Z\times\mathcal{X}_+}A^{\mathrm{fam}}_{-i})) \end{aligned} \tag{4.32}$$

where

$$p_{Z\times\mathbb{A}^1\times Z}\colon (Z \times \mathbb{A}^1) \times_{\mathbb{A}^1} \widehat{\mathcal{X}} \times_{\mathbb{A}^1} (Z \times \mathbb{A}^1) \simeq Z \times \widehat{\mathcal{X}} \times Z \xrightarrow{\operatorname{id}\times\hat{f}\times\operatorname{id}} Z \times \mathbb{A}^1 \times Z.$$

The map (4.32) is a family version of (4.29) and restricts to (4.29) on every fibre different from $0 \in \mathbb{A}^1$.

Let us call S the cone of (4.32). As $(4.32)|_{\mathcal{X}_t} = (4.29)$ for any $t \neq 0$, to conclude the proof of the proposition it is enough to prove that $S_t \simeq 0$ for some $t \in \mathbb{A}^1 \setminus \{0\}$. We claim that this follows if we prove that $S_0 \simeq 0$.

Indeed, let us assume $S_0 \simeq 0$. Then, notice that $p_{Z\times\mathbb{A}^1\times Z}$ is proper because $\widehat{f}$ is. Therefore, as all the Fourier–Mukai kernels involved are perfect objects in the respective derived categories, $S \in \mathrm{D^b}(Z \times \mathbb{A}^1 \times Z)$ and its support is a closed subset of $Z \times \mathbb{A}^1 \times Z$. Assume there existed a point $(z, t, z') \in Z \times \mathbb{A}^1 \times Z$ with $t \neq 0$. Then, as $\widehat{f}$ and $f_\pm$ are trivial outside $0 \in \mathbb{A}^1$ and $\mathrm{Supp}(S)$ is closed, this would mean that $(z, 0, z') \in \mathrm{Supp}(S)$, which is against our assumption.

Hence, if $S_0 \simeq 0$, then $S \simeq 0$, which implies that (4.32) restricts to an isomorphism on every fibre, and in particular on a fibre over $t \neq 0$, which is what we needed to show.

Thus, to conclude the proof of the proposition we are left to show that $S_0 \simeq 0$, *i.e.*, that (4.32) restricts to an isomorphism over 0. We do this in the lemma below. □

Lemma 4.4.7. *With the notation as in the proof of Proposition 4.4.6, we have* $S_0 \simeq 0$.

Proof. Over $0 \in \mathbb{A}^1$ we are in the local situation. Namely, we have $X_+ = \mathrm{Tot}(\mathcal{O}_{\mathbb{P}V_+}(-1) \otimes b_+^* V_-)$ and $\widehat{X} = \mathrm{Tot}(\mathcal{O}_{\mathbb{P}V_- \times_Z \mathbb{P}V_+}(-1, -1))$, and the restriction of (4.32) to this setting is simply the map (4.29) for the local case.

Proving that $S_0 \simeq 0$ is local question in Z. Therefore, we can further assume that Z is affine and that V_- and V_+ are trivial.

In this situation, we have the following description of the Fourier–Mukai kernels for α_{-i}, $\iota_{a,b}$ and $\mathbb{R}_{-j+1}\iota_{0,-j}$, respectively[6]

$$\Delta_*\mathcal{O}_Z \boxtimes \mathcal{O}_{\mathbb{P}^n}(-i) \quad \Delta_*\mathcal{O}_Z \boxtimes \mathcal{O}_{\mathbb{P}^n\times\mathbb{P}^n}(a, b) \quad \Delta_*\mathcal{O}_Z \boxtimes \mathbb{R}_{-j+1}\mathcal{O}_{\mathbb{P}^n\times\mathbb{P}^n}(0, -j).$$

As Z is affine, the derived local homs in (4.29) become derived global homs. Moreover, given the above description of the Fourier–Mukai kernels, these global homs split in a part coming from $Z \times Z$ and a part coming from either $\mathrm{Tot}(\mathcal{O}_{\mathbb{P}^n\times\mathbb{P}^n}(-1, -1))$ or $\mathrm{Tot}(\mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus n+1})$. The parts coming from $Z \times Z$ clearly play no role in the proof of the isomorphism (they are the same on both sides), and therefore it is enough to prove that statement of the lemma for the case $Z = \mathrm{pt}$.

Summing up, we reduced to prove the following statement: the morphism

$$\begin{aligned} &\mathrm{RHom}_{\mathrm{Tot}(\mathcal{O}_{\mathbb{P}^n\times\mathbb{P}^n}(-1,-1))}(\mathbb{R}_{-j+1}\mathcal{O}_{\mathbb{P}^n\times\mathbb{P}^n}(0, -j), \mathbb{R}_{-i+1}\mathcal{O}_{\mathbb{P}^n\times\mathbb{P}^n}(0, -i)) \\ \xrightarrow{(p_+)_*}\ &\mathrm{RHom}_{\mathrm{Tot}(\mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus n+1})}(\mathcal{O}_{\mathbb{P}^n}(-j), \mathcal{O}_{\mathbb{P}^n}(-i)) \end{aligned} \tag{4.33}$$

[6]We suppress the pushforward functors $(j_+)_*$ and l_* to ease the notation.

is an isomorphism for $1 \leq i < j \leq n$.

To prove this statement, we describe the objects $\mathbb{R}_{-j+1}\mathcal{O}_{\mathbb{P}^n\times\mathbb{P}^n}(0,-j)$ in a way that allows us to easily match the two hom spaces appearing in (4.33). For the sake of simplicity we do it for $n = 2$, the general case being analogous.

When $j = 1$, we have $\mathbb{R}_0\mathcal{O}_{\mathbb{P}^n\times\mathbb{P}^n}(0,-1) = \mathcal{O}_{\mathbb{P}^n\times\mathbb{P}^n}(0,-1)$ and there is nothing to do. Let us consider $j = 2$. On $\mathrm{Tot}(\mathcal{O}_{\mathbb{P}^n\times\mathbb{P}^n}(-1,-1))$ we have the following exact sequences

$$\mathcal{O}_{\widehat{X}}(1,1) \xrightarrow{u} \mathcal{O}_{\widehat{X}} \to \mathcal{O}_{\mathbb{P}\times\mathbb{P}} \tag{4.34}$$

$$\mathcal{O}_{\widehat{X}}(-2,-1) \to \mathcal{O}_{\widehat{X}}(-1,-1)^{\oplus 3} \to \mathcal{O}_{\widehat{X}}(0,-1)^{\oplus 3} \to \mathcal{O}_{\widehat{X}}(0,-2) \to \mathcal{O}_{\mathbb{P}\times\mathbb{P}}(0,-2) \tag{4.35}$$

The first one is given by the tautological section $u \in \mathcal{O}_{\mathrm{Tot}(\mathcal{O}_{\mathbb{P}^n\times\mathbb{P}^n}(-1,-1))}(-1,-1)$. The second one is obtained by joining the tensor product of (4.34) with $\mathcal{O}_{\mathrm{Tot}(\mathcal{O}_{\mathbb{P}^n\times\mathbb{P}^n}(-1,-1))}(0,-2)$ and the tensor product of the pull-up of the Koszul complex from the first copy of $\mathbb{P}^n$ with $\mathcal{O}_{\mathrm{Tot}(\mathcal{O}_{\mathbb{P}^n\times\mathbb{P}^n}(-1,-1))}(1,-1)$.

Let us write

$$\pi_\perp = \pi_{p_-^*\mathrm{D}_{\mathrm{qc}}(X_-)^\perp} = \mathrm{cone}(p_-^*(p_-)_* \to \mathrm{id}_{\mathrm{D}_{\mathrm{qc}}(\widehat{X})})$$

for the projection functor to $p_-^*\mathrm{D}_{\mathrm{qc}}(X_-)^\perp$. Then, tensoring the short exact sequence (4.34) with $\mathcal{O}_{\mathrm{Tot}(\mathcal{O}_{\mathbb{P}^n\times\mathbb{P}^n}(-1,-1))}(-1,-1)$, we see that

$$\pi_\perp(\mathcal{O}_{\mathrm{Tot}(\mathcal{O}_{\mathbb{P}^n\times\mathbb{P}^n}(-1,-1))}(-1,-1)) \simeq \mathcal{O}_{\mathbb{P}^n\times\mathbb{P}^n}(-1,-1) \in \mathcal{K}.$$

Similarly, $\pi_\perp(\mathcal{O}_{\mathrm{Tot}(\mathcal{O}_{\mathbb{P}^n\times\mathbb{P}^n}(-1,-1))}(-2,-1)) \in \mathcal{K}$. Therefore, applying $\pi_\perp$ to (4.35) we get the distinguished triangle

$$\begin{aligned}\pi_\perp(\{\mathcal{O}_{\widehat{X}}(0,-1)^{\oplus 3} \to \mathcal{O}_{\widehat{X}}(0,-2)\}) \to \mathcal{O}_{\mathbb{P}\times\mathbb{P}}(0,-2) \to \\ \to \pi_\perp(\{\mathcal{O}_{\widehat{X}}(-2,-1) \to \mathcal{O}_{\widehat{X}}(-1,-1)^{\oplus 3}\})[1]\end{aligned} \tag{4.36}$$

where the last term is in $\mathcal{K}$, and we use the notation $\{-\}$ to denote a complex of coherent sheaves whose rightmost term is in degree 0.

