Spectral analysis of first order systems

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I will report on a project which I have been pursuing over the last 13 years jointly with my collaborators

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Playing field

Let M be a connected closed manifold of dimension $d \geq 2$.

Local coordinates $(x^1, \ldots, x^d)$.

We assume that M is equipped with a positive density $\rho(x)$.

Will work with m-columns of complex-valued scalar functions.

Inner product

$$\langle u, v \rangle := \int_M u^* v \, \rho \, \mathrm{d}x \,,$$

where $*$ stands for Hermitian conjugation and $\mathrm{d}x = \mathrm{d}x^1 \ldots \mathrm{d}x^d$.

Formally self-adjoint first order partial differential operator Q.

Physically meaningful examples

Let M be a connected oriented closed Riemannian 3-manifold.

- Massless Dirac operator acting on 2-component complex-valued spinors. Describes neutrino/antineutrino.
- Massive Dirac operator acting on 4-component complex-valued spinors. Describes electron/positron.
- The operator $\operatorname{curl} := *\mathrm{d}$ acting on real-valued 1-forms. Here d is the exterior derivative and $*$ is the Hodge star. Describes photon.

Single particle living in a closed 3-dimensional universe.

Important assumption number 1

Ellipticity:

$$\det Q_{\mathrm{prin}}(x,\xi) \neq 0\,, \qquad \forall (x,\xi) \in T^*M \setminus \{0\}\,,$$

The operator curl is not elliptic,

$$\det \mathrm{curl}_{\mathrm{prin}}(x,\xi) = 0\,, \qquad \forall (x,\xi) \in T^*M \setminus \{0\}\,,$$

but we know how to deal with this issue. Choose appropriate Hilbert space, that of coexact 1-forms.

Important assumption number 2

Eigenvalues of the principal symbol are simple.

Massless Dirac and curl satisfy this assumption, but massive Dirac does not.

We do not know how to drop this assumption.

Spectrum of Q: the basics

Spectrum of Q is discrete and accumulates both to $+\infty$ and $-\infty$.

The operator Q is not semi-bounded.

Spectrum of Q is asymmetric, so we need two counting functions

$$N^{\pm}(\lambda) := \begin{cases} 0 & \text{for } \lambda \leq 0, \\ \sum_{0<\pm\lambda_k<\lambda} 1 & \text{for } \lambda > 0. \end{cases}$$

Spectral asymptotics

We have

$$N^{\pm}(\lambda) = a\lambda^{d} + O(\lambda^{d-1}) \quad \text{as} \quad \lambda \to +\infty,$$

and under additional geometric assumptions on Hamiltonian trajectories generated by eigenvalues of the principal symbol

$$N^{\pm}(\lambda) = a\lambda^{d} + b^{\pm}\lambda^{d-1} + o(\lambda^{d-1}) \quad \text{as} \quad \lambda \to +\infty.$$

Second Weyl coefficient $b^{\pm}$ comes from the *bulk* of the manifold.

Writing down the first Weyl coefficient a is easy.

Writing down the second Weyl coefficient $b^{\pm}$ is not easy.

Second Weyl coefficient: a historical review

1 V.Ivrii, 1980, Soviet Math. Doklady.

2 V.Ivrii, 1982, Funct. Anal. Appl.

3 G.V.Rozenblyum, 1983, Journal of Mathematical Sciences.

4 V.Ivrii, 1984, Springer Lecture Notes. In 1998 book in Springer. In 2019 another book in 5 volumes, also in Springer.

5 Yu.Safarov, DSc thesis, 1989, Steklov Mathematical Institute.

6 W.J.Nicoll, PhD thesis, 1998, University of Sussex.

7 I.Kamotski and M.Ruzhansky, 2007, Comm. PDEs.

8 O.Chervova, R.J.Downes and D.Vassiliev, 2013, Journal of Spectral Theory.

Second Weyl coefficient: the final solution

Matteo Capoferri and I realised that we have been looking at first order systems the wrong way. Should seek *almost invariant subspaces* and *pseudodifferential projections* onto these subspaces.

Benefit of hindsight.

Invariant subspaces of elliptic systems I: pseudodifferential projections. Journal of Functional Analysis, 2022.

Invariant subspaces of elliptic systems II: spectral theory. Journal of Spectral Theory, 2022.

