Smooth approximations for constant-mean-curvature hypersurfaces with isolated singularities

Costante Bellettini* Konstantinos Leskas†

Abstract

We consider a CMC hypersurface with an isolated singular point at which the tangent cone is regular, and such that, in a neighbourhood of said point, the hypersurface is the boundary of a Caccioppoli set that minimises the standard prescribed-mean-curvature functional. We prove that in a ball centred at the singularity there exists a sequence of smooth CMC hypersurfaces, with the same prescribed mean curvature, that converge to the given one. Moreover, these hypersurfaces arise as boundaries of minimisers. In ambient dimension 8 the condition on the cone is redundant. (When the mean curvature vanishes identically, the result is the well-known Hardt–Simon approximation theorem.)

1 Introduction

It is well known that variational constructions for area-type functionals may lead to singularity formation. Already in the widely studied case of area minimisation, if the ambient dimension is 8 or higher, solutions cannot be expected to be completely smooth. The case of volume-constrained perimeter minimisation, which leads to isoperimetric regions, is analogous: in $\mathbb{R}^{n+1}$, or more generally in an $(n+1)$-dimensional Riemannian manifold, such regions have boundaries that are smoothly embedded away from a possible singular set of dimension at most $(n-7)$; when $n = 7$ the singular set is made more precisely of isolated points. The phenomenon arises yet again in the case of minimax constructions for prescribed-mean-curvature functionals.

Examples show that this singular set is in general unavoidable. The well-known minimal cone $C_{4,4} = \{(x, y) \in \mathbb{R}^4 \times \mathbb{R}^4 \equiv \mathbb{R}^8 : |x|^2 = |y|^2\}$ (shown to be stable by Simons [22]), is smooth away from the isolated singularity at the origin, and is area-minimising, e.g. in any ball $B \subset \mathbb{R}^8$, with respect to the boundary condition $C_{4,4} \cap \partial B$. This was proved by Bombieri–De Giorgi–Giusti ([5], see also a more

*Department of Mathematics, University College London, United Kingdom.
†Department of Mathematics, National and Kapodistrian University of Athens, Greece.

straightforward proof in [9]). This cone is in fact the unique minimiser for said boundary condition. An isoperimetric region with two isolated singular points in an 8-dimensional Riemannian manifold was recently constructed in [18].

On the other hand, it is fruitful to ask whether the appearance of singularities is a generic phenomenon. This question led to very important progress already in the 80s and has received renewed attention in recent years. The fundamental work by Hardt-Simon [15] shows an instance of generic regularity for solutions to the Plateau problem, in the following sense. Let a 7-dimensional area minimiser in $\mathbb{R}^8$ be given, with (prescribed) 6-dimensional smooth boundary Γ, and with an isolated singular point; then a slight perturbation of Γ yields a minimiser that is completely smooth. This type of result lends itself to geometric applications, by shifting the genericity condition onto the Riemannian metric, as exemplified by Smale's proof of generic regularity of area-minimisers in any non-zero homology class [23]. Very recently, the question of generic regularity for area minimisers has found affirmative answer in ambient dimension 9 and 10, in the work by Chodosh–Mantoulidis–Schulze [7], making progress on a long-standing conjecture ([23]).

Our main goal here is to prove a (local) smooth approximation result in the constant-mean-curvature (CMC) case, establishing a generic regularity result for the CMC Plateau problem. (When the mean curvature vanishes identically, the Hardt–Simon theorem gives the result.) The variational setting for CMC hypersurfaces involves an energy that we will denote by J_λ, where $\lambda \in \mathbb{R}$ is the prescribed constant value of the scalar mean curvature. Roughly speaking, J_λ evaluates the n-dimensional area of the hypersurface, from which it subtracts λ times the $(n+1)$-volume enclosed by it. A natural way to formalise this is by working with boundaries of sets with finite perimeter. We briefly recall the relevant notions (with more details in Section 2 below).

Let $E \subset U$ be a set with locally finite perimeter in a bounded open set $U \subset \mathbb{R}^{n+1}$, and let $\lambda \in \mathbb{R}$. We denote by J_λ the functional (defined on any set $D \subset U$ with locally finite perimeter in U),

$$J_\lambda(D) = \mathrm{Per}_U(D) - \lambda|D|,$$

where the notation $|D|$ stands for $\mathcal{L}^{n+1}(D)$. Given $W \subset\subset U$, the set E is said to be a minimiser of J_λ in $W \subset U$ if it attains the following infimum:

$$\inf\{J_\lambda(D) : D \cap (U \setminus W) = E \cap (U \setminus W)\}.$$

In other words, the class of competitors for E is that of sets (with locally finite perimeter in U) that coincide with E outside W. Equalities between sets of locally finite perimeter are always understood to hold in the $\mathcal{L}^{n+1}$-a.e. sense. Prescribing the set in $U \setminus W$ amounts to fixing the boundary condition for the Plateau problem in W (as customary in the setting of Caccioppoli sets).

If E is a minimiser of J_λ in $W \subset\subset U$, it is well-known (see e.g. [17, 14, 3]) that (upon passing to the Lebesgue representative of E) there exists a set $\Sigma \subset W$ with $\dim_{\mathcal{H}}(\Sigma) \leq n-7$, such that $(\partial E \cap W') \setminus \Sigma$ is smoothly embedded in W' for every open set $W' \subset\subset W$, and that $(\partial E \cap W') \setminus \Sigma$ has constant scalar mean curvature equal to λ. (More precisely, the mean curvature vector is $\lambda \nu_E$, where ν_E is the unit normal pointing into E.)

The most immediate instance of our result states the following.

Theorem 1. *Let E be a set with locally finite perimeter in an open set $U \subset \mathbb{R}^8$, and assume that E minimises J_λ in a ball $\hat{B} \subset\subset U$, for a given $\lambda \in \mathbb{R}$. There exists a ball $B \subset \hat{B}$, with the same centre, and a sequence of hypersurfaces T_j smoothly embedded in B, with scalar mean curvature λ, and with $T_j \to \partial^* E$ in B. (The convergence holds in the sense of currents, in the sense of varifolds, as well as in the Hausdorff distance sense.) Moreover, $T_j = \partial^* E_j$, where each E_j is a set with finite perimeter in B and $\partial^* E_j$ stands for the reduced boundary of E_j in B, and we have $E_j \subset E$ and $E_j \to E$ in B.*

We remark that the significance of Theorem 1 lies in the fact that the centre p of $\hat{B}$ may be a singular point of $\overline{\partial^* E}$.

In ambient dimension 8, as in Theorem 1, isolated singular points are the only type of interior singularities that $\overline{\partial^* E}$ may possess. This is no longer the case when the ambient dimension is higher. Just as in [15], we can remove the dimensional restriction in Theorem 1 by (strongly) restricting the singular behaviour of E (Theorem 2 below). We work in a neighbourhood of an isolated (interior) singular point p of $\overline{\partial^* E}$, with the further property that the multiplicity-1 varifold associated to $\partial^* E$, denoted by $|\partial^* E|$, admits a tangent cone at p that is regular. We recall that a cone is regular when it is smooth away from the vertex, and the multiplicity is 1 on the smooth part.

Theorem 2. *Let E be a set with locally finite perimeter in an open set $U \subset \mathbb{R}^{n+1}$, with $n \geq 7$, and assume that E minimises J_λ in a ball $\hat{B} \subset\subset U$, for a given $\lambda \in \mathbb{R}$. Assume furthermore that the centre p of $\hat{B}$ is an isolated singularity of $|\partial^* E|$ and that $|\partial^* E|$ admits a tangent cone at p that is regular (in the sense of varifolds).*

There exists a ball $B \subset \hat{B}$, with the same centre p, and a sequence of hypersurfaces T_j smoothly embedded in B, with scalar mean curvature λ, and with $T_j \to \partial E$ in B. (The convergence holds in the sense of currents, in the sense of varifolds, as well as in the Hausdorff distance sense.) Moreover, $T_j = \partial^* E_j$, where each E_j is a set with finite perimeter in B and $\partial^* E_j$ stands for the reduced boundary of E_j in B, and we have $E_j \subset E$ and $E_j \to E$ in B.

Remark 1.1. By construction, for each j the set E_j is a minimiser, more precisely, it is given by $\hat{E}_j \cap B$ for a set with finite perimeter $\hat{E}_j \subset \hat{B}$ that minimises J_λ in $B \subset \hat{B}$ (among sets that coincide with $\hat{E}_j$ in $\hat{B} \setminus B$). The mean curvature vector of $|\partial^* E_j|$ in B is given by $\lambda \nu_{E_j}$, where ν_{E_j} is the inward pointing unit normal.

Remark 1.2. The regularity theory for $n = 7$ implies not only that the singular set is made of isolated points, but also that any varifold tangent (at a singular point) must be regular, via a standard dimension reduction argument. Therefore Theorem 1 follows from Theorem 2.