Applying $\mathrm{RHom}_{\mathrm{Tot}(\mathcal{O}_{\mathbb{P}^n\times\mathbb{P}^n}(-1,-1))}(\mathcal{O}_{\mathbb{P}^n\times\mathbb{P}^n}(0,-1),-)$ to the distinguished triangle (4.36), as $\mathcal{K} \subset \mathcal{O}_{\mathbb{P}^n\times\mathbb{P}^n}(0,-1)^\perp$ by (4.23), and

$$\mathrm{RHom}_{\mathrm{Tot}(\mathcal{O}_{\mathbb{P}^n\times\mathbb{P}^n}(-1,-1))}(\mathcal{O}_{\mathbb{P}^n\times\mathbb{P}^n}(0,-1), \mathcal{O}_{\mathbb{P}^n\times\mathbb{P}^n}(0,-2)) \simeq 0,$$

we see that

$$\pi_\perp(\{\mathcal{O}_{\widehat{X}}(0,-1)^{\oplus 3} \to \mathcal{O}_{\widehat{X}}(0,-2)\}) \in \mathcal{O}_{\mathbb{P}^n\times\mathbb{P}^n}(0,-1)^\perp.$$

Therefore,

$$\pi_{\perp}(\{\mathcal{O}_{\widehat{X}}(0,-1)^{\oplus 3} \to \mathcal{O}_{\widehat{X}}(0,-2)\}) \in \langle \mathcal{O}_{\mathbb{P}^n\times\mathbb{P}^n}(0,-1), p_-^*\mathrm{D}_{\mathrm{qc}}(X_-)\rangle^{\perp}.$$

As the projections with respect to an SOD are uniquely defined, the distinguished triangle (4.36) and the SOD (4.24) imply

$$\mathbb{R}_{-1}\mathcal{O}_{\mathbb{P}^n\times\mathbb{P}^n}(0,-2) \simeq \pi_{\perp}(\{\mathcal{O}_{\widehat{X}}(0,-1)^{\oplus 3} \to \mathcal{O}_{\widehat{X}}(0,-2)\}).$$

Using this description of $\mathbb{R}_{-1}\mathcal{O}_{\mathbb{P}^n\times\mathbb{P}^n}(0,-2)$ and the adjunctions $p_{\pm}^* \dashv (p_{\pm})_*$, it is easy to see that (4.33) is an isomorphism.

This completes the proof of the lemma. □

4.4.2 Mukai flops

In this section, we consider the example of *Mukai flops.* We keep employing the notation introduced in § 4.4.1, but we restrict to the case $Z = \mathrm{pt}$. Therefore, we have

$$X_{\pm} = \mathrm{Tot}(\mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus n+1}) \quad \text{and} \quad \widehat{X} = \mathrm{Tot}(\mathcal{O}_{\mathbb{P}^n\times\mathbb{P}^n}(-1,-1)).$$

Remark 4.4.8. The reason why we assume that Z is a point is that we want to avoid complications related to $\mathbb{P}$-functors, see [Add16] and [AL19] for the relevant definitions.

Taking global sections, we see that we have a non-trivial surjection of algebras

$$\mathrm{Sym}_k(\mathrm{Hom}_k(k^n,k^n)) \twoheadrightarrow \mathrm{H}^0(\widehat{X},\mathcal{O}_{\widehat{X}}) \simeq \mathrm{H}^0(X_{\pm},\mathcal{O}_{X_{\pm}})$$

Therefore, there exist canonical maps $g_{\pm}\colon X_{\pm} \to \mathbb{A}^1$, $\hat{g}\colon \widehat{X} \to \mathbb{A}^1$ corresponding to the identity id_{k^n} in the left hand side above. We define

$$W_{\pm} = \{g_{\pm} = 0\} \quad \text{and} \quad \widehat{W} = \{\hat{g} = 0\}.$$

Given the above definitions, we get the following diagram whose top row takes the name of *Mukai flop*

$$\begin{array}{ccccc} W_- & \xleftarrow{q_-} & \widehat{W} & \xrightarrow{q_+} & W_+ \\ {\scriptstyle r_-}\downarrow & & \downarrow{\scriptstyle \hat{r}} & & \downarrow{\scriptstyle r_+} \\ X_- & \xleftarrow[p_-]{} & \widehat{X} & \xrightarrow[p_+]{} & X_+ \end{array} \tag{4.37}$$

Notice that $W_{\pm} \simeq \mathrm{Tot}(\Omega^1_{\mathbb{P}^n})$, which is embedded in $X_{\pm}$ via the Euler exact sequence, and that the equation $\hat{g} = 0$ describes $\widehat{W}$ as a normal crossing divisor in $\widehat{X}$ with two irreducible components: one of them is the blow-up of $W_\pm$ along $\mathbb{P}^n$, that we denote by $\widetilde{W}$, and the other is $\mathbb{P}^n \times \mathbb{P}^n$. These two irreducible components are glued along $\mathbb{P}(\Omega^1_{\mathbb{P}^n})$ that is the exceptional locus of the blow-up and that sits inside $\mathbb{P}^n \times \mathbb{P}^n$ via the inclusion $\Omega^1_{\mathbb{P}^n} \hookrightarrow \mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus n+1}$.

The smooth varieties W_- and W_+ are birational, and furthermore the functors $(q_\pm)_* q_\mp^*$ induce derived equivalences $\mathrm{D^b}(W_\pm) \simeq \mathrm{D^b}(W_\mp)$, see [Nam03], [Kaw02]. These equivalences induce equivalences $\mathrm{D_{qc}}(W_\pm) \simeq \mathrm{D_{qc}}(W_\mp)$ by [KL15, Lemma 2.12], and therefore we can apply Theorem 4.1.3. Even more, we can also apply Theorem 4.2.2 because $q_\pm$ are proper and of finite Tor dimension.

As in § 4.4.1, from now on we focus on the description of the spherical functor Ψ_+ and of the categories $\mathcal{S}_+ \subset \mathrm{D_{qc}}(\widehat{W})$, $\mathcal{S}_+^{\mathrm{b}} = \mathcal{S}_+ \cap \mathrm{D^b}(\widehat{W})$, but this is not restrictive by the symmetry of the situation.

Remark 4.4.9. Two remarks are in order here. First, the notation $\mathcal{S}_+$ and $\mathcal{S}_+^{\mathrm{b}}$ will not be ambiguous because even though the varieties $\widehat{X}$ and $X_\pm$ appear in this subsection, the source categories for the spherical functors obtained in § 4.4.1 do not. Second, as it was the case for standard flops, a priori it is not clear that $\mathcal{S}_+^{\mathrm{b}}$ is the correct source category for the spherical functor obtained from Theorem 4.2.2, but Theorem 4.4.13 will prove it is.

In [ADM19], Addington–Donovan–Meachan describe a factorisation of $(q_+)_* q_-^* (q_-)_* q_+^*$ in terms of $\mathbb{P}$-twists around $\mathbb{P}$-objects. We now briefly recall their result and rephrase it in terms of spherical functors.

We have talked about $\mathbb{P}$-objects in the framework of dg-categories in § 3.5. For the convenience of the reader, and to fix the notation, we now quickly survey the concept once more.

$\mathbb{P}$-objects were introduced by Huybrechts–Thomas in [HT06]. Given a smooth, projective variety Y of dimension $2n$, an object $P \in \mathrm{D^b}(Y)$ is called a $\mathbb{P}^n$-object if $P \otimes \omega_Y \simeq P$ and

$$\mathrm{Hom}^\bullet_{\mathrm{D^b}(Y)}(P, P) \simeq \mathrm{H}^\bullet(\mathbb{P}^n, k)$$

as graded algebras. To each such object, Huybrechts and Thomas associate an autoequivalence of $\mathrm{D^b}(Y)$ called the $\mathbb{P}$-twist around P. What Addington–Donovan–Meachan prove is that the objects $\mathcal{O}_{\mathbb{P}^n}(-j) \in \mathrm{D^b}(W_+)$ are $\mathbb{P}^n$-objects for any $j \in \mathbb{Z}$, and that the flop-flop autoequivalence $(q_+)_* q_-^* (q_-)_* q_+^*$ factorises as the composition of inverses of $\mathbb{P}$-twists.

From our perspective, it will be more useful to interpret $\mathbb{P}$-twists as spherical twists according to the construction proposed by Segal in [Seg18] and that we explained in § 3.5.

Let us recall this construction. In [Seg18], Segal notices that the $\mathbb{P}$-object P carries an action of the dg-algebra $k[q]$, where $\deg(q) = 2$. Therefore, if we call t the generator of $\mathrm{Hom}^\bullet_{\mathrm{D^b}(Y)}(P,P)$, by Koszul duality the object

$$\tilde{P} := \mathrm{cone}(t\colon P \to P[2]) \tag{4.38}$$

carries an action of the dg-algebra $k[\varepsilon]/\varepsilon^2$, where $\deg(\varepsilon) = -1$.

Then, Segal proves [Seg18, Proposition 4.2] that the functor

$$\beta_P(-) = - \otimes_{k[\varepsilon]/\varepsilon^2} \tilde{P}\colon \mathrm{D}(k[\varepsilon]/\varepsilon^2) \to \mathrm{D_{qc}}(Y)$$

is spherical and that T_{β_P} is isomorphic to the $\mathbb{P}$-twist around P. With this notation, [ADM19, Theorem B] can be restated saying that we have an isomorphism of functors

$$(q_+)_* q_-^* (q_-)_* q_+^* \simeq T^{-1}_{\beta_{\mathcal{O}_{\mathbb{P}^n}(-1)}} \circ \cdots \circ T^{-1}_{\beta_{\mathcal{O}_{\mathbb{P}^n}(-n)}}. \tag{4.39}$$

Remark 4.4.10. In the description above we have been consciously imprecise because we believe that conveying the basic idea is more important that spelling out every single detail. However, for the proof of Theorem 4.4.13 we will need to perform some choices, and therefore in this remark we explain all the technicalities we hid above.

