# Invariant subspaces of elliptic systems

Dmitri Vassiliev

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Joint work with Matteo Capoferri (Cardiff).

Invariant subspaces of elliptic systems I: pseudodifferential projections, arXiv:2103.14325.

Invariant subspaces of elliptic systems II: spectral theory, arXiv:2103.14334.

# Playing field

Let $M$ be a closed connected manifold of dimension $d \geq 2$. Local coordinates $x = (x^1, \ldots, x^d)$.

Will work with $m$-columns of complex-valued half-densities.

Inner product

$$\langle v, w \rangle := \int_M v^* w \, dx \,,$$

where $dx := dx^1 \ldots dx^d$.

By $\Psi^s$ we denote the space of classical pseudodifferential operators of order $s$. For an operator $P \in \Psi^s$ we denote its matrix-valued principal and subprincipal symbols by $P_{\mathrm{prin}}$ and $P_{\mathrm{sub}}$ respectively. These are invariantly defined matrix-functions on $T^*M \setminus \{0\}$.

**Definition 1** We say that $P \in \Psi^0$ is an *orthogonal pseudodifferential projection* if

$$P^2 = P \mod \Psi^{-\infty},$$

$$P^* = P \mod \Psi^{-\infty}.$$

**Definition 2** We call a set of $m$ orthogonal pseudodifferential projections $\{P_j\}$ an *orthonormal pseudodifferential basis* if their principal symbols are rank 1 matrix-functions and

$$P_j P_k = 0 \mod \Psi^{-\infty} \quad \forall j \neq k,$$

$$\sum_j P_j = \mathrm{Id} \mod \Psi^{-\infty},$$

where $\mathrm{Id} \in \Psi^0$ is the identity operator.

**Question 1** Does there exist a nontrivial operator $P$ satisfying Definition 1?

**Question 2** Assuming that the answer to Question 1 is positive, can we choose the $P_j$'s so that they satisfy Definition 2?

Need to solve an infinite sequence of heavily overdetermined systems of algebraic equations for the homogeneous components of the symbols of the $P_j$'s, and it is not *a priori* clear that these systems have solutions.

Great care is needed in performing this analysis because our operators have matrix-valued symbols which in general do not commute.

Let $A \in \Psi^s$, $s \in \mathbb{R}$, $s > 0$, be an elliptic self-adjoint linear pseudodifferential operator, where ellipticity means that

$$\det A_{\mathrm{prin}}(x,\xi) \neq 0, \qquad \forall (x,\xi) \in T^*M \setminus \{0\}.$$

**Important assumption** The matrix-function $A_{\mathrm{prin}}(x,\xi)$ is assumed to have simple eigenvalues.

We denote by $m^+$ (resp. $m^-$) the number of positive (resp. negative) eigenvalues of $A_{\mathrm{prin}}(x,\xi)$. We denote by $h^{(j)}(x,\xi)$ the eigenvalues of $A_{\mathrm{prin}}(x,\xi)$ enumerated in increasing order, with positive index $j=1,2,\dots,m^+$ for positive $h^{(j)}(x,\xi)$ and negative index $j=-1,-2,\dots,-m^-$ for negative $h^{(j)}(x,\xi)$.

Of course, $m^+ + m^- = m$.

By $P^{(j)}(x,\xi)$ we denote the eigenprojection of $A_{\mathrm{prin}}(x,\xi)$ corresponding to the eigenvalue $h^{(j)}(x,\xi)$. The matrix-functions $P^{(j)}(x,\xi)$ are rank 1.

**Question 3** Assuming that the answer to Question 2 is positive, can we choose the $P_j$'s so that they commute with the operator $A$

$$[A, P_j] = 0 \mod \Psi^{-\infty}$$

and

$$(P_j)_{\mathrm{prin}} = P^{(j)}?$$

**Question 4** Can we exploit the pseudodifferential projections $P_j$ to advance the current understanding of spectral asymptotics for elliptic systems?

**Question 5** Can we exploit the pseudodifferential projections $P_j$ to advance the current understanding of propagation of singularities for hyperbolic systems?

# Existing literature on pseudodifferential projections

Integration of $(A - \lambda\,\mathrm{Id})^{-1}$ over a careful chosen contour in the complex plane.

