Spectral theory of differential operators: what's it all about and what is its use
Part II

Dmitri Vassiliev

University College London

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Higher order operators

Let M be a compact d-dimensional manifold with boundary ∂M. Consider the spectral problem for an elliptic self-adjoint semi-bounded from below differential operator of even order $2n$:

$$Au = \lambda u \quad \text{on} \quad M, \qquad (B^{(j)}u)\Big|_{\partial M} = 0, \quad j = 1, \ldots, n.$$

Has been proven (by many authors over many years) that

$$N(\lambda) = a\lambda^{d/(2n)} + o(\lambda^{d/(2n)}) \quad \text{as} \quad \lambda \to +\infty,$$

where

$$a = \frac{1}{(2\pi)^d} \int_{A_{2n}(x,\xi)<1} dx\, d\xi$$

and $A_{2n}(x, \xi)$ is the principal symbol of the differential operator A.

For a general partial differential operator of order $2n$ Weyl's Conjecture reads

$$N(\lambda) = a\lambda^{d/(2n)} + b\lambda^{(d-1)/(2n)} + o(\lambda^{(n-1)/(2n)}) \quad \text{as} \quad \lambda \to +\infty,$$

where the constant b can also, hopefully, be written down explicitly.

Theorem (Vassiliev, 1984) Under certain geometric assumptions on the branching Hamiltonian billiards, Weyl's conjecture holds for higher order operators. Furthermore, I have an explicit algorithm for the evaluation of the second Weyl coefficient.

Branching Hamiltonian billiards

Define the Hamiltonian $h(x,\xi) := (A_{2n}(x,\xi))^{1/(2n)}$.

Consider Hamiltonian trajectories

$$\dot{x} = h_\xi(x,\xi), \quad \dot{\xi} = -h_x(x,\xi).$$

Reflection law at moment T:

- momentum experiences a jump $\xi|_{t=T^+} - \xi|_{t=T^-}$ which is conormal to the boundary, and
- the value of the Hamiltonian is preserved.

Geometric assumptions

- Periodic billiard trajectories have measure zero.
- Dead-end billiard trajectories have measure zero.

Evaluation of second Weyl coefficient

- Fix a point on the boundary and 'freeze' all coefficients in differential operator and boundary conditions.
- Leave only higher order derivatives in differential operator and boundary conditions.
- Replace all differentiations along the boundary $\partial/\partial x^\alpha$, $\alpha = 1, \ldots, d-1$, by $i\xi_\alpha$. This gives a spectral problem for an ordinary differential operator with constant coefficients on the half-line $x^d \in [0, +\infty)$.
- Introduce *spectral shift*: regularised trace of the spectral projection.
- Evaluate spectral shift using tools from scattering theory. Key word: *scattering phase* or *phase shift*.
- Integrate over $T^*\partial M$.

Proving the existence of a two-term asymptotics expansion

Introduce time t and study the equation

$$Au = \left(i \frac{\partial}{\partial t} \right)^{2n} u.$$

Construct the *propagator*

$$U(t) := e^{-itA^{1/(2n)}}.$$

Problem: equation not hyperbolic, so cannot pose Cauchy problem.

Solution: the propagator can be constructed without posing a Cauchy problem. Explained in my book + revisited in a recent paper jointly with Matteo Capoferri.

Yu.Safarov and D.Vassiliev, *The asymptotic distribution of eigenvalues of partial differential operators*, American Mathematical Society, 1997 (hardcover), 1998 (softcover).

"In the reviewer's opinion, this book is indispensable for serious students of spectral asymptotics". Lars Hörmander for the Bulletin of the London Mathematical Society.

Example: vibrations of a plate with clamped edge

$$\Delta^2 u = \lambda u \quad \text{in} \quad \Omega \subset \mathbb{R}^2, \qquad u|_{\partial\Omega} = \partial u/\partial n|_{\partial\Omega} = 0.$$

My formula (1987):

$$N(\lambda) = \frac{S}{4\pi}\lambda^{1/2} + \frac{\beta L}{4\pi}\lambda^{1/4} + o(\lambda^{1/4}) \quad \text{as} \quad \lambda \to +\infty$$

where S is area of the plate, L is length of the boundary and

$$\beta = -1 - \frac{\Gamma(3/4)}{\sqrt{\pi}\,\Gamma(5/4)} \approx -1.763.$$

The first asymptotic term was derived by Courant (1922).