Given a $\mathbb{P}^n$-object P by definition we have an isomorphism $\mathrm{Hom}^\bullet_{\mathrm{D^b}(Y)}(P,P) \simeq k[t]/t^{n+1}$ as graded algebras, where $\deg(t) = 2$. To define a $k[q]$-module structure on P we fix an h-injective[7] resolution I_P of P. Then, $\mathrm{Hom}_{\widehat{X}}(I_P, I_P)$ is a dg-algebra whose cohomology is given by $\mathrm{Hom}^\bullet_{\mathrm{D^b}(Y)}(P,P)$, and we can fix a lift $t_P\colon I_P \to I_P[2]$ of t so that we make q act on I_P as t_P.

The next step in giving a formal definition of β_P is to fix a lift of $\tilde{P}$. Let us consider the cone of t_P in the category of complexes of $\mathcal{O}_Y$-modules. Namely, we consider

$$\tilde{I}_P = \mathrm{cone}(t_P) = I_P[1] \oplus I_P[2]$$

with the differential given by the cone construction. Then, we have a closed, degree zero morphism $\tilde{I}_P \to \tilde{I}_P[-1]$ given by

$$\tilde{I}_P = I_P[1] \oplus I_P[2] \ni (a,b) \mapsto (0,a) \in I_P \oplus I_P[1] = \tilde{I}_P[-1]$$

that endows $\tilde{I}_P$ with the structure of a $k[\varepsilon]/\varepsilon^2$-dg-module. Hence, we can consider $\tilde{I}_P$ as a Fourier–Mukai kernel $\tilde{I}_P \in \mathrm{D_{qc}}(\mathrm{Spec}(k[\varepsilon]/\varepsilon^2) \times Y)$. The functor β_P is the functor associated to this Fourier–Mukai kernel. Notice that it does not depend on the choices performed.

Remark 4.4.11. The notion of a $\mathbb{P}$-object has been generalised to that of a split $\mathbb{P}$-functor,

[7] See [Spa88] for these complexes, which in *ibidem* are called K-injective.

[Add16], [Cau12b], and more generally to a $\mathbb{P}$-functor [AL19]. Using these more general notions, [ADM19, § 3] and [AL19, § 7.4] prove the factorisation of the flop-flop autoequivalence for Mukai flops in terms of $\mathbb{P}$-twists for any base Z.

Once again, given Theorem 3.1.4, the isomorphism of functors (4.39) is for us a hint that a glued spherical functor might be hiding in plain sight. This is indeed the case, as we are about to show.

Let us define $\mathrm{D}^{\mathrm{b}}(k[\varepsilon]/\varepsilon^2)$, $\deg(\varepsilon) = -1$, to be the smallest triangulated subcategory of $\mathrm{D}(k[\varepsilon]/\varepsilon^2)$ generated by $k \in \mathrm{D}(k[\varepsilon]/\varepsilon^2)$.

Remark 4.4.12. Notice that $\mathrm{D}^{\mathrm{b}}(k[\varepsilon]/\varepsilon^2)$ coincides with the subcategory of $\mathrm{D}(k[\varepsilon]/\varepsilon^2)$ given by those complexes whose total cohomology is finite dimensional. See e.g. [KS22, Proposition 2.2].

Theorem 4.4.13. *Let us write $\mathcal{S}_+$ and Ψ_+ for the source category and the spherical functor, respectively, obtained by applying Theorem 4.1.3 to the setup of Mukai flops. Then, the category $\mathcal{S}_+$ has an SOD*

$$\mathcal{S}_+ = \langle \mathrm{D}(k[\varepsilon]/\varepsilon^2), \dots, \mathrm{D}(k[\varepsilon]/\varepsilon^2)\rangle \tag{4.40}$$

where n copies of $\mathrm{D}(k[\varepsilon]/\varepsilon^2)$ appear. Moreover, the functor Ψ_+ restricted to the i-th copy of $\mathrm{D}(k[\varepsilon]/\varepsilon^2)$ (counting right to left) is identified with $\beta_{\mathcal{O}_{\mathbb{P}^n}(-i)}$, and for any $1 \le i < j \le n$ the functor $\beta^R_{\mathcal{O}_{\mathbb{P}^n}(-j)}\beta_{\mathcal{O}_{\mathbb{P}^n}(-i)}[1]$ is the right gluing functor for the couple formed by the j-th and i-th copy of $\mathrm{D}(k[\varepsilon]/\varepsilon^2)$ (counting right to left).

Furthermore, the SODs obtained by applying Theorem 4.1.3 to the setup of Mukai flops induce SODs of ${}^{\perp}\mathcal{K} \cap \mathrm{D}^{\mathrm{b}}(\widehat{W})$, and thus the category $\mathcal{S}^{\mathrm{b}}_+$ has an SOD

$$\mathcal{S}^{\mathrm{b}}_+ = \langle \mathrm{D}^{\mathrm{b}}(k[\varepsilon]/\varepsilon^2), \dots, \mathrm{D}^{\mathrm{b}}(k[\varepsilon]/\varepsilon^2)\rangle \tag{4.41}$$

and the quotient functor $\pi\colon \mathrm{D}_{\mathrm{qc}}(\widehat{W}) \to \mathrm{D}_{\mathrm{qc}}(\widehat{W})/\mathcal{K}$ induces an equivalence $\pi\colon \mathcal{S}^{\mathrm{b}}_+ \xrightarrow{\simeq} \ker(\overline{q}_-)_* \subset \mathrm{D}^{\mathrm{b}}(\widehat{W})/\mathcal{K}^{\mathrm{b}}$.

We will obtain this theorem by base changing the results we proved in § 4.4.1. Because of the many parts it is made of, we split the proof of Theorem 4.4.13 in various parts.

For $1 \le j \le n$, let us write

$$O_{-j} := \mathbb{R}_{-j+1}\mathcal{O}_{\mathbb{P}^n \times \mathbb{P}^n}(0,-j) \in \mathrm{D}^{\mathrm{b}}(\widehat{X}), \qquad \mathcal{R}_{-j} = \langle \hat{r}^* O_{-j}\rangle^{\oplus}, \quad \text{and} \quad \mathcal{R}^{\mathrm{b}}_{-j} = \mathcal{R}_{-j} \cap \mathrm{D}^{\mathrm{b}}(\widehat{W}).$$

First, we prove that $\mathcal{S}_+$ and $\mathcal{S}^{\mathrm{b}}_+$ have SODs in terms of the subcategories $\mathcal{R}_{-j}$ and $\mathcal{R}^{\mathrm{b}}_{-j}$, respectively.

Lemma 4.4.14. *The subcategory $\mathcal{S}_+$ has the SOD $\mathcal{S}_+ = \langle \mathcal{R}_{-n}, \dots, \mathcal{R}_{-1}\rangle$.*

Proof. Notice that $\hat{g} \in \mathcal{O}_{\widehat{X}}$ is a regular section because $\widehat{X}$ is smooth. Therefore, the diagram

$$\begin{array}{ccc} \widehat{W} & \xrightarrow{\hat{r}} & \widehat{X} \\ \downarrow & & \downarrow \hat{g} \\ \{0\} & \xrightarrow{i_0} & \mathbb{A}^1 \end{array} \tag{4.42}$$

is Tor independent, and we can apply base change for SODs as developed in [Kuz11]. More precisely, let us write $\mathcal{K}_{\widehat{X}} \subset \mathrm{D}_{\mathrm{qc}}(\widehat{X})$ for the common kernel of $(p_\pm)_*$. Then, by (4.24) we have the SOD

$$\mathrm{D}_{\mathrm{qc}}(\widehat{X}) = \langle \mathcal{K}_{\widehat{X}}, \mathcal{O}_{-n}, \dots, \mathcal{O}_{-1}, p_-^* \mathrm{D}_{\mathrm{qc}}(X_-) \rangle \tag{4.43}$$

Now, [Kuz11, Proposition 4.2] allows us to base change (4.43) along (4.42) to obtain the SOD

$$\mathrm{D}_{\mathrm{qc}}(\widehat{W}) = \langle \langle \hat{r}^* \mathcal{K}_{\widehat{X}} \rangle^\oplus, \mathcal{R}_{-n}, \dots, \mathcal{R}_{-1}, q_-^* \mathrm{D}_{\mathrm{qc}}(W_-) \rangle \tag{4.44}$$

If we write $\mathcal{K}_{\widehat{W}}$ for the common kernel of $(q_\pm)_*$, to prove that we have the SOD $\mathcal{S}_+ = \langle \mathcal{R}_{-n}, \dots, \mathcal{R}_{-1} \rangle$ it is now enough to prove that $\mathcal{K}_{\widehat{W}} = \langle \hat{r}^* \mathcal{K}_{\widehat{X}} \rangle^\oplus$. Indeed, if we have $\mathcal{K}_{\widehat{W}} = \langle \hat{r}^* \mathcal{K}_{\widehat{X}} \rangle^\oplus$, then Theorem 4.1.3 and (4.44) imply

$$\mathcal{S}_+ = {}^\perp \mathcal{K}_{\widehat{W}} \cap \ker(q_-)_* = {}^\perp \mathcal{K}_{\widehat{W}} \cap (q_-^* \mathrm{D}_{\mathrm{qc}}(W_-))^\perp = \langle \mathcal{R}_{-n}, \dots, \mathcal{R}_{-1} \rangle,$$

as we wanted.