Birman and Solomyak 1982.

Bolte and Glaser 2005. Semiclassical setting + additional assumption on the separation of the eigenvalues of $A_{\mathrm{prin}}(x, \xi)$.

Problem: difficult to carry out explicit calculations beyond the principal symbol.

Example: abstract formula for the second Weyl coefficient of a system 1984 (Ivrii), actual formula 2013 (Vassiliev and co-authors).

# Existing literature on microlocal diagonalisation

Construct an almost-unitary operator $U$ such that $U^*AU$ is a diagonal matrix operator, modulo $\Psi^{-\infty}$.

Numerous publications, starting from Taylor 1975 and Cordes 1983.

Problem: the almost-unitary operator $U$ is not defined uniquely, not even at the level of the principal symbol, but only up to gauge transformations. These gauge transformations generate curvature.

Source of problem: a normalised eigenvector of an Hermitian matrix is not uniquely defined, one can multiply it by $e^{i\phi}$, $\phi \in \mathbb{R}$.

**Theorem 1** Given a family of $m$ orthonormal rank 1 projections

$$P^{(j)} \in C^\infty(T^*M \setminus \{0\}; \mathrm{Mat}(m; \mathbb{C}))$$

positively homogeneous in momentum of degree zero, there exists an orthonormal pseudodifferential basis $\{P_j\} \subset \Psi^0$ as per Definition 2 with $(P_j)_{\mathrm{prin}} = P^{(j)}$.

**Proof** Explicit algorithm leading to the determination of the full symbols of the pseudodifferential projections $P_j$'s. Algorithm is global and does not use local coordinates.

**Remark** The full symbols of the pseudodifferential projections $P_j$'s are not uniquely defined. Remaining degrees of freedom are described by explicit formulae.

**Remark** One can turn approximate projections into exact projections, i.e. drop the modulo $\Psi^{-\infty}$.

**Theorem 2** There exist $m$ pseudodifferential operators $P_j \in \Psi^0$ satisfying Definition 2 and conditions

$$[A, P_j] = 0 \quad \text{mod } \Psi^{-\infty},$$

$$(P_j)_{\mathrm{prin}} = P^{(j)},$$

and these are uniquely determined, modulo $\Psi^{-\infty}$, by the operator $A$.

**Proof** Application of the algorithm from Theorem 1.

**Remark** We cannot drop the modulo $\Psi^{-\infty}$ in the commutation condition $[A, P_j] = 0$.

**Theorem 3** The explicit formula for the subprincipal symbol of the pseudodifferential projection $P_j$ reads

$$(P_j)_{\mathrm{sub}} = \frac{i}{2}\{P^{(j)}, P^{(j)}\} - i\,P^{(j)}\{P^{(j)}, P^{(j)}\}P^{(j)} + \sum_{l\neq j} \frac{P^{(j)}(A_{\mathrm{sub}} - iQ^{(j)})P^{(l)} + P^{(l)}(A_{\mathrm{sub}} + iQ^{(j)})P^{(j)}}{h^{(j)} - h^{(l)}},$$

where

$$Q^{(j)} := \frac{1}{2}\big(\{A_{\mathrm{prin}}, P^{(j)}\} - \{P^{(j)}, A_{\mathrm{prin}}\}\big).$$

Here curly brackets denote the matrix-valued Poisson bracket

$$\{B, C\} := \sum_{\alpha=1}^{d}(B_{x^\alpha}\,C_{\xi_\alpha} - B_{\xi_\alpha}\,C_{x^\alpha}).$$

**Remark** The trace of the matrix $(P_j)_{\mathrm{sub}}(x,\xi)$ has the geometric meaning of scalar curvature generated by gauge transformations of the $j$th eigenvector of $A_{\mathrm{prin}}(x,\xi)$.

**Remark** Safarov mistakenly assumed (DSc thesis, 1989) that

$$\operatorname{tr}(P_j)_{\mathrm{sub}} = 0.$$

**Remark** In 1989 Safarov and I did not think in terms of invariant subspaces and pseudodifferential projections. We also had no idea about gauge transformations and curvature that they generate. Took decades to get a clear understanding and fix mistakes: Nicoll's PhD thesis 1998, Chervova's PhD thesis 2012, Downes' PhD thesis 2014 and Fang's PhD thesis 2017. The idea of thinking in terms of invariant subspaces and pseudodifferential projections emerged in 2020, after Capoferri submitted his PhD thesis.