Remark 1.3. In the special case $\lambda = 0$ Theorems 1 and 2 were proved in [15] (see also [7]). Our proof relies on the result for $\lambda = 0$.

Remark 1.4. In both Theorems 1 and 2, the convergence $T_j \to \partial E$ is strong (graphical and C^2) in $B \setminus \{p\}$, thanks to Allard's regularity theorem and standard elliptic PDE theory.

Remark 1.5. Theorems 1 and 2 lend themselves applications in geometry, such as the surgery procedure in [2] (where a generic existence result for smooth CMC closed hypersurfaces in compact Riemannian 8-dimensional manifolds is proved).

In proving Theorem 2 we establish a result of independent interest on the existence and regularity of minimisers of J_λ, for the CMC Plateau problem. We present here a simplified version (sufficient for its scope within the proof of Theorem 2). The more general result requires some notation and will be given in Theorem 4 of Section 2.

Theorem 3. *Let E_0 be a set with finite perimeter in $U = B^{n+1}_R(p)$. Let $\lambda \in (0, \infty)$ and $r \in (0, \frac{n}{\lambda})$. Assume that ∂E_0 is smooth in a neighbourhood of $\partial B^{n+1}_r(p)$ and that it intersects $\partial B^{n+1}_r(p)$ transversely; let T_0 denote (the $(n-1)$-dimensional submanifold) $\partial E_0 \cap \partial B^{n+1}_r(p)$.*

There exists a set E, with finite perimeter in $B^{n+1}_R(p)$, that coincides a.e. with E_0 in $B^{n+1}_R(p) \setminus B^{n+1}_r(p)$, that is a minimiser of J_λ in $B^{n+1}_r(p) \subset B^{n+1}_R(p)$, and with the following properties:

- *there exists $\Sigma \subset B^{n+1}_r(p)$, closed in $B^{n+1}_r(p)$, with $\dim_{\mathcal{H}}(\Sigma) \leq n-7$ such that $\left(\overline{\partial^* E} \cap B^{n+1}_r(p)\right) \setminus \Sigma$ is a smoothly embedded hypersurface with mean curvature $\lambda \nu_E$, where ν_E is the inward unit normal to E; more precisely, $\Sigma = \emptyset$ if $n \leq 6$, and Σ is discrete if $n = 7$.*
- *$\overline{\partial^* E} \cap \partial B^{n+1}_r(p) = T_0$.*

The 'boundary condition' is set by prescribing the coincidence a.e. with a reference set E_0 (the condition $r < R$ provides an annulus in which E_0 is non-trivial). The submanifold T_0 acts as prescribed boundary condition. The last conclusion of the theorem states that the solution does not touch $\partial B^{n+1}_r(p)$ except at T_0. So $\overline{\partial^* E} \setminus (B^{n+1}_R(p) \setminus \overline{B^{n+1}_r(p)}) \setminus \Sigma$ is a smooth hypersurface with boundary in the open set $B^{n+1}_R(p) \setminus \Sigma$. (Since T_0 is smooth, Σ does not accumulate onto T_0 by Allard's boundary regularity theorem, [1]; this property is not needed in our forthcoming arguments.) The condition $\lambda < \frac{n}{r}$ is essential for the last conclusion of Theorem 3, as we will point out in Section 2. On the other hand, existence alone follows for any λ. In Theorem 4 below we will drop smoothness and tranversality conditions.

Theorem 3 (and Theorem 4 below) and its proof are close in spirit to the results in Duzaar–Fuchs [11] (and Duzaar [10]). We highlight that our last conclusion in Theorem 3 is sharper than the corresponding statement in [10, 11], since we are able to rule out any interior touching of the solution with the "obstacle" $\partial B^{n+1}_r(p)$ in which the boundary condition T_0 lies (the only touching is the necessary one at T_0 itself). The results in [10, 11], while establishing the validity of the CMC condition, would only prevent touching of the solution with larger spheres. The sharper conclusion we obtain is ultimately due to our use of the regularity theory for stable CMC hypersurfaces developed in [3, 4] (with the sheeting theorem therein being the key ingredient in our proof). The same reasoning that we employ to that end (see Lemma 2.4 and the discussion preceding it) can be applied to sharpen the corresponding conclusion in [11].

We are now ready to present an outline of the proof of Theorem 2, setting $p = 0$. By fairly standard arguments, there exists a sufficiently small ball centred at 0, which we denote by $B_{2R}(0)$, such that E is the unique minimiser of J_λ in $B_R(0) \subset B_{2R}(0)$, and with the further requirements that $\lambda < \frac{n}{R}$ and that ∂E meets $\partial B_R(0)$ smoothly and transversely.

Then we perturb E towards its interior (keeping it fixed outside an annulus that contains $\partial B_r(0)$) and use the resulting set as 'boundary condition' in $B_{2R}(0)\setminus B_R(0)$ for a CMC Plateau problem. The perturbation is indexed on j and tends to the identity as $j \to \infty$, and we denote the deformed set by $E_j \subset E$. For each j we find a minimiser of J_λ with said boundary condition; note that Theorem 3 applies here. Theorem 2 follows by showing the existence of a sufficiently small ball centred at 0 in which, for all sufficiently large j, ∂E_j are smooth. Arguing by contradiction, we assume the existence of singular points $p_j \in \overline{\partial^* E_j}$, $p_j \to 0$. If the condition $p_j \neq 0$ is valid (for all sufficiently large j) then we dilate E_j around 0 by scaling $B_R(0)$ to $B_{\frac{R}{|p_j|}}(0)$. Using [15], we check that the limit of these rescalings of $\partial^* E_j$ has to be either one of the leaves of the Hardt–Simon foliation, or the tangent cone C to E at p: in either case we find a contradiction to the smoothness respectively of the leaves, or of the cone (at points at distance 1 from the origin).

Therefore we have to establish the condition $p_j \neq 0$. By construction $E_j \subsetneq E$ and both boundaries are hypersurfaces with the same scalar mean curvature, and with mean curvature vectors both pointing inwards. We thus show that the inclusion is strict everywhere by proving an instance of a singular maximum principle for CMC hypersurfaces, see Proposition 4.1 below. Its proof (by contradiction) relies on a linearisation argument that yields a non-trivial Jacobi field on the cone C (an analogous argument appears in [15] in the minimal case), combined with Simon's result [21], which gives a quantitative decay of ∂E towards C at small scales. The resulting behaviour of the Jacobi field is in contradiction with the ones that are known ([6]) to be permitted by the stability of the cone (stability follows from the minimising condition for E).

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2 Prescribed CMC Plateau problem

In the following we denote by B_R the open ball $B_R^{n+1}(0) \subset \mathbb{R}^{n+1}$. Let E_0 be a set of finite perimeter in B_2, that is, $E_0 \subset B_2$ is measurable and the perimeter of E_0 in B_2 is finite,

$$\mathrm{Per}_{B_2}(E_0) = \sup\left\{ \int_{E_0} \mathrm{div} T \, d\mathcal{L}^{n+1} \ : T \in C_c^1(B_2; \mathbb{R}^{n+1}), \sup |T| \leq 1 \right\} < \infty,$$

where $\mathcal{L}^{n+1}$ denotes the Lebesque measure on $\mathbb{R}^{n+1}$. This is equivalent to the requirement that $\chi_{E_0} \in \mathrm{BV}(B_2)$, that is, the distributional gradient $D\chi_{E_0}$ is a vector valued Radon measure with finite total variation in B_2.

For $\lambda \geq 0$ we will be interested in the following energy, defined on the class of sets of finite perimeter in B_2 that coincide with the given E_0 in $B_2 \setminus B_1$:

$$J_\lambda(E) = \mathrm{Per}_{B_2}(E) - \lambda |E|,$$

where $|E| = \mathcal{L}^{n+1}(E)$ is the $(n+1)$-volume of the Caccioppoli set $E \subset B_2$. This class is non-empty, since E_0 is one such set, and $J_\lambda(E_0) < \infty$, hence it makes sense to seek a minimiser of J_λ in this class.