We now prove that $\mathcal{K}_{\widehat{W}} = \langle \hat{r}^* \mathcal{K}_{\widehat{X}} \rangle^\oplus$. Take $K \in \mathrm{D}_{\mathrm{qc}}(\widehat{W})$, then $K \in \langle \hat{r}^* \mathcal{K}_{\widehat{X}} \rangle^\oplus$ if and only if

$$K \in \langle \mathcal{R}_{-n}, \dots, \mathcal{R}_{-1}, p_-^* \mathrm{D}_{\mathrm{qc}}(W_-) \rangle^\perp,$$

which in turn is equivalent to

$$\hat{r}_* K \in \langle \mathcal{O}_{-n}, \dots, \mathcal{O}_{-1}, p_-^* \mathrm{D}_{\mathrm{qc}}(X_-) \rangle^\perp = \mathcal{K}_{\widehat{X}}.$$

However, $0 = (p_\pm)_* \hat{r}_* K = (r_\pm)_* (q_\pm)_* K$ if and only if $(q_\pm)_* K = 0$ because r_- and r_+ are closed embeddings, and we get $\mathcal{K}_{\widehat{W}} = \langle \hat{r}^* \mathcal{K}_{\widehat{X}} \rangle^\oplus$, as we wanted. □

Given Lemma 4.4.14, we are now in the position to prove

Lemma 4.4.15. *The SODs obtained by applying Theorem 4.1.3 to the setup of Mukai flops induce SODs of* ${}^\perp \mathcal{K} \cap \mathrm{D}^{\mathrm{b}}(\widehat{W})$*, and thus the category* $\mathcal{S}_+^{\mathrm{b}}$ *has an SOD*

$$\mathcal{S}_+^{\mathrm{b}} = \langle \mathcal{R}_{-n}^{\mathrm{b}}, \dots, \mathcal{R}_{-1}^{\mathrm{b}} \rangle$$

and the quotient functor $\pi \colon \mathrm{D}_{\mathrm{qc}}(\widehat{W}) \to \mathrm{D}_{\mathrm{qc}}(\widehat{W})/\mathcal{K}$ *induces an equivalence* $\pi \colon \mathcal{S}_+^{\mathrm{b}} \xrightarrow{\simeq}$

$\ker(\overline{q}_-)_* \subset \mathrm{D^b}(\widehat{W})/\mathcal{K}^{\mathrm{b}}$.

Proof. This lemma is actually just an easy consequence of the following observation: as $\widehat{X}$ is smooth, the projection functors of the SOD (4.43) have finite cohomological amplitude, see [Kuz11] for the definition of this notion, and therefore we can also use [Kuz11, Theorem 5.6] to base change the SOD

$$\mathrm{D^b}(\widehat{X}) = \langle \mathcal{K}^{\mathrm{b}}_{\widehat{X}}, O_{-n}, \dots, O_{-1}, p_-^* \mathrm{D^b}(X_-) \rangle$$

of Remark 4.4.3 to the SOD

$$\mathrm{D^b}(\widehat{W}) = \langle \mathcal{K}_{\widehat{W}} \cap \mathrm{D^b}(\widehat{W}), \mathcal{R}^{\mathrm{b}}_{-n}, \dots, \mathcal{R}^{\mathrm{b}}_{-1}, q_-^* \mathrm{D^b}(W_-) \rangle, \tag{4.45}$$

and we get the desired induced SOD of ${}^\perp\mathcal{K}_{\widehat{W}} \cap \mathrm{D^b}(\widehat{W})$. Similarly, one shows that also the other three SODs of Theorem 4.1.3 induce SODs of $\mathrm{D^b}(\widehat{W})$.

As a consequence of (4.45), we get

$$\mathcal{S}^{\mathrm{b}}_+ = \mathcal{S}_+ \cap \mathrm{D^b}(\widehat{W}) = \langle \mathcal{R}^{\mathrm{b}}_{-n}, \dots, \mathcal{R}^{\mathrm{b}}_{-1} \rangle,$$

as we wanted.

Now that we proved that the SODs obtained by applying Theorem 4.1.3 to the setup of Mukai flops induce SODs of ${}^\perp\mathcal{K}_{\widehat{W}} \cap \mathrm{D^b}(\widehat{W})$, the proof that the quotient functor induces an equivalence $\pi\colon \mathcal{S}^{\mathrm{b}}_+ \xrightarrow{\simeq} \ker(\overline{q}_-)_* \subset \mathrm{D^b}(\widehat{W})/\mathcal{K}^{\mathrm{b}}$ is the same as in Proposition 4.4.5. □

Summing up, thanks to Lemma 4.4.14 and Lemma 4.4.15 we know that we have the SODs

$$\mathcal{S}_+ = \langle \mathcal{R}_{-n}, \dots, \mathcal{R}_{-1} \rangle \quad \text{and} \quad \mathcal{S}^{\mathrm{b}}_+ = \langle \mathcal{R}^{\mathrm{b}}_{-n}, \dots, \mathcal{R}^{\mathrm{b}}_{-1} \rangle.$$

Thus, to prove the existence the SODs (4.40) and (4.41) it is enough to prove

$$\mathcal{R}_{-j} \simeq \mathrm{D}(k[\varepsilon]/\varepsilon^2) \quad \text{and} \quad \mathcal{R}^{\mathrm{b}}_{-j} \simeq \mathrm{D^b}(k[\varepsilon]/\varepsilon^2)$$

for $j = 1, \dots, n$. We do this in the following

Lemma 4.4.16. *For any $j = 1, \dots, n$, we have equivalences*

(i) $\mathcal{R}_{-j} \simeq \mathrm{D}(k[\varepsilon]/\varepsilon^2)$

(ii) $\mathcal{R}^{\mathrm{b}}_{-j} \simeq \mathrm{D^b}(k[\varepsilon]/\varepsilon^2)$

Proof of (i). By definition the category $\mathcal{R}_{-j}$ is generated by the compact object $\hat{r}^* O_{-j}$. Therefore, to prove that $\mathcal{R}_{-j} \simeq \mathrm{D}(k[\varepsilon]/\varepsilon^2)$ it is enough to prove that the derived endomorphism algebra of $\hat{r}^*O_{-j}$ is isomorphic to $k[\varepsilon]/\varepsilon^2$. Namely, we want to prove

$$\mathrm{RHom}_{\widehat{W}}(\hat{r}^*O_{-j}, \hat{r}^*O_{-j}) \simeq k[\varepsilon]/\varepsilon^2. \tag{4.46}$$

Recall that, by construction, we have isomorphisms

$$\begin{array}{ccc} \mathrm{RHom}_{\widehat{X}}(O_{-j}, O_{-j}) & & \mathrm{RHom}_{\widehat{X}}(\mathcal{O}_{\mathbb{P}^n\times\mathbb{P}^n}(0,-j), \mathcal{O}_{\mathbb{P}^n\times\mathbb{P}^n}(0,-j)) \\ & \searrow^{\simeq} \quad {}^{\simeq}\swarrow & \\ & \mathrm{RHom}_{\widehat{X}}(O_{-j}, \mathcal{O}_{\mathbb{P}^n\times\mathbb{P}^n}(0,-j)) & \end{array} \tag{4.47}$$

given by postcomposition and precomposition, respectively, with the canonical morphism $O_{-j} \to \mathcal{O}_{\mathbb{P}^n\times\mathbb{P}^n}(0,-j)$ coming from the definition of right mutation. Notice that all the hom spaces in (4.47) carry an action of $\mathrm{H}^0(\mathbb{A}^1, \mathcal{O}_{\mathbb{A}^1})$ via $\hat{g}$, and that the isomorphisms in (4.47) are $\mathrm{H}^0(\mathbb{A}^1, \mathcal{O}_{\mathbb{A}^1})$-linear. This means that we can rewrite the diagram (4.47) as follows

$$\begin{array}{ccc} \hat{g}_*\mathrm{R}\mathcal{H}om_{\widehat{X}}(O_{-j}, O_{-j}) & & \hat{g}_*\mathrm{R}\mathcal{H}om_{\widehat{X}}(\mathcal{O}_{\mathbb{P}^n\times\mathbb{P}^n}(0,-j), \mathcal{O}_{\mathbb{P}^n\times\mathbb{P}^n}(0,-j)) \\ & \searrow^{\simeq} \quad {}^{\simeq}\swarrow & \\ & \hat{g}_*\mathrm{R}\mathcal{H}om_{\widehat{X}}(\mathcal{O}_{\mathbb{P}^n\times\mathbb{P}^n}(0,-j), O_{-j}) & \end{array} \tag{4.48}$$

It is easy to show that $\hat{g}_*\mathrm{R}\mathcal{H}om_{\widehat{X}}(\mathcal{O}_{\mathbb{P}^n\times\mathbb{P}^n}(0,-j), \mathcal{O}_{\mathbb{P}^n\times\mathbb{P}^n}(0,-j)) \simeq (i_0)_*\mathcal{O}_{\{0\}}$, where $i_0\colon \{0\} \hookrightarrow \mathbb{A}^1$, and therefore (4.48) tells us that we have an isomorphism

$$\hat{g}_*\mathrm{R}\mathcal{H}om_{\widehat{X}}(O_{-j}, O_{-j}) \simeq (i_0)_*\mathcal{O}_{\{0\}}.$$

Applying flat base change along (4.42) to this isomorphism, we obtain the isomorphism[8]

$$\mathrm{RHom}_{\widehat{W}}(\hat{r}^*O_{-j}, \hat{r}^*O_{-j}) \simeq i_0^*\hat{g}_*\mathrm{R}\mathcal{H}om_{\widehat{X}}(O_{-j}, O_{-j}) \simeq i_0^*(i_0)_*\mathcal{O}_{\{0\}} \simeq k[\varepsilon]/\varepsilon^2,$$

which is exactly (4.46), as we wanted.