**Definition 3** We say that a symmetric pseudodifferential operator $B$ is *nonnegative (resp. nonpositive) modulo* $\Psi^{-\infty}$ and write

$$B \geq 0 \mod \Psi^{-\infty} \qquad (\text{resp. } B \leq 0 \mod \Psi^{-\infty})$$

if there exists a symmetric operator $C \in \Psi^{-\infty}$ such that $B + C \geq 0$ (resp. $B + C \leq 0$).

**Theorem 4** We have

$$P_j^* A P_j \geq 0 \quad \text{mod } \Psi^{-\infty} \quad \text{for} \quad j = 1, \dots, m^+,$$
$$P_j^* A P_j \leq 0 \quad \text{mod } \Psi^{-\infty} \quad \text{for} \quad j = -1, \dots, -m^-.$$

**Remark** The operators $P_j^* A P_j$ in Theorem 4 are not elliptic,

$$\det(P_j^* A P_j)_{\mathrm{prin}}(x, \xi) = 0 \qquad \forall (x, \xi) \in T^*M \setminus \{0\},$$

therefore, proving that they are sign semidefinite modulo $\Psi^{-\infty}$ is a delicate matter. The fact that their principal symbols are sign semidefinite does not, on its own, imply that the operators are sign semidefinite — it does not even imply that they are semibounded.

**Remark** We have $A = \sum_j P_j^* A P_j \mod \Psi^{-\infty}$, i.e. we decomposed our operator $A$ into a sum of $m$ sign semidefinite operators.

**Theorem 5** The operator $|A|$ is pseudodifferential and

$$|A| = \sum_{j=1}^{m^+} AP_j - \sum_{j=1}^{m^-} AP_{-j} \mod \Psi^{-\infty}.$$

Furthermore, the explicit formula for the subprincipal symbol of the operator $|A|$ reads

$$|A|_{\mathrm{sub}} = \sum_{j,k} \frac{h^{(j)} + h^{(k)}}{|h^{(j)}| + |h^{(k)}|} P^{(j)} A_{\mathrm{sub}} P^{(k)}$$

$$+ \frac{i}{2} \sum_{j,k} \frac{1}{|h^{(j)}| + |h^{(k)}|} P^{(j)} (\{A_{\mathrm{prin}}, A_{\mathrm{prin}}\} - \{|A|_{\mathrm{prin}}, |A|_{\mathrm{prin}}\}) P^{(k)}.$$

Let $\theta : \mathbb{R} \to \mathbb{R}$,

$$\theta(z) := \begin{cases} 0 & \text{if} \quad z \le 0, \\ 1 & \text{if} \quad z > 0 \end{cases}$$

be the Heaviside function.

**Theorem 6** The operator $\theta(A)$ is pseudodifferential and

$$\theta(A) = \sum_{j=1}^{m^+} P_j \quad \text{mod } \Psi^{-\infty}.$$

**Remark** Theorems 6 and 3 give us an explicit formula for $[\theta(A)]_{\mathrm{sub}}$.

# Spectral results

Let $m^+ \geq 2$ and let

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k \leq \cdots \to +\infty$$

be the positive eigenvalues of $A$ enumerated in increasing order with account of multiplicity.

**Task:** partition the positive eigenvalues $\lambda_k$ of the operator $A$ into $m^+$ separate series corresponding to the $m^+$ different positive eigenvalues $h^{(j)}(x,\xi)$ of the matrix-function $A_{\mathrm{prin}}(x,\xi)$.

eigenvalues of operator $A \overset{?}{\longleftrightarrow}$ eigenvalues of matrix-function $A_{\mathrm{prin}}(x,\xi)$

Naive approach: look at eigenvalues of $P_j^* A P_j$, $j = 1, \ldots, m^+$. Doesn't work because the operators $P_j^* A P_j$ are not elliptic.