Lemma 2.1. *There exists a minimiser F of J_λ in the class of sets with finite perimeter that coincide with the given E_0 in $B_2 \setminus B_1$.*

Proof. We will use the direct method. Let E_j, for $j \in \mathbb{N} \setminus \{0\}$, be a minimising sequence (of sets in the admissible class), that is

$$\lim_{j\to\infty} J_\lambda(E_j) = \inf\left\{ J_\lambda(E) : \chi_E \in \mathrm{BV}(B_2),\ \chi_E|_{B_2\setminus B_1} = \chi_{E_0}|_{B_2\setminus B_1} \right\}.$$

For all sufficiently large j we must then have

$$J_\lambda(E_j) = \mathrm{Per}_{B_2}(E_j) - \lambda|E_j| \leq J_\lambda(E_0) + 1 = \mathrm{Per}_{B_2}(E_0) - \lambda|E_0| + 1,$$

from which

$$\mathrm{Per}_{B_2}(E_j) \leq \mathrm{Per}_{B_2}(E_0) - \lambda|E_0| + \lambda|E_j| + 1 \leq \mathrm{Per}_{B_2}(E_0) + \lambda|B_2| + 1.$$

Therefore $\mathrm{Per}_{B_2}(E_j)$ are uniformly bounded above and there exist (by BV compactness) a set of finite perimeter F in B_2 and a subsequence (that we do not relabel) E_j such that $\chi_{E_j} \to \chi_F$ in $\mathrm{BV}(B_2)$. In particular, $\chi_{E_j} \to \chi_F$ in $L^1(B_2)$, so that $|E_j| \to |F|$; moreover, by the hypothesis that $E_j = E_0$ on $B_2 \setminus B_1$, we have also that $F = E_0$ on $B_2 \setminus B_1$. The lower semi-continuity of perimeters then gives $J_\lambda(F) \leq \liminf_{j\to\infty} J_\lambda(E_j)$, therefore F minimises J_λ in the admissible class. $\square$

The energy J_λ is relevant in many variational problems. The geometric significance of J_λ lies in the fact that it should select, as its critical points, sets whose boundary is a hypersurface with constant mean curvature λ. With the set up above, we are using E_0 to prescribe a boundary condition (in the sense of the Plateau problem). If ∂E_0 is smooth and intersects ∂B_1 transversely, then the set up amounts to fixing $\partial E_0 \cap \partial B_1$ as $(n-1)$-dimensional boundary data, and looking for a (n-dimensional) CMC hypersurface-with-boundary, with mean curvature λ, and whose boundary is $\partial E_0 \cap \partial B_1$. The hope is to obtain this hypersurface-with-boundary as $\partial F \setminus \big(\partial E_0 \cap (B_2 \setminus \overline{B_1})\big)$ (if ∂F is smooth).

Remark 2.1. If $\lambda < 0$ and F is a minimiser of $J_{|\lambda|}$ in $B_1 \subset B_2$, then $U \setminus F$ is a minimiser of J_λ in $B_1 \subset B_2$ (and vice versa), so we only treat the case $\lambda \geq 0$ (and all results extend in a straightforward manner to $\lambda < 0$). This follows from the fact that complementary sets have the same perimeter (in an open set).

A well-known consequence of the minimising property is that the integral varifold V (in B_2) defined by

$$V = \big|\partial^* F \setminus \big(\partial^* E_0 \cap (B_2 \setminus \overline{B_1})\big)\big|$$

(the notation $|\quad|$ denotes the multiplicity-1 varifold associated to a rectifiable set) has first variation in B_1 represented by the vector-valued measure

$$\lambda(\mathcal{H}^n \llcorner (\partial^* F \cap B_1))\nu_F,$$

where ν_F is the (measure theoretic) inward unit normal ($\mathcal{H}^n$-a.e. well-defined on $\partial^* F$). Indeed, given any vector field $X \in C^1_c(B_1; \mathbb{R}^{n+1})$, we can consider, for $\delta > 0$ sufficiently small, the one-parameter family of diffeomorphisms $\Phi_t = Id + tX$ for $t \in (-\delta, \delta)$. For every such t, we have $\Phi_t = Id$ on $B_2 \setminus B_1$ and therefore the set $\Phi_t(F)$ remains in the admissible class for every t. The image of V under Φ_t is $\big|\partial^* \Phi_t(F) \setminus \big(\partial^* E_0 \cap (B_2 \setminus \overline{B_1})\big)\big|$.

This permits to write the stationarity condition for V with respect to the energy J_λ, which gives (see e.g. [17, Chapters 17 and 19])

$$\int \mathrm{div}_{\partial^* F} X \, dV + \lambda \int (\nu_F \cdot X) \, dV = 0$$

and the desired conclusion. The candidate V thus has the correct mean curvature in B_1.

Next we are going to examine when it is possible to conclude this same condition away from the prescribed boundary: the missing analysis at this stage is the behaviour at points that potentially lie on ∂B_1 but are not part of the prescribed boundary. We begin by pointing out that, if the vector field X is non-zero somewhere on ∂B_1, then the above argument breaks down, since a one-parameter family of diffeomorphisms with initial speed X may map F to a set that is not in the admissible class (no matter how small δ is). In fact, the minimiser may just fail to have mean curvature λ when $\lambda > n$, as the following examples show.

Remark 2.2. Let H be the half-space $\{x_{n+1} < 0\}$ and $E_0 = H \cap B_2$. Then for any given $\lambda > n$ the minimisation procedure fails to produce a set whose boundary is a

CMC hypersurface-with-boundary with mean curvature λ and boundary condition $\partial H \cap \partial B_1$. (In fact, the unique minimiser F is given by $E_0 \cup B_1$ for all $\lambda \geq n$.) To see that, we observe that, for any given possible value $v \in \left[\frac{|B_1|}{2}, |B_1|\right]$, the (unique) perimeter-minimiser with volume v in B_1, that coincides with E_0 in $B_2 \setminus B_1$, is given by the set $E_0 \cup E_v$, where E_v is the ball of radius r centred at the point $(0, \ldots, 0, -\sqrt{r^2-1})$, where $r \geq 1$ is chosen so that $|E_v \cap B_1| = v$. Similarly, for any given possible value $v \in |E \cap B_1| \in \left[0, \frac{|B_1|}{2}\right]$, the perimeter-minimiser with volume v in B_1, and that coincides with E_0 in $B_2 \setminus B_1$, is given by the set $E_0 \setminus \tilde{E}_v$, where $\tilde{E}_v$ is the ball of radius r centred at the point $(0, \ldots, 0, \sqrt{r^2-1})$, where $r \geq 1$ is chosen so that $|\tilde{E}_v \cap B_1| = |B_1| - v$. The minimisation property just claimed is checked by a calibration argument, using the fact that $\partial E_v \cap B_1$ (and, similarly, $\partial \tilde{E}_v \cap B_1$) is a CMC graph on $B_1^n \subset \mathbb{R}^n \equiv \mathbb{R}^n \times \{0\}$. (See e. g. [3, Appendix B].) With this understood, the minimiser of J_λ (for any λ) has to be one of the minimising sets that have been exhibited for each possible value of v. Each of these minimisers has scalar mean curvature in $[-n, n]$ (away from $B_2 \setminus \overline{B_1}$). Hence for any $\lambda > n$ the minimisation procedure will not produce the desired CMC hypersurface of mean curvature λ. (By direct computation, one can check that the lowest value of J_λ for $\lambda > n$ is attained by $E_0 \cup B_1$.)

In the case $\lambda = n+1$ one can alternatively see that the minimiser is $E_0 \cup B_1$ by arguing as follows. Given any Caccioppoli set D that coincides with E_0 in $B_2 \setminus B_1$, consider the $(n+1)$-current $C = [\![E_0 \cup B_1]\!] - [\![D]\!]$ and the n-form $\beta = \iota_T(dx^1 \wedge \ldots \wedge dx^{n+1})$. where $T = (x_1, \ldots, x_{n+1})$. Then $d\beta = (\mathrm{div} T) dx^1 \wedge \ldots \wedge dx^{n+1} = (n+1) dx^1 \wedge \ldots \wedge dx^{n+1}$. We note that C is supported in $\overline{B_1}$, so it can act on $d\beta$ (by introducing a cut off function that is 1 on $\overline{B_1}$ and vanishes outside B_2). Then the equality $C(d\beta) = (\partial C)(\beta)$ gives $\partial [\![E_0 \cup B_1]\!](\beta) - (n+1)|E_0 \cup B_1| = \partial [\![D]\!](\beta) - (n+1)|D|$. Finally we note that $\partial [\![E_0 \cup B_1]\!](\beta) = \mathrm{Per}_{B_2}(E_0 \cup B_1) - \mathrm{Per}_{B_2 \setminus \overline{B_1}} H + \mathcal{H}^n(\partial B_2)$, while $\partial [\![D]\!](\beta) \leq \mathrm{Per}_{B_2}(D) - \mathrm{Per}_{B_2 \setminus \overline{B_1}}(H) + \mathcal{H}^n(\partial B_2)$, which gives that $J_{n+1}(E_0 \cup B_1) \leq J_{n+1}(D)$, that is, $E_0 \cup B_1$ is a minimiser. In fact, the inequality is not strict if and only if $\partial^* D \setminus (B_2 \setminus \overline{B_1})$ is a.e. orthogonal to T and contained in ∂B_1, which shows that $E_0 \cup B_1$ is the unique minimiser.