Given the isomorphism (4.46), by e.g. [LS16, Proposition B.1], we conclude that $\mathcal{R}_{-j} \simeq \mathrm{D}(k[\varepsilon]/\varepsilon^2)$, concluding the proof. □

Proof of (ii). To prove the equivalence $\mathcal{R}^{\mathrm{b}}_{-j} \simeq \mathrm{D}^{\mathrm{b}}(k[\varepsilon]/\varepsilon^2)$, we have to study the equivalence $\mathcal{R}_{-j} \simeq \mathrm{D}(k[\varepsilon]/\varepsilon^2)$ more carefully and answer the following question: under the equivalence $\mathcal{R}_{-j} \simeq \mathrm{D}(k[\varepsilon]/\varepsilon^2)$ to what subcategory does $\mathrm{D}^{\mathrm{b}}(\widehat{W})$ correspond? The equiv-

[8]Notice that we are doing three steps at once: by flat base change we obtain that the underlying graded algebra of the RHom in (4.46) is $k \oplus k[1]$. However, there is only one structure of graded algebra on this graded vector space, which is given by $k[\varepsilon]/\varepsilon^2$. Finally, for degree reasons, the dg-algebra $k[\varepsilon]/\varepsilon^2$ is intrinsically formal, see also [KS22, Lemma 2.1], which means that any dg-algebra with the same cohomology is already quasi-isomorphic to it. Thus, we can conclude that (4.46) holds.

alence $\mathcal{R}_{-j} \simeq \mathrm{D}(k[\varepsilon]/\varepsilon^2)$ is given by

$$\nu_{-j}(-) = - \otimes_{k[\varepsilon]/\varepsilon^2} \hat{r}^* O_{-j} \colon \mathrm{D}(k[\varepsilon]/\varepsilon^2) \to \mathrm{D}_{\mathrm{qc}}(\widehat{W}).$$

What we want to show is that $\nu_{-j}(\mathrm{D}^{\mathrm{b}}(k[\varepsilon]/\varepsilon^2)) = \mathcal{R}^{\mathrm{b}}_{-j}$.

Notice that as $\hat{r}^* O_{-j} \in \mathrm{D}(\widehat{W})^{\mathrm{c}}$, it follows formally that $\nu_{-j}(\mathrm{D}(k[\varepsilon]/\varepsilon^2)^{\mathrm{c}}) = \mathcal{R}^{\mathrm{c}}_{-j}$. However, we cannot formally deduce anything about $\nu_{-j}(\mathrm{D}^{\mathrm{b}}(k[\varepsilon]/\varepsilon^2))$ because this subcategory is controlled by the action of ν_{-j} on k, which in general can be recovered by the action of ν_{-j} on $k[\varepsilon]/\varepsilon^2$ only as an homotopy colimit.

We begin by showing that if $\nu_{-j}(A) \in \mathcal{R}^{\mathrm{b}}_{-j}$, then $A \in \mathrm{D}^{\mathrm{b}}(k[\varepsilon]/\varepsilon^2)$. Take $A \in \mathrm{D}(k[\varepsilon]/\varepsilon^2)$ and assume that $A' = \nu_{-j}(A) \in \mathcal{R}^{\mathrm{b}}_{-j}$. As ν_{-j} is fully-faithful, we have

$$A \simeq \nu^R_{-j}\nu_{-j}(A) \simeq \mathrm{RHom}_{\widehat{W}}(\hat{r}^* O_{-j}, A') \simeq \mathrm{RHom}_{\widehat{X}}(O_{-j}, \hat{r}_* A').$$

As A' is bounded, so is $\hat{r}_* A'$, and thus, as $O_{-j} \in \mathrm{D}^{\mathrm{b}}(\widehat{X})$ is set-theoretically supported on $\mathbb{P}^n \times \mathbb{P}^n$, the rightmost term above has bounded and finite dimensional cohomology. Therefore, by Remark 4.4.12, we get $A \in \mathrm{D}^{\mathrm{b}}(k[\varepsilon]/\varepsilon^2)$.

Conversely, we now prove $\nu_{-j}(\mathrm{D}^{\mathrm{b}}(k[\varepsilon]/\varepsilon^2)) \subset \mathrm{D}^{\mathrm{b}}(\widehat{W})$. By definition of $\mathrm{D}^{\mathrm{b}}(k[\varepsilon]/\varepsilon^2)$, it is enough to prove that $\nu_{-j}(k) \in \mathrm{D}^{\mathrm{b}}(\widehat{W})$, and this is what we show.

Let us write $\pi_{O_{-j}}$ for the projection functor to the subcategory generated by O_{-j} for the SOD (4.43), and $\pi_{\mathcal{R}_{-j}}$ for the projection functor to the subcategory $\mathcal{R}_{-j}$ for the SOD (4.44). Here $1 \leq j \leq n$.

[Kuz11, Proposition 4.2, Theorem 5.6] tell us that $\pi_{O_{-j}}$ and $\pi_{\mathcal{R}_{-j}}$ are compatible. Namely, they tell us that for any $1 \leq j \leq n$ we have an isomorphism of functors

$$\hat{r}_* \pi_{\mathcal{R}_{-j}} \simeq \pi_{O_{-j}} \hat{r}_*. \tag{4.49}$$

We will use this isomorphism of functors to prove that $\nu_{-j}(k) \in \mathrm{D}^{\mathrm{b}}(\widehat{W})$.

Notice that

$$\hat{r}_* \mathcal{O}_{\mathbb{P}^n \times \mathbb{P}^n}(0, -j) \in \langle O_{-j+1}, \dots, O_{-1}, p_-^* \mathrm{D}_{\mathrm{qc}}(X_-) \rangle^{\perp} = \langle \mathcal{K}_{\widehat{X}}, O_{-n}, \dots, O_{-j} \rangle. \tag{4.50}$$

Therefore, we have[9]

$$\pi_{O_{-j}}(\hat{r}_* \mathcal{O}_{\mathbb{P}^n \times \mathbb{P}^n}(0, -j)) \simeq \mathrm{RHom}_{\widehat{X}}(O_{-j}, \hat{r}_* \mathcal{O}_{\mathbb{P}^n \times \mathbb{P}^n}(0, -j)) \otimes_k O_{-j} \simeq O_{-j},$$

[9] Recall that by Remark 2.3.3 the projection functor to the rightmost subcategory in an SOD is the right adjoint to the inclusion of that subcategory. In this case, the inclusion of the category generated by O_{-j} is the given by the functor $- \otimes_k O_{-j}$, and its right adjoint is $\mathrm{RHom}_{\widehat{W}}(O_{-j}, -)$.

and thus

$$\hat{r}_*\pi_{\mathcal{R}_{-j}}(\mathcal{O}_{\mathbb{P}^n\times\mathbb{P}^n}(0,-j)) \overset{(4.49)}{\simeq} \pi_{O_{-j}}(\hat{r}_*\mathcal{O}_{\mathbb{P}^n\times\mathbb{P}^n}(0,-j)) \simeq O_{-j} \tag{4.51}$$

Now notice that (4.50) implies

$$\mathcal{O}_{\mathbb{P}^n\times\mathbb{P}^n}(0,-j) \in \langle \mathcal{R}_{-j+1},\dots,\mathcal{R}_{-1}, q_-^*\mathrm{D}_{\mathrm{qc}}(W_-)\rangle^{\perp} = \langle \mathcal{K}_{\widehat{W}}, \mathcal{R}_{-n},\dots,\mathcal{R}_{-j}\rangle$$

and therefore

$$\begin{aligned}\pi_{\mathcal{R}_{-j}}(\mathcal{O}_{\mathbb{P}^n\times\mathbb{P}^n}(0,-j)) &\simeq \mathrm{RHom}_{\widehat{W}}(\hat{r}^*O_{-j}, \mathcal{O}_{\mathbb{P}^n\times\mathbb{P}^n}(0,-j)) \otimes_{k[\varepsilon]/\varepsilon^2} \hat{r}^*O_{-j} \\ &\simeq k \otimes_{k[\varepsilon]/\varepsilon^2} \hat{r}^*O_{-j} = \nu_{-j}(k).\end{aligned}$$

Plugging this isomorphism into (4.51), we obtain $\hat{r}_*\nu_{-j}(k) \simeq O_{-j}$. As $\hat{r}$ is a closed embedding, $\hat{r}_*$ reflects boundedness and coherence, and therefore $\nu_{-j}(k) \in \mathrm{D}^{\mathrm{b}}(\widehat{W})$, as we wanted. □

Having proved Lemma 4.4.14, Lemma 4.4.15, and Lemma 4.4.16, we know that we have SODs

$$\mathcal{S}_+ = \langle \mathrm{D}(k[\varepsilon]/\varepsilon^2),\dots,\mathrm{D}(k[\varepsilon]/\varepsilon^2)\rangle \quad \text{and} \quad \mathcal{S}_+^{\mathrm{b}} = \langle \mathrm{D}^{\mathrm{b}}(k[\varepsilon]/\varepsilon^2),\dots,\mathrm{D}^{\mathrm{b}}(k[\varepsilon]/\varepsilon^2)\rangle,$$

and to conclude the proof of Theorem 4.4.13 we have to prove that the functor Ψ_+ restricted to the i-th copy of $\mathrm{D}(k[\varepsilon]/\varepsilon^2)$ (counting right to left) is identified with $\beta_{\mathcal{O}_{\mathbb{P}^n}(-i)}$, and that for any $1 \le i < j \le n$ the functor $\beta^R_{\mathcal{O}_{\mathbb{P}^n}(-j)}\beta_{\mathcal{O}_{\mathbb{P}^n}(-i)}[1]$ is the right gluing functor for the couple formed by the j-th and i-th copy of $\mathrm{D}(k[\varepsilon]/\varepsilon^2)$ (counting right to left).