Given an $m \times m$ first order differential operator Q, seek m zero order pseudodifferential operators P_j, $j = 1, \ldots, m$, such that

$$P_j^2 = P_j \mod \Psi^{-\infty},$$

$$P_j^* = P_j \mod \Psi^{-\infty},$$

$$P_j P_k = 0 \mod \Psi^{-\infty} \quad \text{for } j \neq k,$$

$$\sum_{j=1}^m P_j = \mathrm{Id} \mod \Psi^{-\infty},$$

$$[Q, P_j] = 0 \mod \Psi^{-\infty},$$

$(P_j)_{\mathrm{prin}}(x,\xi)$ are projections onto eigenspaces of $(Q)_{\mathrm{prin}}(x,\xi)$.

An orthonormal basis of pseudodifferential projections.

Number of projections equals number of equations in our system.

Theorem An orthonormal basis of pseudodifferential projections exists and is unique modulo $\Psi^{-\infty}$.

Moreover, we provide an explicit algorithm for the construction of the orthonormal basis of pseudodifferential projections.

Algorithm is global and does not use local coordinates. Magic!

Implementation of 'magic' algorithm benefits from the use of the computer algebra package Mathematica©.

What about diagonalisation?

Construct an almost-unitary operator U such that U^*QU is a diagonal operator, modulo $\Psi^{-\infty}$.

Numerous publications, starting from Taylor 1975 and Cordes 1983.

- The diagonalisation approach is different from our projections approach.
- Diagonalisation doesn't always work.

Capoferri, Rozenblum, Saveliev and Vassiliev, *Topological obstructions to the diagonalisation of pseudodifferential systems*. Proceedings of the AMS, 2022.

Examples: the massless Dirac operator and the operator curl cannot be diagonalised, not even at a single point.

Source of topological obstructions to diagonalisation

Consider an Hermitian matrix whose eigenvalues are simple.

We have two different types of objects.

- Projections onto eigenspaces.
- Eigenvectors.

Projections onto eigenspaces are defined uniquely.

Eigenvectors are not uniquely defined. One can multiply them by $e^{i\phi}$, $\phi \in \mathbb{R}$. This $\mathrm{U}(1)$ gauge leads to topological obstructions when the matrix depends on parameters (x, ξ).

For diagonalisation one needs eigenvectors themselves. Projections onto eigenspaces are not enough.

What about higher Weyl coefficients?

Let $\widehat{\mu} : \mathbb{R} \to \mathbb{C}$ be a smooth function such that $\widehat{\mu} = 1$ in some neighbourhood of the origin and $\operatorname{supp} \widehat{\mu} \subset (-T_0, T_0)$, where T_0 is the infimum of lengths of all the Hamiltonian loops originating from all the points of the manifold. Let μ be the inverse Fourier transform of $\widehat{\mu}$. Then

$$((N^{\pm})' * \mu)(\lambda) = c_{d-1}\,\lambda^{d-1} + c^{\pm}_{d-2}\,\lambda^{d-2} + c^{\pm}_{d-3}\,\lambda^{d-3} + \dots$$

as $\lambda \to +\infty$. Here the star stands for convolution in the variable λ.

Compare with

$$N^{\pm}(\lambda) = a\lambda^{d} + b^{\pm}\lambda^{d-1} + o(\lambda^{d-1}) \quad \text{as} \quad \lambda \to +\infty.$$

Here

$$a = \frac{c_{d-1}}{d}, \qquad b^{\pm} = \frac{c^{\pm}_{d-2}}{d-1}.$$

Classical approach to spectral asymmetry

Definition of *eta function:*

$$\eta(s) := \sum_{\lambda_k \neq 0} \frac{\operatorname{sgn} \lambda_k}{|\lambda_k|^s} .$$

Series converges absolutely for $\operatorname{Re} s > d$.

Meromorphic continuation to $\mathbb{C}$.

Residues of the eta function are expressed via Weyl coefficients:

$$\operatorname{Res}(\eta, n) = c^+_{n-1} - c^-_{n-1} , \qquad n = d-1, d-2, \ldots .$$

Eta function generalises the more familiar zeta function.

Definition of *eta invariant:* $\eta(0)$.

Mathematicians who contributed to this subject area: Atiyah, Patodi, Singer, Hitchin, Gilkey, Pontryagin, Hirzebruch, Chern, Simons, Seeley ...