Before proceeding further we set up some notation. The integral $(n+1)$-current $[\![E_0]\!]$ in B_2 admits a well-defined (outer) slice $\langle [\![E_0]\!], |x| = 1^+ \rangle = -\partial [\![E_0 \cap (B_2 \setminus \overline{B_1})]\!] + (\partial [\![E_0]\!]) \llcorner (B_2 \setminus \overline{B_1})$. (See e.g. [12, Section 2.5].) The (outer) slice also coincides with $\langle [\![F]\!], |x| = 1^+ \rangle$. Let T_0 denote the $(n-1)$-dimensional current

$$T_0 = -\partial \langle [\![E_0]\!], |x| = 1^+ \rangle = -\partial \Big((\partial [\![E_0]\!]) \llcorner (B_2 \setminus \overline{B_1}) \Big),$$

then the Plateau problem under consideration seeks an integral n-current with boundary T_0. Note that $\partial [\![F]\!] = \partial [\![F \cap B_1]\!] + \partial [\![E_0 \cap (B_2 \setminus B_1)]\!]$ so

$$S := \partial [\![F \cap B_1]\!] - \langle [\![F]\!], |x| = 1^+ \rangle = \partial [\![F]\!] - (\partial [\![E_0]\!]) \llcorner (B_2 \setminus \overline{B_1})$$

has boundary $\partial S = T_0$. The integral n-current S is our candidate (hypersurface-with-boundary) solution to the Plateau problem. We let

$$\mathcal{S} = \partial^* F \setminus \big(\partial^* E_0 \cap (B_2 \setminus \overline{B_1})\big),$$

then $S = (\mathcal{S}, 1, -\star\nu_F)$, where $\star$ is the Hodge star (so $\nu_F \wedge \star\nu_F$ gives the positive orientation of $\mathbb{R}^{n+1}$) and ν_F is the unit inward (measure theoretic) normal for F on its reduced boundary. Also note that $V = \underline{v}(\mathcal{S}, 1)$ is the associated varifold (with notation from [19]).

We turn our attention to the analysis of the first variation (with respect to J_λ) of V on $B_2 \setminus \mathrm{spt}T_0$. Combining Lemma 2.1 with Lemmas 2.2, 2.3, 2.4 below, we will in particular prove the following overall result.

Theorem 4. *With the above setting and notation, let $\lambda \in (0, n)$. In the class of sets with finite perimeter that coincide with the given E_0 in $B_2 \setminus B_1$ there exists a minimiser F of J_λ, and there exists a set $\Sigma \subset B_1$ with $dim_{\mathcal{H}}\Sigma \leq n-7$, such that $(sptV \setminus sptT_0) \setminus \Sigma$ is a smoothly embedded CMC hypersurface with mean curvature vector $\lambda\nu_F$. If $n = 7$, more precisely, Σ is made of isolated points (possibly accumulating onto $sptT_0$).*

Remark 2.3. By scaling and translating, the theorem can be stated replacing B_1, B_2 and $(0, n)$ respectively with $B_r^{n+1}(p)$, $B_{2r}^{n+1}(p)$, $(0, \frac{n}{r})$. Moreover, the role of $B_{2r}^{n+1}(p)$ is only to provide an annulus in which E_0 is non-trivial, so $2r$ can be replaced by any radius $R > r$. Theorem 3 is thus a special case of Theorem 4, and in the case of Theorem 3 the accumulation of Σ onto T_0 is ruled out by [1].

Our first result on the first variation (with respect to J_λ) Lemma 2.2, is valid for any λ and yields a sign condition and an upper bound. The analysis has to be carried out only in a neighbourhood of an arbitrary $p \in \partial B_1 \setminus \mathrm{spt}T_0$ (since $\mathrm{spt}(V) \subset \overline{B_1}$ and we have established that the first variation is 0 in B_1). This result is the analogue of [11, Theorem 4.1].

Lemma 2.2. *Let $X \in C_c^1(B_2 \setminus sptT_0; \mathbb{R}^{n+1})$. Then the first variation with respect to J_λ of V evaluated on the vector field X (equal to the left-hand-side of the following expression) satisfies*

$$\int div_{\mathcal{S}} X d\mathcal{H}^n \llcorner \mathcal{S} + \lambda \int (\nu_F \cdot X) d\mathcal{H}^n \llcorner \mathcal{S} = \int (X \cdot N) d\mathcal{M},$$

where $\mathcal{M}$ is a positive Radon measure supported in ∂B_1 and $N = -\frac{x}{|x|}$ (for $x \neq 0$). Moreover, $\mathcal{M} \leq \Big(div_{\mathcal{S}} N + \lambda(\nu_F \cdot N)\Big) d\mathcal{H}^n \llcorner (\mathcal{S} \cap \partial B_1)$ (as measures).

Proof. Let $p \in \partial B_1 \setminus \mathrm{spt}T_0$ and consider $B_r(p) \subset B_{\frac{5}{4}} \setminus \mathrm{spt}T_0$. In the first part of the proof, we analyse the action of the first variation on a vector field of the type ηN, where $\eta \in C_c^1(B_r(p))$, $\eta \geq 0$. Let $d(\cdot) = \mathrm{dist}(\cdot, \partial B_1)$, where dist is the signed distance, taken to be positive in B_1 and negative in $B_2 \setminus \overline{B_1}$. Note that in any tubular neighbourhood of ∂B_1 we have that d is smooth and its gradient is N. Given $\epsilon > 0$, let $f_\epsilon : \mathbb{R} \to \mathbb{R}$ be a C^1 function such that $f_\epsilon \equiv 0$ on $[2\epsilon, \infty)$,

$f_\epsilon \equiv 1$ on $(-\infty, \epsilon]$ and $f' \leq 0$. We consider the following one-sided ($s \in [0, s_0]$) one-parameter family of diffeomorphisms:

$$\phi_s(z) = z + s\,\eta(z) f_\epsilon(d(z))\,N(z).$$

The reason for the one-sided restriction, $s \geq 0$, is that we need to ensure that we stay in the admissible class when deforming via ϕ_s, which we check next.

Since $\partial S = T_0$, and $\mathrm{spt} S \subset \overline{B_1}$, by the conditions on ϕ_s we also have $\partial(\phi_s)_\sharp S = T_0$ and $\mathrm{spt}(\phi_s)_\sharp S \subset \overline{B_1}$. On one hand we have $S + \langle [\![F]\!], |x| = 1^+\rangle = \partial [\![F \cap B_1]\!]$, therefore

$$(\phi_\sigma)_\sharp S + (\phi_\sigma)_\sharp \langle [\![F]\!], |x| = 1^+\rangle = \partial (\phi_\sigma)_\sharp [\![F \cap B_1]\!].$$

On the other hand, letting $\Phi(s,z) = \phi_s(z)$ for $s \in [0,\sigma]$ (this is a homotopy between the identity ϕ_0 and ϕ_σ on B_2) we obtain, from the homotopy formula,

$$(\phi_\sigma)_\sharp \langle [\![F]\!], |x| = 1^+\rangle - \langle [\![F]\!], |x| = 1^+\rangle = \partial\Big(\Phi_\sharp([0,\sigma] \times \langle [\![F]\!], |x| = 1^+\rangle)\Big).$$

Next we check that $-\Phi_\sharp([0,\sigma] \times \langle [\![F]\!], |x| = 1^+\rangle)$ is a Caccioppoli set. Note that Φ only acts on $z \in \partial B_1$ in this case. The map $\Phi|_{[0,\sigma]\times\partial B_1} : [0,\sigma] \times \partial B_1 \to B_1$ is Lipschitz and orientation-reversing wherever its differential is injective. Therefore, since $[0,\sigma] \times \langle [\![F]\!], |x| = 1^+\rangle$ is a Caccioppoli set in $\mathbb{R} \times \partial B_1$, so is its negative pushforward. We finally note that $-\Phi_\sharp([0,\sigma] \times \langle [\![F]\!], |x| = 1^+\rangle)$ is disjoint from $(\phi_\sigma)_\sharp [\![F \cap B_1]\!]$. Indeed, $\Phi\Big([0,\sigma] \times \partial B_1\Big)$ is contained in $\big\{x \in B_1 : \big|x - \frac{x}{|x|}\big| \leq \sigma\eta(\frac{x}{|x|})\big\}$, while the image $\phi_\sigma(B_1)$ is contained in $\big\{x \in B_1 : \big|x - \frac{x}{|x|}\big| > \sigma\eta(\frac{x}{|x|})\big\}$. We can therefore conclude that

$$(\phi_\sigma)_\sharp S + \langle [\![F]\!], |x| = 1^+\rangle = \partial [\![\tilde{F}_\sigma]\!]$$

where $\tilde{F}_\sigma$ is the Caccioppoli set

$$\tilde{F}_\sigma = (\phi_\sigma)_\sharp [\![F \cap B_1]\!] - \Phi_\sharp([0,\sigma] \times \langle [\![F]\!], |x| = 1^+\rangle).$$

Recalling that F and E_0 agree in $B_2 \setminus B_1$, and since $\tilde{F}_\sigma \subset B_1$, we set

$$F_\sigma = \tilde{F}_\sigma \cup \big(F \cap (B_2 \setminus B_1)\big)$$

and conclude that

$$(\phi_\sigma)_\sharp S + (\partial [\![E_0]\!]) \llcorner (B_2 \setminus B_1) = \partial [\![F_\sigma]\!],$$

with F_σ a set of finite perimeter in B_2 that coincides with E_0 in $B_2 \setminus B_1$ (that is, it is in the admissible class).