We begin by proving

Lemma 4.4.17. *With the notation of Lemma 4.4.16, we have an isomorphism of functors $\Psi_+\nu_{-j} \simeq \beta_{\mathcal{O}_{\mathbb{P}^n}(-j)}$ for any $j = 1,\dots,n$.*

Proof. Recall that in Remark 4.4.10 we explained that the functor $\beta_{\mathcal{O}_{\mathbb{P}^n}(-j)}$ is defined by performing some choices. However, in the same remark we also noticed that the resulting functor does not depend on the choices made, and therefore proving the isomorphism $\Psi_+\nu_{-j} \simeq \beta_{\mathcal{O}_{\mathbb{P}^n}(-j)}$ amounts to prove the following: write A for the Fourier–Mukai kernel defining $\Psi_+\nu_{-j}$, then A has the structure of a $k[\varepsilon]/\varepsilon^2$-dg-module and we have an isomorphism

$$A \simeq \mathrm{cone}(\mathcal{O}_{\mathbb{P}^n}(-j) \to \mathcal{O}_{\mathbb{P}^n}(-j)[2])$$

under which the two actions of ε correspond (the action of $k[\varepsilon]/\varepsilon^2$ on the right hand side was defined in Remark 4.4.10).

We now prove that the Fourier–Mukai kernel of $\Psi_+\nu_{-j}$ does indeed satisfy these properties. Let us write $\tilde{I}_{-j}$ for an h-injective resolution of $\hat{r}^*O_{-j}$. Then, we can lift the $k[\varepsilon]/\varepsilon^2$-

dg-module structure on $\hat{r}^* O_{-j}$ to a $k[\varepsilon]/\varepsilon^2$-dg-module structure on $\tilde{I}_{-j}$ in such a way that the isomorphism $\tilde{I}_{-j} \simeq \hat{r}^* O_{-j}$ becomes an isomorphism in $\mathrm{D}_{\mathrm{qc}}(\mathrm{Spec}(k[\varepsilon]/\varepsilon^2) \times \widehat{W})$.

Having made this choice of an h-injective resolution, we get that ν_{-j} is the functor associated to the Fourier–Mukai kernel $\tilde{I}_{-j} \in \mathrm{D}_{\mathrm{qc}}(\mathrm{Spec}(k[\varepsilon]/\varepsilon^2) \times \widehat{W})$, and that $\Psi_+ \nu_{-j}$ is the functor associated to the Fourier–Mukai kernel

$$(\mathrm{id} \times q_+)_*(\tilde{I}_{-j}) \in \mathrm{D}_{\mathrm{qc}}(\mathrm{Spec}(k[\varepsilon]/\varepsilon^2) \times W_+).$$

Now notice that the underlying $\mathcal{O}_{W_+}$-module of $(\mathrm{id} \times q_+)_*(\tilde{I}_{-j})$ is

$$(q_+)_*(\tilde{I}_{-j}) \simeq (q_+)_* \hat{r}^* O_{-j} \simeq r_+^* \mathcal{O}_{\mathbb{P}^n}(-j). \tag{4.52}$$

We claim that we have an isomorphism

$$r_+^* \mathcal{O}_{\mathbb{P}^n}(-j) \simeq \mathrm{cone}(\mathcal{O}_{\mathbb{P}^n}(-j) \to \mathcal{O}_{\mathbb{P}^n}(-j)[2]) \tag{4.53}$$

and that the isomorphism obtained by composing (4.52) and (4.53) is an isomorphism in the category $\mathrm{D}_{\mathrm{qc}}(\mathrm{Spec}(k[\varepsilon]/\varepsilon^2) \times W_+)$.

We begin by proving the isomorphism (4.53). First, notice that $\mathcal{O}_{\mathbb{P}^n} \in \mathrm{D}^{\mathrm{b}}(X_+)$ can be resolved using a Koszul resolution. Namely, $\mathbb{P}^n \subset X_+$ is the zero locus of a regular section $s \in \mathcal{O}_{X_+}(-1)^{\oplus n+1}$ and the Koszul complex associated to s is a resolution of $\mathcal{O}_{\mathbb{P}^n}$. Then, if we use (the twist by $\mathcal{O}_{X_+}(-j)$ of) this complex to compute $r_+^* \mathcal{O}_{\mathbb{P}^n}(-j)$, we obtain the (twist by $\mathcal{O}_{W_+}(-j)$) of the Koszul complex associated to the restriction of s to W_+. The latter complex is not exact because because $\mathbb{P}^n$ has codimension n in W_+, while $s|_{W_+}$ is a section a vector bundle of rank $n+1$. However, the dimension is off only by one, which means that the Koszul complex associated to $s|_{W_+}$ has cohomology only in degree 0 and -1. In degree zero the cohomology is $\mathcal{O}_{\mathbb{P}^n}$, while in degree -1 it is given by the kernel of the map

$$\mathcal{O}_{\mathbb{P}^n}(1)^{\oplus n+1} \simeq \mathcal{O}_{W_+}(1)^{\oplus n+1}|_{\mathbb{P}^n} \to N_{\mathbb{P}^n/W_+} \simeq (\Omega^1_{\mathbb{P}^n})^\vee,$$

which is $\mathcal{O}_{\mathbb{P}^n}$. Therefore, we have a distinguished triangle

$$\mathcal{O}_{\mathbb{P}^n}(-j)[1] \to r_+^* \mathcal{O}_{\mathbb{P}^n}(-j) \to \mathcal{O}_{\mathbb{P}^n}(-j),$$

which proves the isomorphism (4.53).

To prove that the isomorphism obtained by composing (4.52) and (4.53) respects the $k[\varepsilon]/\varepsilon^2$-dg-module structure on both sides, we notice that by our choice of the $k[\varepsilon]/\varepsilon^2$-dg-module structure on $\tilde{I}_{-j}$ and the diagram (4.47), we have the following commutative diagram

$$\begin{array}{ccccc} \tilde{I}_{\nu_{-j}} & \xrightarrow{\simeq} & \hat{r}^*O_{-j} & \longrightarrow & \hat{r}^*\mathcal{O}_{\mathbb{P}^n\times\mathbb{P}^n}(0,-j) \\ \downarrow{\scriptstyle\varepsilon} & & \downarrow{\scriptstyle\varepsilon} & & \downarrow{\scriptstyle\varepsilon} \\ \tilde{I}_{\nu_{-j}}[-1] & \xrightarrow{\simeq} & \hat{r}^*O_{-j}[-1] & \longrightarrow & \hat{r}^*\mathcal{O}_{\mathbb{P}^n\times\mathbb{P}^n}(0,-j)[-1] \end{array}$$

Applying $(q_+)_*$ to the above diagram, and using the isomorphism (4.53), we see that the isomorphism $(q_+)_*(\tilde{I}_{-j}) \to \mathrm{cone}(\mathcal{O}_{\mathbb{P}^n}(-j) \to \mathcal{O}_{\mathbb{P}^n}(-j)[2])$ is an isomorphism of $k[\varepsilon]/\varepsilon^2$-dg-modules. Therefore, we proved $\Psi_+\nu_{-j} \simeq \beta_{\mathcal{O}_{\mathbb{P}^n}(-j)}$, as required. □

Finally, we are in the position to complete the proof of Theorem 4.4.13 by proving the statement about the right gluing functors of the SOD (4.40).

Lemma 4.4.18. *With the notation of Lemma 4.4.16, we have an isomorphism of functors* $\nu_{-j}^R\nu_{-i} \simeq \beta^R_{\mathcal{O}_{\mathbb{P}^n}(-j)}\beta_{\mathcal{O}_{\mathbb{P}^n}(-i)}$ *for any* $1 \leq i < j \leq n$.

Proof. We proved in Lemma 4.4.17 that we have an isomorphism $\Psi_+\nu_{-j} \simeq \beta_{\mathcal{O}_{\mathbb{P}^n}(-j)}$, therefore we only have to prove

$$\nu_{-j}^R\nu_{-i} \simeq (\Psi_+\nu_{-j})^R\Psi_+\nu_{-i}.$$

Notice that we have a natural transformation

$$\nu_{-j}^R\nu_{-i} \to \nu_{-j}^R\Psi_+^R\Psi_+\nu_{-i} = (\Psi_+\nu_{-j})^R\Psi_+\nu_{-i} \tag{4.54}$$

and that all the functors appearing are cocontinuous. Thus, to prove that the natural transformation is an isomorphism it is enough to prove that it is an isomorphism on $k[\varepsilon]/\varepsilon^2 \in \mathrm{D}(k[\varepsilon]/\varepsilon^2)$. We have

$$\nu_{-j}^R\nu_{-i}(k[\varepsilon]/\varepsilon^2) = \mathrm{RHom}_{\widehat{W}}(\hat{r}^*O_{-j}, \hat{r}^*O_{-i}) \simeq i_0^*\hat{g}_*\mathrm{R}\mathcal{H}om_{\widehat{X}}(O_{-j}, O_{-i})$$

and

$$\begin{aligned}(\Psi_+\nu_{-j})^R\Psi_+\nu_{-i}(k[\varepsilon]/\varepsilon^2) &= \mathrm{RHom}_{W_-}((q_+)_*(\tilde{I}_{-j}), (q_+)_*(\tilde{I}_{-j})) \\ &\simeq \mathrm{RHom}_{W_-}(r_+^*\mathcal{O}_{\mathbb{P}^n}(-j), r_+^*\mathcal{O}_{\mathbb{P}^n}(-i)) \\ &\simeq i_0^*(g_+)_*\mathrm{R}\mathcal{H}om_{X_+}(\mathcal{O}_{\mathbb{P}^n}(-j), \mathcal{O}_{\mathbb{P}^n}(-i)).\end{aligned}$$

Under this identification the natural transformation (4.54) becomes the image via i_0^* of the morphism

$$\begin{aligned}\hat{g}_*\mathrm{R}\mathcal{H}om_{\widehat{X}}(O_{-j}, O_{-i}) = (g_+)_*(p_+)_*\mathrm{R}\mathcal{H}om_{\widehat{X}}(O_{-j}, O_{-i}) \to \\ \to (g_+)_*\mathrm{R}\mathcal{H}om_{X_+}(\mathcal{O}_{\mathbb{P}^n}(-j), \mathcal{O}_{\mathbb{P}^n}(-i)).\end{aligned}$$

However, in Lemma 4.4.7 we proved that this last morphism is an isomorphism, and therefore the natural transformation $\nu_{-j}^R\nu_{-i} \to (\Psi_+\nu_{-j})^R\Psi_+\nu_{-i}$ is an isomorphism when evaluated at $k[\varepsilon]/\varepsilon^2$, as we wanted.