Key words: Hirzebruch L-polynomials, Hirzebruch $\hat{A}$-polynomials, Pontryagin forms, Pontryagin classes ...

Our approach to spectral asymmetry

Argue as an analyst as opposed to geometer.

Put

$$P_\pm := \theta(\pm Q)\,,$$

where

$$\theta(x) := \begin{cases} 1, & x > 0, \\ 0, & x \le 0, \end{cases}$$

is the Heaviside step function.

We know how to construct, modulo $\Psi^{-\infty}$, the projections $P_\pm$.

Then, morally,

$$\eta(0) = \mathrm{Tr}(P_+ - P_-)\,.$$

Elephant in the room

How do we calculate the trace of the operator $P_+ - P_-$?

The operator $P_+ - P_-$ is of order 0. Trace class requires order to be less than $-d$, where d is the dimension of the manifold.

Main idea

Idea: split the process of calculating trace into two separate steps.

- Take matrix trace first, i.e. contract indices in the integral kernel (Schwartz kernel) of the operator $P_+ - P_-$. This gives a *scalar* pseudodifferential operator which we denote by A and call *the asymmetry operator*.
- Attempt to calculate the trace of the asymmetry operator the usual way, by taking the value of the integral kernel on the diagonal $x = y$ and integrating over the manifold M.

Implementation of our approach for the operator curl

Miracle: for the operator curl the corresponding asymmetry operator A is a pseudodifferential operator of order -3.

Prima facie one would have expected the order of A to be zero.

Reason for miracle: symmetries of the Riemann curvature tensor.

The asymmetry operator is *almost* trace class.

The asymmetry operator is a self-adjoint scalar pseudodifferential operator which is completely determined by the Riemannian 3-manifold (M, g) and its orientation.

Singularity of the integral kernel of the asymmetry operator

Theorem The principal symbol of the asymmetry operator reads

$$A_{\mathrm{prin}}(x,\xi) = -\frac{\varepsilon^{\alpha\beta\gamma}}{2\,\rho(x)\,\|\xi\|^5}\,\nabla_\alpha\,\mathrm{Ric}_\beta{}^\delta(x)\,\xi_\gamma\,\xi_\delta\,,$$

where Ric is the Ricci curvature tensor, $\nabla\,\mathrm{Ric}$ is its covariant derivative and ε is the totally antisymmetric symbol (Levi-Civita symbol), $\varepsilon^{123} := +1$.

Corollary The singularity of the integral kernel $\mathfrak{a}(x,y)$ of the asymmetry operator is very weak. Namely, $\mathfrak{a}(x,y)$ is a bounded function, smooth outside the diagonal and discontinuous on the diagonal: for any $x \in M$ the limit $\lim_{y\to x}\mathfrak{a}(x,y)$ depends on the direction along which y tends to x.

The regularised local trace of the asymmetry operator

Denote by $\mathbb{S}_r(x)$ the geodesic sphere of radius $r > 0$ centred at the point $x \in M$.

Theorem For any $x \in M$ the limit

$$\lim_{r \to 0^+} \frac{1}{4\pi r^2} \int_{\mathbb{S}_r(x)} \mathfrak{a}(x, y)\, \mathrm{d}S_y$$

exists and defines a scalar continuous function

$$\psi^{\mathrm{loc}}(x)\,, \quad \psi^{\mathrm{loc}} : M \to \mathbb{R}\,.$$

Definition We call $\psi^{\mathrm{loc}}(x)$ *the regularised local trace of the asymmetry operator*.

The regularised global trace of the asymmetry operator

Definition We call the number

$$\psi := \int_M \psi^{\mathrm{loc}}(x)\,\rho(x)\,\mathrm{d}x$$

the regularised global trace of the asymmetry operator.

Our results for curl are presented in

Capoferri and Vassiliev, *Beyond the Hodge Theorem: curl and asymmetric pseudodifferential projections*.
`https://arxiv.org/abs/2309.02015`

Reconciling our approach with the classical one

Working with the operator curl and using microlocal techniques, we have defined a differential geometric invariant ψ, a measure of the asymmetry of our Riemannian 3-manifold under change of orientation.

Is it true that $\psi = \eta(0)$?

Yes, it is.

Capoferri and Vassiliev, *A microlocal pathway to spectral asymmetry: curl and the eta invariant*, in preparation.