The previous conclusion permits to use the minimising property of F, as we are allowed to compare the energy with that of F_σ (for any $\sigma \in [0, s_0]$ — s_0 depends on ϵ). For $\epsilon > 0$ fixed, we can write (from the minimising property)

$$0 \leq \lim_{\sigma \to 0^+} \frac{J_\lambda(F_\sigma) - J_\lambda(F)}{\sigma} = \int_{\mathcal{S}} \mathrm{div}_{\mathcal{S}}\big(\eta f_\epsilon(d) N\big) d\mathcal{H}^n + \lambda \int_{\mathcal{S}} \nu_F \cdot \big(\eta f_\epsilon(d) N\big) d\mathcal{H}^n. \tag{1}$$

This equality is justified as follows. First, as by construction

$$\mathrm{Per}_{B_2}(F_\sigma) - \mathrm{Per}_{B_2}(F) = \mathbb{M}((\phi_\sigma)_\sharp S) - \mathbb{M}(S),$$

we can use the well-known formula for the first variation of n-area, which gives the first term on the right-hand-side of (1). Next we observe that, denoting by dx the $(n+1)$-form $dx^1 \wedge \ldots \wedge dx^{n+1}$ and by $x = (x_1, \ldots, x_{n+1})$, and since $d(\iota_x dx) = \mathcal{L}_x dx = (n+1)dx$, we have

$$|F_\sigma| - |F| = \big([\![F_\sigma]\!] - [\![F]\!]\big)(dx) = \frac{1}{n+1}\partial\big([\![F_\sigma]\!] - [\![F]\!]\big)(\iota_x dx) = \frac{1}{n+1}\big((\phi_\sigma)_\sharp S - S\big)(\iota_x dx)$$

$$= \frac{1}{n+1}\partial\big(\Phi_\sharp([0,\sigma] \times S)\big)(\iota_x dx) = \big(\Phi_\sharp([0,\sigma] \times S)\big)(dx).$$

Then by direct computation

$$\frac{d}{d\sigma}\Big|_{\sigma=0^+}\big(\Phi_\sharp([0,\sigma] \times S)\big)(dx) = \big(\Phi_\sharp(\{0\} \times S)\big)(\iota_{d\Phi(\frac{\partial}{\partial s})} dx) =$$

$$= S(\iota_{\eta f_\epsilon(d) N} dx) = -\int_{\mathcal{S}} \nu_F \cdot (\eta f_\epsilon N) d\mathcal{H}^n,$$

which completes the proof of (1).

The next argument follows [11, Theorem 4.1] verbatim. We check that the right-hand-side of (1) is independent of ϵ. Indeed, for $\epsilon' < \epsilon$ we consider

$$\psi_s(z) = z + s\eta(z)\big(f_\epsilon(d(z)) - f_{\epsilon'}(d(z))\big)N(z).$$

This is (for $s \in (-\delta, \delta)$ with $\delta > 0$ sufficiently small, depending on ϵ') a (two-sided) one-parameter family of diffeomorphisms, equal to the identity in a neighbourhood of ∂B_1. We can then use the vanishing of the first variation under the deformation induced by ψ_s. As the first variation is linear in the vector field, and the initial speed of ψ_s is the difference of the two vector fields appearing on the right-hand-side of (1) for ϵ and ϵ' respectively, we conclude that said right-hand-side is independent of ϵ.

By the sign condition in (1), and viewing the right-hand-side of (1) as the action of a distribution on C^1_c, there exists a (positive) Radon measure $\mathcal{M}$ in B_2 such that the right-hand-side of (1) is given by $\int \eta d\mathcal{M}$. (A priori this distribution should depend on ϵ, however we have proved that the action is independent of ϵ.)

On the other hand, sending $\epsilon \to 0$ on the right-hand-side of (1) (denoting by $\nabla_{\mathcal{S}} = \mathrm{proj}_{T\mathcal{S}}\nabla$ the gradient on $\mathcal{S}$, a.e. well-defined), we obtain:

$$\int_{\mathcal{S}} f_\epsilon(d)\, \nabla_{\mathcal{S}}\eta \cdot N d\mathcal{H}^n \to \int_{\mathcal{S}\cap\partial B_1} \nabla_{\mathcal{S}}\eta \cdot N\, d\mathcal{H}^n = 0,$$

where the last equality follows from the fact that $\nabla_{\mathcal{S}}\eta \cdot N = 0$ a.e. on $\mathcal{S} \cap \partial B_1$;

$$\int_{\mathcal{S}} f_\epsilon(d)\eta\, \mathrm{div}_{\mathcal{S}} N d\mathcal{H}^n \to \int_{\mathcal{S}\cap\partial B_1} \eta\, \mathrm{div}_{\mathcal{S}} N\, d\mathcal{H}^n;$$

$$\int_{\mathcal{S}} \eta \nabla_{\mathcal{S}} f_\epsilon(d) \cdot N d\mathcal{H}^n = \int_{\mathcal{S}} \eta f'_\epsilon(d) |\nabla_{\mathcal{S}} d|^2 d\mathcal{H}^n \leq 0,$$

where we used $\nabla d = N$ on the support of f_ϵ;

$$\int_{\mathcal{S}} \nu_F \cdot \big(\eta f_\epsilon(d) N\big) d\mathcal{H}^n \to \int_{\mathcal{S}} \eta\, \nu_F \cdot N\, d\mathcal{H}^n.$$

These imply (expanding the divergence in (1))

$$\int \eta d\mathcal{M} \leq \int_{\mathcal{S}\cap\partial B_1} \eta \mathrm{div}_{\mathcal{S}} N d\mathcal{H}^n + \lambda \int_{\mathcal{S}\cap\partial B_1} \eta(\nu_F \cdot N) d\mathcal{H}^n$$

is valid for all $\eta \in C^1_c(B_r(p))$, $\eta \geq 0$, hence

$$\hat{\mathcal{M}} = \Big(\mathrm{div}_{\mathcal{S}} N + \lambda(\nu_F \cdot N)\Big) d\mathcal{H}^n \llcorner (\mathcal{S} \cap \partial B_1)$$

is a (positive) Radon measure. The first variation of V (with respect to J_λ) computed on the test vector field ηN can be decomposed as the sum of the first variation computed on $\eta f_\epsilon(d) N$ and on $\eta(1 - f_\epsilon(d))N$. The latter contribution gives 0 since $\eta(1 - f_\epsilon(d))N \in C^1_c(B_1; \mathbb{R}^{n+1})$. Therefore the first variation of V on ηN gives just (1), that is, is given by $\int \eta d\mathcal{M}$, and we have seen that $0 \leq \mathcal{M} \leq \hat{\mathcal{M}}$.

In the second part of the proof we consider a vector field $Y \in C^1_c(B_r(p); \mathbb{R}^{n+1})$ such that $Y \cdot N = 0$. Let ψ_s be the flow of Y, that is, the one-parameter (two-sided) family of diffeomorphisms obtained by solving the ODE for each trajectory, $\frac{d}{ds}\Psi(s,x) = Y(x)$, with initial condition $\Psi(0,x) = x$, and setting $\psi_s(x) = \Psi(s,x)$. Then $\psi_s(B_1) \subset B_1$ and we consider $\tilde{F}_s = \psi_s(F \cap B_1)$. These are Caccioppoli sets with support in $\overline{B_1}$ and such that $\partial^* \tilde{F}_s = \psi_s(\partial^* F)$ is a.e. contained in B_1. The Caccioppoli set $F_s = \tilde{F}_s \cup \big(F \cap (B_2 \setminus B_1)\big)$ is in the admissible class. We need to show that its boundary (as a current) is $(\psi_s)_\sharp S + (\partial[\![E_0]\!]) \llcorner (B_2 \setminus B_1)$. The immediate expression for this boundary is $(\psi_s)_\sharp(\partial[\![F \cap B_1]\!]) + \partial[\![E_0 \cap (B_2 \setminus B_1)]\!]$. Recalling that $S = \partial[\![F \cap B_1]\!] - \langle [\![E_0]\!], |x| = 1^+\rangle$ we arrive at

$$(\psi_s)_\sharp S + (\psi_s)_\sharp \langle [\![E_0]\!], |x| = 1^+\rangle + (\partial[\![E_0]\!]) \llcorner (B_2 \setminus B_1) - \langle [\![E_0]\!], |x| = 1^+\rangle.$$