The proof of the lemma, and thus of Theorem 4.4.13, is now complete. □

Remark 4.4.19. Notice that by Koszul duality we have $\mathrm{D^b}(k[\varepsilon]/\varepsilon^2) \simeq \mathrm{D}(k[q])^{\mathrm{c}}$, thus we also recover the other construction of [Seg18]. The thick generators of the copies of $\mathrm{D}(k[q])^{\mathrm{c}}$ are the objects

$$k \otimes_{k[\varepsilon]/\varepsilon^2} \hat{r}^* \mathbb{R}_{-j+1}\mathcal{O}_{\mathbb{P}^n\times\mathbb{P}^n}(0,-j),$$

which in the terminology of [KS22] are $\mathbb{P}^\infty[2]$-objects.

4.4.3 Other examples

Let us walk through a few more examples where we can apply the theory of § 4 but where an explicit description of the category $\mathcal{S}_+$ eludes our understanding.

Grassmannian flops

Let V and S be two vector spaces of dimension n and r, respectively, with $r < n$. Consider the quotient stack

$$\mathfrak{X}^{r,n} = [\mathrm{Hom}(S,V) \oplus \mathrm{Hom}(V,S) \,/\mathrm{GL}(S)]$$

where the action is given by $M \cdot (a,b) = (aM^{-1}, Mb)$. The GIT quotient associated to the linearisations $\mathcal{O}(1) := \det S^\vee$ and $\mathcal{O}(-1)$ are, respectively,

$$\mathfrak{X}_+^{ss}\,/\mathrm{GL}(S) = \{(a,b) : b \text{ is surjective}\}\,/\mathrm{GL}(S) = \mathrm{Tot}\left(\mathrm{Hom}(S,V) \to \mathrm{Gr}(V,r)\right) =: X_+,$$

which is the total space of a vector bundle over the grassmannian of r dimensional quotients in V, and

$$\mathfrak{X}_-^{ss}\,/\mathrm{GL}(S) = \{(a,b) : a \text{ is injective}\}\,/\mathrm{GL}(S) = \mathrm{Tot}\left(\mathrm{Hom}(V,S) \to \mathrm{Gr}(r,V)\right) =: X_-,$$

where $\mathrm{Gr}(r,V)$ is the grassmannian of r dimensional subspaces of V.

The two varieties X_- and X_+ are birational and derived equivalent, see [DS14], [HL15]. The fibre product over the common singularity is given by

$$\widehat{X} = \mathrm{Tot}\left(\mathrm{Hom}(Q,S) \to \mathrm{Gr}(r,V) \times \mathrm{Gr}(V,r)\right)$$

where Q is the tautological quotient bundle and S is the tautological subbundle.

In [BCF⁺19] it was proved that $\widehat{X}$ gives a derived equivalence between $\mathrm{D^b}(X_-)$ and $\mathrm{D^b}(X_+)$. As everything is smooth, by [KL15, Lemma 2.12] we obtain an equivalence

$\mathrm{D_{qc}}(X_-) \simeq \mathrm{D_{qc}}(X_+)$, and we can apply Theorem 4.1.3. In [BCF+19] it is also (implicitly) proved that the flop-flop functor is given by a composition of $n-r$ window shifts, see [*ibidem*, Corollary 5.2.9, Corollary 5.2.10]. Moreover, by [DS14] we know that every window shift is realised as the spherical twist around a spherical functor whose source category is given by $\mathrm{D_{qc}}(\mathfrak{X}^{r-1,n})$. Therefore, as in all the examples we considered, one would expect $\mathcal{S}_+$ to have an SOD reflecting this factorisation of the flop-flop autoequivalence. However, the picture is more complicated, and we are not able to match up the construction of § 3 with the spherical functor constructed via Theorem 4.1.3.

Abuaf flop

Consider V a symplectic vector space of dimension 4. Then, define

$$X_- = \mathrm{Tot}(L^\perp/L \otimes L^2 \to \mathbb{P}V) \quad \text{and} \qquad X_+ = \mathrm{Tot}(S(-1) \to \mathrm{LGr}(2,V))$$

where L is the tautological subbundle on $\mathbb{P}V$, S is the tautological subbundle on $\mathrm{LGr}(2,V)$, and $\mathcal{O}_{\mathrm{LGr}}(-1) = \bigwedge^2 S$. The varieties X_- and X_+ are birational and derived equivalent [Seg16], and furthermore they resolve the same singularity $Y = \operatorname{Spec} \mathrm{H}^0(X_\pm, \mathcal{O}_{X_\pm})$. In [Har17], Hara proves that the structure sheaf of the fibre product $X_- \times_Y X_+$ gives an equivalence from $\mathrm{D^b}(X_-)$ to $\mathrm{D^b}(X_+)$. He proves this statement identifying the flop functor with the equivalence constructed by Segal, who used tilting bundles to prove $\mathrm{D^b}(X_-) \simeq \mathrm{D^b}(X_+)$. Furthermore, Hara constructs another family of tilting bundles, and one can show that (one of) the equivalence(s) produced by this new family is identified with the flop functor $\mathrm{D^b}(X_+) \to \mathrm{D^b}(X_-)$. Thus, the diagram $X_- \leftarrow X_- \times_Y X_+ \to X_+$ induces a flop-flop diagram, and we can apply Theorem 4.1.3 and Theorem 4.2.2.

Studying the families of tilting bundles in further detail, one can show that the flop-flop autoequivalence is given by

$$T^{-1}_{j_*S} T^{-1}_{j_*\mathcal{O}_{\mathrm{LGr}}(-1)} T^{-1}_{j_*S(-1)} T^{-1}_{j_*\mathcal{O}_{\mathrm{LGr}}(-2)} T^{-1}_{j_*S(-2)} \tag{4.55}$$

where $j : \mathrm{LGr}(2,V) \hookrightarrow X_+$ is the inclusion of the zero section. Given this decomposition of the flop-flop functor, we might expect the category $\mathcal{S}_+$ to have a full exceptional collection of length five, so to match the spherical functor of Theorem 4.1.3 with the glued spherical functor of Theorem 3.1.4, as it happened for standard flops.

However, there is a fundamental difference between this example and standard flops: the objects j_*S, $i_*\mathcal{O}_{\mathrm{LGr}}(-1)$, $j_*S(-1)$, $i_*\mathcal{O}_{\mathrm{LGr}}(-2)$, and $j_*S(-2)$ are not “independent” in the derived category: we have the following short exact sequence on $\mathrm{LGr}(2,V)$

$$S(-1) \to V^* \otimes \mathcal{O}_{\mathrm{LGr}}(-1) \to S. \tag{4.56}$$

Hence, the functor obtained by gluing the twists in (4.55) using the construction of Theorem 3.1.4 is not conservative, while the spherical functor Ψ_+ is. We can guess what $\mathcal{S}_+$ should look like as follows; the fibre product $\widehat{X}$ is the gluing of $\mathrm{Bl}_{\mathrm{LGr}} X_+$ and $\mathbb{P}V \times \mathrm{LGr}(2, V)$ along $\mathbb{P}(S(-1))$, which is embedded in the latter via the exact sequence (4.56). We can prove that $\mathcal{S}_+$ is generated by the objects $i_*\mathcal{O}_{\mathrm{LGr}}(-1)$, $i_*\mathcal{O}_{\mathrm{LGr}}(-2)$, $i_*S(-1)$ where $i : \mathbb{P}V \times \mathrm{LGr}(2, V) \hookrightarrow \widehat{X}$ is the closed embedding, and therefore one might conjecture that $\mathcal{S}_+$ is the quotient of the source category obtained by gluing the spherical twists in (4.55) by the kernel of the associated glued spherical functor. However, at the moment we do not know how to prove whether this is right or wrong.

Chapter 5

Concluding remarks and further directions

We want to conclude this thesis by presenting some possible future applications of the novel mathematics we developed in the previous chapters. The author does not know whether the suggestions that he is about to make will turn into precise theorems, but he hopes that others will share his interest in the approach proposed.

5.1 Glued spherical functors

We have spent the entire § 3 to understand how to glue spherical functors, and then we spent a big part of § 4 to find geometric examples of such construction. However, the reader might wonder why should they be interested in such construction. What are the new information we can access that we could not grasp without glued spherical functors?

