# Dirac and Weyl equations from Cosserat elasticity

James Burnett, Olga Chervova
and Dmitri Vassiliev

13 February 2009

Mathematical Foundations meeting

**Describing a deformable continuous medium**

(a) Classical elasticity: displacements only.

(b) Cosserat elasticity: displacements and rotations. See

E. Cosserat and F. Cosserat, *Théorie des Corps Déformables*, 1909. Available from Amazon.

(c) Teleparallelism (absolute parallelism, fernparallelismus): rotations only.

## Teleparallelism in Euclidean 3-space

Work in $\mathbb{R}^3$ equipped with standard metric and Cartesian coordinates $x^\alpha$, $\alpha = 1, 2, 3$.

A *frame* $\vartheta$ is a triplet $\vartheta^j$, $j = 1, 2, 3$, of orthonormal vector fields. Each vector field $\vartheta^j$ has a hidden tensor index: $\vartheta^j = \vartheta^j_\alpha$, $\alpha = 1, 2, 3$.

Same in plain English: a frame is a field of orthonormal bases.

**NB.** Frame lives separately from Cartesian coordinates (not aligned with coordinate lines).

## My model

The frame $\vartheta$ is an unknown quantity (dynamical variable).

The other dynamical variable is a density $\rho$.

My Lagrangian density $L(\vartheta, \rho)$ is chosen from the condition of conformal invariance.

Action (variational functional) $\int L(\vartheta, \rho)\, dx^1 dx^2 dx^3$.

Vary action with respect to frame $\vartheta$ and density $\rho$ to get Euler–Lagrange equations.

## Difference with existing models

1. I assume metric to be fixed (prescribed).

2. My Lagrangian has never been considered.

## Introducing time into my model

Standard Newtonian construction: write down angular velocity, write down kinetic energy etc.

My model remains conformally invariant, only now in the Lorentzian sense.

## Solving Euler–Lagrange equations

Switch to spinors:

frame $\vartheta$ and density $\rho > 0$

$\Updownarrow$

nonvanishing spinor field $\xi$ modulo sign

My Lagrangian density $L(\xi)$ is a rational function of $\xi$, $\bar{\xi}$ and partial derivatives of $\xi$, $\bar{\xi}$.

## Quasi-stationary solutions

Look first for quasi-stationary solutions

$$\xi(t, x^1, x^2, x^3) = e^{-i\omega t}\eta(x^1, x^2, x^3), \qquad \omega \neq 0.$$

**Theorem 1** *In the quasi-stationary case my Euler–Lagrange equation is equivalent to a pair of Weyl equations.*

"Weyl equation" = "massless Dirac equation".

**Proof** Amazing fact: my Lagrangian factorises into a product of two Weyl Lagrangians! Result follows from factorisation. □

## Plane waves

Special case of quasi-stationary solution

$$\xi(t, x^1, x^2, x^3) = e^{-i(\omega t + k\cdot x)}\eta$$

where $\eta$ is constant spinor. This is *plane wave*.

**Corollary 1** *Plane wave solutions in my model are the same as for a pair of Weyl equations.*

## Perturbations of plane waves

Idea: seek spinor field in the form

$$\text{slowly varying amplitude } \times e^{-i(\omega t + k\cdot x)}.$$

**Theorem 2** *Perturbations of plane wave solutions in my model are described by a pair of Weyl equations.*

## Relativistic version of my model

Work in Minkowski 4-space instead of Euclidean 3-space. Write $x^0$ instead of $t$. Frame now has 4 elements, each of these has 4 components.

## Comparing the relativistic and nonrelativistic models

Relativistic model has 3 extra degrees of freedom (Lorentz boosts in 3 directions) and, consequently, 3 extra field equations.

**Theorem 3** *At the perturbative level the 3 extra field equations are automatically satisfied.*

*Conclusion:* my nonrelativistic model possesses relativistic invariance at the perturbative level.

## Incorporating mass $m$ and electromagnetic (co)vector potential $A$

Introduce 5th coordinate: $(x^0, x^1, x^2, x^3, \underline{x^4})$.

O.Klein (1926): prescribe oscillation $\sim e^{-imx^4}$ along extra coordinate, then separate variables.

T.Kaluza (1921): perturb extended metric

$$\begin{pmatrix} g_{\alpha\beta} & 0 \\ 0 & -1 \end{pmatrix} \rightarrow \begin{pmatrix} g_{\alpha\beta} - \frac{1}{m^2} A_\alpha A_\beta & \frac{1}{m} A_\alpha \\ \\ \frac{1}{m} A_\beta & -1 \end{pmatrix}.$$

This is *shear*.

**NB.** Kaluza–Klein extension destroys conformal invariance!

**Theorem 4** *In special case with no dependence on $x^3$ my model is equivalent to the massive Dirac equation with electromagnetic field.*

## Summary

New mathematical model for fermions.

- Spacetime viewed as Cosserat continuum.
- Lagrangian chosen from the condition of conformal invariance.
- Mass and electromagnetic field incorporated via Kaluza–Klein extension.