As $\Psi(t,z)$, for $(t,z) \in [0,s] \times B_2$ is a homotopy joining the identity ψ_0 to ψ_s, we will use the homotopy formula. We note that $\Psi(t,z) = z$ in a neighbourhood of $T_0 = -\partial\langle [\![E_0]\!], |x| = 1^+\rangle$, so that $\Psi_\sharp([0,s] \times \partial\langle [\![E_0]\!], |x| = 1^+\rangle) = 0$. Moreover, $\Psi([0,s] \times \partial B_1) \subset \partial B_1$, so that $\Psi_\sharp([0,s] \times \langle [\![E_0]\!], |x| = 1^+\rangle) = 0$ (as an n-current). The homotopy formula then gives $(\psi_s)_\sharp \langle [\![E_0]\!], |x| = 1^+\rangle = \langle [\![E_0]\!], |x| = 1^+\rangle$ and therefore

$$\partial[\![F_s]\!] = (\psi_s)_\sharp S + (\partial[\![E_0]\!]) \llcorner (B_2 \setminus B_1).$$

We can therefore use the minimising condition to write the standard condition for the vanishing of the first variation (with respect to J_λ) as

$$\int_{\mathcal{S}} \mathrm{div}_{\mathcal{S}} Y d\mathcal{H}^n + \lambda \int_{\mathcal{S}} \nu_F \cdot Y d\mathcal{H}^n = 0. \tag{2}$$

For the third (and final) part of the proof, given an arbitrary vector field $X \in C^1_c(B_r(p);\mathbb{R}^{n+1})$ we write the orthogonal decomposition $X = X^T + X^N$, where $X^N = (X\cdot N)N$ and both X^T and X^N are $C^1_c(B_r(p);\mathbb{R}^{n+1})$. Then the first variation of J_λ on X is given by the sum of the two actions on X^T and X^N. For the former, in view of (2) the action is 0. For the latter, we have that $X^N = \eta_+ N - \eta_- N$, where $\eta_+, \eta_- \geq 0$ and $\eta_+ = (X\cdot N)^+$, $\eta_- = (X\cdot N)^-$. By the conclusion in the first part (applied separately to $\eta_+ N$ and $\eta_- N$, using the linearity of the first variation), we then have that the action is given by $\int(\eta_+ - \eta_-)d\mathcal{M} = \int(X\cdot N)d\mathcal{M}$. $\square$

Lemma 2.2 holds for any λ and, in the example given in Remark 2.2, for $\lambda > n$ one actually has $\mathcal{M} \neq 0$. If $\lambda \leq n$ we obtain the following result (this is analogous to [11, Theorem 7.1]).

Lemma 2.3. *Let $\lambda \leq n$. Then $\mathcal{M} = 0$, that is, V is stationary (with respect to J_λ) in $B_2 \setminus sptT_0$.*

Proof. We have $N = \nu_F$ a.e. on $\mathcal{S}\cap\partial B_1$ and $\mathrm{div}_{\mathcal{S}} N = \mathrm{div}_{\partial B_1} N$ a.e. on $\mathcal{S}\cap\partial B_1$. By explicit computation $\mathrm{div}_{\partial B_1} N = -n$ (where n is the mean curvature of ∂B_1). Then the inequality $0 \leq \mathcal{M} \leq \Big(\mathrm{div}_{\mathcal{S}} N + \lambda(\nu_F\cdot N)\Big)d\mathcal{H}^n \llcorner(\mathcal{S}\cap\partial B_1)$ obtained in Lemma 2.2 becomes $0 \leq \mathcal{M} \leq (\lambda - n)d\mathcal{H}^n\llcorner(\mathcal{S}\cap\partial B_1)$. Thus with $\lambda \leq n$ we must have $\mathcal{M} = 0$ (and if $\lambda < n$ also $\mathcal{H}^n\big(\mathcal{S}\cap\partial B_1\big) = 0$). $\square$

Having established this stationarity property, we move on to the regularity of the minimiser, focusing for simplicity on the case $\lambda < n$. We note immediately that the regularity in B_1 follows from the theory of minimisers, however we may a priori have that $\mathrm{spt}V \cap \partial B_1 \neq \emptyset$. We make the following observations.

If $p \in \partial B_1 \cap \mathrm{spt}V \setminus \mathrm{spt}T_0$ is a point in gen-reg V (using the terminology in [3], [4] to denote by this the C^2 immersed part of $\mathrm{spt}V$), then by definition there exists an embedded disc $D \subset \mathrm{spt}V \setminus \mathrm{spt}T_0 \subset \overline{B_1}$ of class C^2 with $p \in D$. As gen-reg V is CMC with mean curvature λ, the maximum principle gives a contradiction if $\lambda < n$ (since n is the mean curvature of ∂B_1 with respect to the inward normal to B_1). This means that if $\lambda < n$ then gen-reg $V \cap (\partial B_1 \setminus \mathrm{spt}T_0) = \emptyset$. We are able to verify the stability hypothesis in [3] or [4]: indeed, by the conclusion just obtained we only have to establish its validity for test functions with compact support in B_1, and this is immediate from the minimisation property.

We further note that for $p \in \partial B_1 \cap \mathrm{spt}V \setminus \mathrm{spt}T_0$ the varifold V has a unique tangent cone at p, given by the hyperplane that is tangent to ∂B_1 at p, possibly counted with integer multiplicity. The existence of tangent cones, and the fact that any such cone is a stationary varifold, both follow from the monotonicity-type formula for the mass, valid thanks to the stationarity with respect to J_λ. Since $\mathrm{spt}V \subset \overline{B_1}$, any such tangent cone must be contained in a half-space (whose boundary is the tangent to ∂B_1 at p), and thus is has to be supported on that tangent hyperplane itself (see e.g. [19]), from which the claim follows (thanks to the constancy theorem [19]).

Finally, we note the absence of classical singularities in $\text{spt}V \setminus \text{spt}T_0$. In B_1, this is a consequence of the minimising property, while at any $p \in \partial B_1 \cap \text{spt}V \setminus \text{spt}T_0$ we have proved that the tangent has to be supported on a hyperplane (which rules out that p could be a classical singularity).

Lemma 2.4. *Let $\lambda < n$ and V, F as above. There exists $\Sigma \subset B_1$ with $\dim_{\mathcal{H}}\Sigma \leq n-7$ such that $(\text{spt}V \setminus \text{spt}T_0) \setminus \Sigma$ is a smoothly embedded CMC hypersurface (with mean curvature vector $\lambda\nu_F$). If $n = 7$, more precisely, Σ is made of isolated points (possibly accumulating onto $\text{spt}T_0$).*

Proof. All hypotheses needed to apply the theory from [3], [4] are in place, thanks to the observations made. We conclude (by the sheeting result in [4]) that if $p \in \text{spt}V \cap \partial B_1 \setminus \text{spt}T_0$ then $\text{spt}V$ is a union of C^2 discs with constant mean curvature λ in a neighbourhood of p. This violates the maximum principle (since $\lambda < n$ and ∂B_1 has mean curvature n) and permits to conclude, in a first instance, that $\text{spt}V \setminus \text{spt}T_0 \subset B_1$. At this stage one may either use the regularity theory for minimisers, or alternatively [3] or [4] again, to conclude. □

Remark 2.4. We expect that the same regularity conclusions should hold for $\lambda = n$, albeit with the possibility that open subsets of ∂B_1 may be contained in $\text{spt}V \setminus \text{spt}T_0$, as in the example of Remark 2.2.

3 Regular minimal cones, graphs, Jacobi operator

In Section 4 we will prove Proposition 4.1, an instance of a singular maximum principle for CMC hypersurfaces, which will then be needed in Section 5. In this section we collect some preliminaries on stable minimal cones and their Jacobi fields that will be needed in Section 4.

In what follows let C be a regular cone that is also minimal. We recall that the notion of regular cone means that $C = \{ry : r \geq 0, y \in \Sigma\}$, where Σ (the link of C) is a smooth embedded compact $(n-1)$-dimensional submanifold of the unit sphere S^n. The minimality condition is the vanishing of the mean curvature of $C \setminus \{0\}$ (as a submanifold of $\mathbb{R}^{n+1}$). (This requirement is equivalent to the minimality of Σ as a submanifold of S^n, see [22]). We first recall some facts about graphs over C and their mean curvature operator.