Let us introduce the operators

$$A_j := A - 2 \sum_{\substack{l=1,\dots,m^+ \\ l \neq j}} P_l^* A P_l \,, \qquad j = 1, \dots, m^+ .$$

Each operator $A_j$ is elliptic and is 'simpler' than our original operator $A$ in that the principal symbol of $A_j$ has only one positive eigenvalue, namely, $h^{(j)}(x, \xi)$.

Let

$$0 < \lambda_1^{(j)} \le \lambda_2^{(j)} \le \cdots \le \lambda_k^{(j)} \le \cdots \to +\infty$$

be the positive eigenvalues of $A_j$ enumerated in increasing order with account of multiplicity.

**Theorem 7** For each $j = 1, \dots, m^+$ we have

$$\operatorname{dist}\big(\lambda_k^{(j)}, \sigma^+(A)\big) = O(k^{-\infty}) \quad \text{as} \quad k \to +\infty.$$

**Theorem 8** We have

$$\operatorname{dist}\big(\lambda_k \,, \textstyle\bigcup_{j=1}^{m^+} \sigma^+(A_j)\big) = O(k^{-\infty}) \quad \text{as} \quad k \to +\infty.$$

Theorems 7 and 8 do not quite achieve the sought after partition of the spectrum of $A$ in that they do not establish a one-to-one correspondence between the positive eigenvalues of the operator $A$ and the positive eigenvalues of the operators $A_j$ , $j = 1, \dots, m^+$. These theorems establish asymptotic closeness of the spectra but do not provide sufficient information on the closeness of individual eigenvalues enumerated in our particular way.

Let us combine the sequences $\{\lambda_k^{(j)}\}_{k\in\mathbb{N}}$, $j = 1, \dots, m^+$, into one sequence and denote it by

$$0 < \mu_1 \le \mu_2 \le \cdots \le \mu_k \le \cdots \to +\infty.$$

Here we combine them with account of multiplicities.

**Theorem 9** For any $\alpha > 0$ there exists an $r_\alpha \in \mathbb{Z}$ such that

$$\lambda_k = \mu_{k+r_\alpha} + O(k^{-\alpha}) \quad \text{as} \quad k \to +\infty.$$

**Remark** We are unable to replace $\alpha$ by $\infty$ in Theorem 9.

# Propagation of singularities for hyperbolic systems

Let $A \in \Psi^1$ and let $U(t) := e^{-itA}$ be the propagator.

It is known that the propagator can be written, modulo an infinitely smoothing operator, as the sum of precisely $m$ oscillatory integrals

$$U(t) = \sum_j U^{(j)}(t) \mod C^\infty(\mathbb{R}; \Psi^{-\infty})$$

where each $U^{(j)}(t)$ is a Fourier integral operator whose Schwartz kernel is a Lagrangian distribution associated with the Lagrangian submanifold of $T^*\mathbb{R} \times T^*M \times T^*M$ generated by the Hamiltonian flow of the $j$th eigenvalue of the principal symbol of $A$.

**Theorem 10** We have

$$U^{(j)}(t) = P_j\, U(t) = U(t)\, P_j \mod C^\infty(\mathbb{R}; \Psi^{-\infty}).$$

Let $A \in \Psi^2$ be a nonnegative operator and let $U(t) := e^{-it\sqrt{A}}$ be the propagator.

In this case we also have

$$U(t) = \sum_j U^{(j)}(t) \mod C^\infty(\mathbb{R}; \Psi^{-\infty})$$

where each $U^{(j)}(t)$ is a Fourier integral operator whose Schwartz kernel is a Lagrangian distribution associated with the Lagrangian submanifold of $T^*\mathbb{R} \times T^*M \times T^*M$ generated by the Hamiltonian flow of the $j$th eigenvalue of the principal symbol of $A$.

**Theorem 11** We have

$$U^{(j)}(t) = P_j\, U(t) = U(t)\, P_j \mod C^\infty(\mathbb{R}; \Psi^{-\infty}).$$

# Applications

- Massless Dirac operator on a Riemannian 3-manifold.
- Elasticity operator on a Riemannian 2-manifold.
- The Dirichlet-to-Neumann map $DN$ for elasticity. Here the elastic body is assumed to occupy a domain $\Omega \subset \mathbb{R}^3$ and the operator $DN$ acts on $\partial\Omega$.