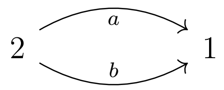
Theorem 3.1.4 has already found applications in the joint work of the author with Dr. Jongmeyong Kim [BK21] to compute the categorical entropy of the composition of two spherical twists around spherical objects. However, here we want to present another possible application of Theorem 3.1.4.

In § 1, we said that sometimes having different presentations of a fixed autoequivalence as a spherical twist around a spherical functor can be an advantage, rather than a disadvantage. Glued spherical functors are able to harness this opportunity and shed light on some relations that might result a priori unexpected.

Let us consider the following example. Let $E \in \mathcal{C}$ be a d-spherical object in a proper triangulated category with a Serre functor $\mathbb{S}_{\mathcal{C}}$. By Theorem 3.1.4, we know that the autoequivalence $T_E^2 \in \mathrm{Aut}(\mathcal{C})$ can be realised as the spherical twist around the spherical functor

$$\Psi(-) = - \overset{L}{\otimes}_R (E \oplus E) \colon \mathrm{D}(R) \to \mathcal{C}$$

where R is the path dg-algebra of the quiver



$$\deg(a) = 0, \deg(b) = d$$

Here, the vertex labelled 2 corresponds to the left copy of E in $E \oplus E$, the vertex labelled 1 corresponds to the right copy of E, and a and b correspond to the identity and the unique extension of E by itself, respectively. Notice that instead of the dg-algebra constructed in § 3.4 we consider its associated graded algebra, which in this case we can do because there are no relations among the morphisms and therefore the algebra (3.41) is formal, *i.e.*, isomorphic to its cohomology.

What we want to point to the attention of the reader is that the functor Ψ defined above is not conservative. Namely, if we consider the SOD $\mathrm{D}(R) = \langle \mathrm{D}(k), \mathrm{D}(k) \rangle$ constructed in Proposition 2.4.25 and explained in Example 2.4.30, then we see that

$$\Psi(\mathrm{cone}(a\colon k \otimes_2 R \to k \otimes_1 R)) \simeq \mathrm{cone}(\mathrm{id}\colon E \to E) \simeq 0.$$

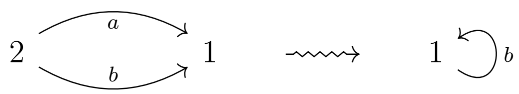
Therefore, $\ker \Psi \neq 0$, and actually $\ker \Psi = \langle \mathrm{cone}(a\colon k \otimes_2 R \to k \otimes_1 R) \rangle$.

Thus, we can consider another spherical functor whose twist is isomorphic to T_E^2, namely, the functor

$$\overline{\Psi}\colon \mathrm{D}(R)/\ker \Psi \to \mathcal{C}.$$

The question now is: can we say something about $\mathrm{D}(R)/\ker \Psi$ and this new functor?

To answer this question we will proceed pictorially, but the we reassure the reader that every step can be formalised. In passing from $\mathrm{D}(R)$ to $\mathrm{D}(R)/\ker \Psi$ we are imposing that the morphism a should become an isomorphism. Therefore, what we are saying is that the two vertices in the path-algebra presentation of R should become isomorphic. Namely, the process of taking the quotient acts as follows



The algebra corresponding to the quiver on the right is the algebra $k[b]$, where $\deg(b) = d$.

For compactness reasons, this answer is not quite right, and we have to pass the algebra $k[b]$ through Koszul duality. Thus, we obtain

$$\mathrm{D}(R)/\ker \Psi \simeq \mathrm{D}(k[\varepsilon]/\varepsilon^2) \quad \deg(\varepsilon) = 1 - d$$

and the spherical functor $\overline{\Psi}$ is identified with the spherical functor

$$\Psi_\varepsilon(-) = - \overset{L}{\otimes}_{k[\varepsilon]/\varepsilon^2} \operatorname{cone}(E \to E[d]) \colon \mathrm{D}(k[\varepsilon]/\varepsilon^2) \to \mathcal{C}.$$

Hence, we proved that $T_E^2 = T_{\Psi_\varepsilon}$.

The spherical functor Ψ_ε has already appeared in [Seg18] as the spherical functor whose twist gives the $\mathbb{P}$-twist around a $\mathbb{P}$-object. The claim we make here and the statement of *ibidem* tie up once we recall that the square of the spherical twist around a 1-spherical object is isomorphic to the $\mathbb{P}$-twist around the same object considered as a $\mathbb{P}^1$-object, see [HT06], and more generally that the square of the spherical twist around a d-spherical object is isomorphic to the $\mathbb{P}$-twist around E considered as a $\mathbb{P}^1[d]$-object, see [Kru18] for this notion.

However, the point we want to stress is that even if we did not know the result of [HT06], we could use the formula $T_E^2 = T_{\Psi_\varepsilon}$ to prove that the square of the spherical twist around a spherical object is the $\mathbb{P}$-twist around the same object.

This is the insight we want to provide to the reader: the gluing construction of Theorem 3.1.4 gives us many new spherical functors with the same twist because the glued spherical functor is not necessarily conservative. The question the author has in mind is: can we harness the failure of conservativity to find new, interesting relations among (a priori) different autoequivalences? For example, along the same lines of what we showed above, one can recover the well known fact that an A_n-configuration of spherical objects gives rise to an action of the braid group, see [ST01].

5.2 Four periodic SODs

Throughout the thesis we motivated the research that led to the mathematics presented in § 4 by a search for geometric examples of glued spherical functors as constructed in § 3. We want to use this concluding section to show that the mathematics of § 4 has also the potential to be used to prove new, interesting statements.

In § 4 we started from what we called a flop-flop diagram, see Definition 4.1.1,

$$\mathcal{B}_- \xleftarrow{\alpha_-} \mathcal{A} \xrightarrow{\alpha_+} \mathcal{B}_+$$

and we showed that left orthogonal to the category $\mathcal{K} = \ker \alpha_- \cap \ker \alpha_+$ admits a four periodic SOD whose induced spherical functors have twists whose inverses are isomorphic to the flop-flop autoequivalences.

What we want to point out is that it is easy to prove that, under the assumption that α_-^L and α_+^L are fully faithful, the existence of the four periodic SOD described in

Theorem 4.1.3 is equivalent to the flop functors being equivalences.

This observation can be used, for example, to prove that if the pull-push functors via $\widehat{X}$ for the standard flops are equivalences, then so are the pull-push functors via $\widehat{W}$ for the Mukai flops. See § 4.4.1 and § 4.4.2 for the notation we are employing.

Indeed, once we know that the flop functors for the standard flops are equivalences, we can use Theorem 4.1.3 to claim the existence of four SODs

$$\begin{aligned} \mathrm{D}_{\mathrm{qc}}(\widehat{X}) &= \langle \mathcal{K}_{\widehat{X}}, \mathcal{S}^X_+, p_-^* \mathrm{D}_{\mathrm{qc}}(X_-) \rangle = \langle \mathcal{K}_{\widehat{X}}, p_-^* \mathrm{D}_{\mathrm{qc}}(X_-), \mathcal{S}^X_- \rangle = \\ &= \langle \mathcal{K}_{\widehat{X}}, \mathcal{S}^X_-, p_+^* \mathrm{D}_{\mathrm{qc}}(X_+) \rangle = \langle \mathcal{K}_{\widehat{X}}, p_+^* \mathrm{D}_{\mathrm{qc}}(X_+), \mathcal{S}^X_+ \rangle \end{aligned}$$

where $\mathcal{K}_{\widehat{X}} = \ker(p_-)_* \cap \ker(p_+)_*$. Then, using the theory of base change for SODs as developed in [Kuz11], we obtain four SODs

$$\begin{aligned} \mathrm{D}_{\mathrm{qc}}(\widehat{W}) &= \langle \mathcal{K}_{\widehat{W}}, \mathcal{S}^W_+, q_-^* \mathrm{D}_{\mathrm{qc}}(W_-) \rangle = \langle \mathcal{K}_{\widehat{W}}, q_-^* \mathrm{D}_{\mathrm{qc}}(W_-), \mathcal{S}^W_- \rangle = \\ &= \langle \mathcal{K}_{\widehat{W}}, \mathcal{S}^W_-, q_+^* \mathrm{D}_{\mathrm{qc}}(W_+) \rangle = \langle \mathcal{K}_{\widehat{W}}, q_+^* \mathrm{D}_{\mathrm{qc}}(W_+), \mathcal{S}^W_+ \rangle \end{aligned} \tag{5.1}$$

Here we are implicitly using a series of facts that we either proved in § 4.4.2, or that are easy to prove. Namely, that

$$\langle \hat{g}^* \mathcal{K}_{\widehat{X}} \rangle^{\oplus} = \mathcal{K}_{\widehat{W}} \quad \text{and} \quad \langle \hat{g}^* p_\pm^* \mathrm{D}_{\mathrm{qc}}(X_\pm) \rangle^{\oplus} = q_\pm^* \mathrm{D}_{\mathrm{qc}}(W_\pm).$$

Given the four SODs (5.1), we obtain a four periodic SOD of ${}^\perp\mathcal{K}_{\widehat{W}}$, and using the fully faithfulness of q_-^* and q_+^* (which implicitly we already used because we wrote $q_-^* \mathrm{D}_{\mathrm{qc}}(W_-)$ and $q_+^* \mathrm{D}_{\mathrm{qc}}(W_+)$ rather than $\langle q_-^* \mathrm{D}_{\mathrm{qc}}(W_-) \rangle^{\oplus}$ and $\langle q_-^* \mathrm{D}_{\mathrm{qc}}(W_-) \rangle^{\oplus}$), we conclude that the flop functors for the Mukai flops are equivalences.

What the author wonders is: can we use this strategy to construct new examples in support of the Bondal–Orlov–Kawamata conjecture?

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