Let $C = \partial[\![E]\!]$, for a set[1] of locally finite perimeter $E \subset \mathbb{R}^{n+1}$. The graph of $u \in C^2(C_1;\mathbb{R})$ over $C_1 = (C \setminus \{0\}) \cap B_1$ is defined to be

$$\text{gr}_C u = \{x + u(x)N(x) : x \in C_1\},$$

[1] In the forthcoming sections, any minimal regular cone C will arise automatically as a boundary. However any regular minimal cone has connected link Σ (by a standard application of the maximum principle) and using this one shows that $S^n \setminus \Sigma$ has two connected components (by Alexander's duality), thus so does $\mathbb{R}^{n+1} \setminus C$, therefore there always exists E such that $C = \partial[\![E]\!]$.

where N is the inward pointing unit normal on C. We will be interested in functions u that satisfy the following radial decay

$$\frac{|u(x)|}{|x|} + |\nabla u(x)| + |x||\nabla^2 u(x)| \xrightarrow[|x|\to 0]{} 0, \tag{3}$$

where ∇ denotes the Levi-Civita connection on C with respect to the Riemannian metric induced on C by the Euclidean one in $\mathbb{R}^{n+1}$, and $|\cdot|$ is taken with respect to the Euclidean inner product.

We remark that there exists $M = M_\Sigma$ such that, if

$$\frac{|u(x)|}{|x|} + |\nabla u(x)| \leq M \tag{4}$$

is valid for all $x \in C_1$ then $\mathrm{gr}_C u$ is an embedded hypersurface, with $\{0\} = (\overline{\mathrm{gr}_C u} \setminus \mathrm{gr}_C u) \cap B_1$ an isolated singularity when C is not a hyperplane. We will assume in this section that (4) is satisfied on C_1. We further note that (3) implies the validity of (4) for all $0 < |x| < r$ for sufficiently small r, and therefore, after rescaling, $\tilde{u}(x) = u\left(\frac{x}{r}\right)$ satisfies $\frac{|\tilde{u}(x)|}{|x|} + |\nabla \tilde{u}(x)| \leq M$ on C_1. (This fact will be implicitly used in Section 4.)

Assume now that the associated current to $\mathrm{gr}_C u$ is of the form $\partial[\![F]\!] \llcorner B_1$ [2], where F is a set of finite perimeter and that F is a critical point of J_λ thus in particular we have that

$$\left.\frac{d}{dt}\right|_{t=0} J_\lambda(F_t) = 0,$$

where F_t is the set of finite perimeter whose boundary is $\mathrm{gr}_C(u + tv)$ and $v \in C^2_c(C_1;\mathbb{R})$. We recall that the mean curvature operator $\mathcal{M}_C$ of the cone is defined as follows by defining in duality its action on $u \in C^2(C_1;\mathbb{R})$:

$$\left.\frac{d}{dt}\right|_{t=0} \mathcal{H}^n\big(\mathrm{gr}_C(u+tv)\big) = - < \mathcal{M}_C u, v >_{L^2},$$

where $<,>_{L^2}$ denotes the L^2-inner product on C and $v \in C^2_c(C_1;\mathbb{R})$. The PDE that the function u satisfies is given in terms of $\mathcal{M}_C$ as we prove in the following:

Lemma 3.1. *Let u and $gr_C u$ be as above then*

$$\mathcal{M}_C u = \lambda det(Id - uA_C), \tag{5}$$

where A_C denotes the second fundamental form of C.

[2] In what follows every graph of the form $\mathrm{gr}_C u$ will arise as a boundary of a set of finite perimeter. However, since $\mathrm{gr}_C u$ is embedded the map $G(x) = x + u(x)N(x)$ is a diffeomorphism to its image and, since C_1 is a boundary, $\mathbb{R}^{n+1} \setminus C_1$ has two connected components thus so does $\mathbb{R}^{n+1} \setminus G(C_1)$ therefore there always exist a set F such that the associated current to $\mathrm{gr}_C u$ is of the form $\partial[\![F]\!] \llcorner B_1$.

Proof. Let $G(x) = x + u(x)N(x)$ and consider an extension $\hat{N}$ of N. Then for any $v \in C^2_c(C_1;\mathbb{R})$ we have that

$$0 = \frac{d}{dt}\bigg|_{t=0} J_\lambda(F_t) = - < \mathcal{M}_C u, v >_{L^2} + \lambda \int_{gr_C u} v\hat{N} \cdot \hat{\nu} d\mathcal{H}^n,$$

where F_t is the associated set to $\mathrm{gr}_C(u + tv)$, $\hat{\nu}$ the inward pointing unit normal of $\mathrm{gr}_C u$ and the last term is the derivative of the volume term. Using the area formula the latter can be written as $\int_C v\hat{N} \cdot \hat{\nu}|J_G|d\mathcal{H}^n$, where $|J_G|$ denotes the Jacobian of G. Thus it suffices to compute $\hat{N} \cdot \hat{\nu}|J_G|$. Let (τ_i) be an orthonormal basis of C then

$$D_{\tau_i} G \cdot \tau_j = \delta_{ij} - uA_{ij},$$

$$D_{\tau_i} G \cdot \hat{N} = D_{\tau_i} u,$$

where $D_{\tau_i} G$ denotes the differential of G in the direction of τ_i and (A_{ij}) is the matrix that corresponds to the second fundamental form of C with respect to the chosen basis. Consider the matrix

$$B = \begin{pmatrix} D_{\tau_1} G \cdot \hat{N}, & D_{\tau_1} G \cdot \tau_1, & \dots & D_{\tau_1} G \cdot \tau_n \\ \vdots & \vdots & \vdots & \vdots \\ D_{\tau_n} G \cdot \hat{N}, & D_{\tau_n} G \cdot \tau_1, & \dots & D_{\tau_n} G \cdot \tau_n \end{pmatrix}$$

Let $B^{(k)}$ denote the $n \times n$ minor of the matrix B for $2 \leq k \leq n+1$ that comes from erasing the k-th row of the matrix B then

$$\hat{\nu} = \left(\det(Id - uA_C)\hat{N} + \sum_{k=2}^{n+1} (-1)^{k-1} B^{(k)} \tau_{k-1} \right) |J_G|^{-1}.$$

In particular $\hat{N} \cdot \hat{\nu}|J_G| = \det(Id - uA_C)$ and this finishes the proof. $\square$

In view of (5), we recall some properties of the operator $\mathcal{M}_C$, referring to [8, Lemma 2.26] for details (see also [6], [15]). Under assumption (4) the operator $\mathcal{M}_C$ has the form

$$\mathcal{M}_C u = L_C u + N\left(x, \frac{u}{|x|}, \nabla u\right) \cdot \nabla^2 u(x) + \frac{1}{|x|} P\left(x, \frac{u}{|x|}, \nabla u(x)\right),$$

where $L_C u = \Delta_C u + |A_C|^2 u$ is the Jacobi field operator of the cone C and P, S have a C^2-dependency on the arguments $(x, z, p) \in C_1 \times \mathbb{R} \times TC_1$. Moreover $\mathcal{M}_C$ is a quasilinear elliptic PDE and for $|z|, |p| \leq 1$ we have the following inequalities at (x, z, p)

$$
\begin{aligned}
&|N(x,z,p)| \le M_\Sigma(|z|+|p|),\\
&|P(x,z,p)| \le M_\Sigma(|z|+|p|)^2,\\
&|P_z|+|P_p|+|x|(|P_{xz}|+|P_{xp}|) \le M_\Sigma(|z|+|p|),\\
&|x|(|N_x|+|P_x|+|N_{xz}|+|N_{xp)}|)+|N_z|+|N_p|+|N_{zz}|+\\
&\quad+|N_{zp}|+|N_{pp}|+|P_{zz}|+|P_{zp}|+|P_{pp}| \le M_\Sigma,
\end{aligned}
\tag{6}
$$

where the subscripts denote partial differentiation and M_Σ is a constant that depends on the dimension n and the link Σ of the cone.

The estimates in (6) along with the radial decay assumption (3) allow us to prove that the linearisation of the PDE (5) has the following form:

Lemma 3.2. *Let $u, v \in C^2(C_1;\mathbb{R})$ that satisfy (3) and $\mathcal{M}_C u = \lambda\, det(Id - uA_C)$, $\mathcal{M}_C v = \lambda\, det(Id - vA_C)$. Then $h = v - u$ satisfies the following linear PDE*

$$
L_C h = A_1 \cdot \nabla^2 h + \frac{1}{|x|} A_2 \cdot \nabla h + \frac{1}{|x|^2} A_3 h,
\tag{7}
$$

where $A_1 : C_1 \to End(TC_1)$, $A_2 : C_1 \to TC_1$, $A_3 : C_1 \to \mathbb{R}$ and $A_1, A_2, A_3 \xrightarrow[|x|\to 0]{} 0$. Moreover if $u, v \in C^3(C_1;\mathbb{R})$ then the coefficients of the PDE are in $C^{0,\alpha}(U;\mathbb{R})$ for some $\alpha \in (0,1)$ and any $U \subset\subset C_1$.

Proof. We first compute the operator $\mathcal{L}$ such that $\mathcal{L}h = \mathcal{M}_C v - \mathcal{M}_C u$. Let N, P be the operators associated to $\mathcal{M}_C$. Then, since L_C is linear,

$$
\begin{aligned}
\mathcal{M}_C v - \mathcal{M}_C u = L_C h + N_v \cdot \nabla^2 v - N_u \cdot \nabla^2 u +\\
+ \frac{1}{|x|}(P_v - P_u),
\end{aligned}
$$

where the subscripts on N, P denote the dependency on v, u respectively.

We recall the standard method. We rewrite the term $N_{ij}(x, \frac{v}{|x|}, \nabla v)\nabla_{ij} v - N_{ij}(x, \frac{u}{|x|}, \nabla u)\nabla_{ij} u$ as

$$
\int_0^1 \frac{d}{dt}\left(N_{ij}\left(x, \frac{u}{|x|} + t\frac{(v-u)}{|x|}, \nabla u + t(\nabla v - \nabla u)\right)(\nabla_{ij} u + t(\nabla_{ij} v - \nabla_{ij} u))\right) dt,
$$

where note that repeated indices are summed. We differentiate with respect to t to get the following terms

$$
\begin{aligned}
&\left(\int_0^1 N_{ij} dt\right)\nabla_{ij} h + \left(\int_0^1 |x| N_{ij,z}(\nabla_{ij} u + t\nabla_{ij} h) dt\right)\frac{h}{|x|^2}\\
&+ \left(\int_0^1 |x| N_{ij,p}(\nabla_{ij} u + t\nabla_{ij} h) dt\right)\cdot \frac{\nabla h}{|x|},
\end{aligned}
$$

where the subscripts z, p denote partial differentiation. A similar computation gives that

$$\frac{1}{|x|}(P_v - P_u) = \left(\int_0^1 P_z dt\right)\frac{h}{|x|^2} + \left(\int_0^1 P_p dt\right)\cdot \frac{\nabla h}{|x|}.$$

Putting together we get that

$$\mathcal{M}_C v - \mathcal{M}_C u = A_1 \cdot \nabla^2 h + \frac{1}{|x|}A_2 \cdot \nabla h + \frac{1}{|x|^2}A_3 h,$$

and using the estimates (6) we have that

$$|A_1| \leq M_\Sigma\left(\frac{|u|}{|x|} + \frac{|h|}{|x|} + |\nabla u| + |\nabla h|\right),$$

$$|A_2| \leq M_\Sigma\left(\frac{|u|}{|x|} + \frac{|h|}{|x|} + |x||\nabla^2 u| + |x||\nabla^2 h|\right),$$

$$|A_3| \leq M_\Sigma\left(\frac{|u|}{|x|} + \frac{|h|}{|x|} + |x||\nabla^2 u| + |x||\nabla^2 h|\right),$$

where M_Σ denotes a constant that depends on the link Σ of the cone C. Thus from (3) we have that $A_1, A_2, A_3 \to 0$ as $|x| \to 0$.

Finally, we compute the difference of the terms involving the determinants in a similar way and making use of the Jacobi formula for the derivative of the determinant we get that

$$\begin{aligned}\det(Id - vA_C) - \det(Id - uA_C) &= \int_0^1 \frac{d}{dt}\det(Id - (u+th)A_C)dt \\ &= \int_0^1 \det(Id - (u+th)A_C)\mathrm{tr}((Id - (u+th)A_C)^{-1}hA_C)dt.\end{aligned}$$

Note that $\mathrm{tr}((Id - (u+th)A_C)^{-1}hA_C) = h\mathrm{tr}((Id - (u+th)A_C)^{-1}A_C)$ thus the difference gives us a term of the form $\frac{1}{|x|^2}A_4 h$ where the coefficient A_4 is given by $\int_0^1 |x|^2\det(Id - (u+th)A_C)\mathrm{tr}((Id - (u+th)A_C)^{-1}A_C)dt$. From (3) we have that $Id - (u+th)A_C \to Id$ as $|x| \to 0$ thus A_4 converges to 0 as $|x| \to 0$ as well. $\square$

From (7) we see that L_C becomes the leading term of the PDE as $|x| \to 0$. This crucial fact will allow us, in Proposition 4.1 below, to construct a non-trivial positive Jacobi field of C. In view of that, we recall some well-known properties of the Jacobi operator L_C.

For $x \in C \setminus \{0\}$ let $r = |x|$ and $\omega = \frac{x}{|x|} \in \Sigma$ denote spherical coordinates on C. Then the metric of the cone is given by $g = dr^2 + r^2 g_\Sigma$ where g_Σ is the pull-back on

Σ of the round metric on S^n (via the inclusion map). The operator L_C is expressed in spherical coordinates as

$$L_C f = r^{-2} L_\Sigma f + r^{1-n} \partial_r (r^{n-1} \partial_r f), \tag{8}$$

where $L_\Sigma = \Delta_\Sigma + |A_\Sigma|^2$ and A_Σ is the second fundamental form of Σ in S^n. Since L_Σ is a a linear elliptic operator on a smooth compact manifold, we consider the spectrum $\lambda_1 < \lambda_2 \le \dots, \to +\infty$ of $-L_\Sigma$.

The first eigenvalue λ_1 is simple and it is known from [6] that C is stable if and only if

$$\max\{-\lambda_1, 0\} \le \frac{(n-2)^2}{4}.$$

In particular, if C is stable (which will be the case in forthcoming sections) we define $\gamma^\pm = \dfrac{n-2}{2} \pm \sqrt{\frac{(n-2)^2}{4} + \lambda_1}$ and we have $\gamma^+ \ge \gamma^- \ge 0$.

Remark 3.1. Unless C is a hyperplane, one always has $\gamma^- > 0$. Indeed if $\gamma^- = 0$ then $\lambda_1 = 0$ and from the variational characterisation of the first eigenvalue of L_Σ, if we take as a test function a constant function, we get that $|A_\Sigma| \equiv 0$ thus $|A_C| \equiv 0$ and C is a plane.

Any positive solution of $L_C f = 0$ is of the form (see e.g. [15], and Lemma A.2 below)

$$f(r\omega) = \left(\frac{c_1}{r^{\gamma^+}} + \frac{c_2}{r^{\gamma^-}} \right) \phi_1(\omega), \tag{9}$$

where $\phi_1 > 0$ is the first eigenfunction of L_Σ, that is $L_\Sigma \phi_1 = -\lambda_1 \phi_1$ and c_1, c_2 are non-negative constants.

4 A singular maximum principle

We first state and prove the following fact regarding the convergence of minimisers of J_λ. Analogous results hold (with similar arguments that require building competitors) for area-minimising currents (see e.g. [19, Chapter 7, Theorem 2.4]) and for perimeter minimisers or almost-minimisers (see e.g. [17, Theorem 21.14]).

Lemma 4.1. *Let E_j be sets with finite perimeter in B_2. For each j we assume that E_j minimises J_λ among sets that coincide with E_j in $B_2 \setminus B_1$. Let E be a set with finite perimeter in B_2 and assume that $[\![E_j]\!] \to [\![E]\!]$ (as currents) in B_2. Then E minimises J_λ among sets that coincide with E in $B_2 \setminus B_1$. Moreover, $|\partial^* E_j| \to |\partial^* E|$ in B_1 (as varifolds).*

Remark 4.1. Let D be a set with finite perimeter in B_2. The outer and inner slices $\langle [\![D]\!], |x| = 1^+ \rangle$ and $\langle [\![D]\!], |x| = 1^- \rangle$ are n-dimensional integral currents supported in ∂B_1 (which is n-dimensional), therefore there exist integer valued BV-functions θ_D^+ and θ_D^- such that $\langle [\![D]\!], |x| = 1^+ \rangle = \theta_D^+ (\mathcal{H}^n \llcorner \partial B_1) \vec{\xi}$ and $\langle [\![D]\!], |x| = 1^- \rangle = \theta_D^- (\mathcal{H}^n \llcorner \partial B_1) \vec{\xi}$, where $\vec{\xi}$ is the orientation of ∂B_1 corresponding (in Hodge duality) to the choice of outward pointing unit normal. In fact, $\theta^+_D, \theta^-_D \in \{0,1\}$ since $[\![D]\!]$ is the current of integration on a Caccioppoli set.

Proof. We remark that $\langle [\![E_j]\!], |x| = 1^+\rangle \to \langle [\![E]\!], |x| = 1^+\rangle$ as currents (since by definition $\langle [\![E_j]\!], |x| = 1^+\rangle = -\partial [\![E_j \cap (B_2 \setminus \overline{B_1})]\!] + (\partial [\![E_j]\!]) \llcorner (B_2 \setminus \overline{B_1})$, and $[\![E_j]\!] \to [\![E]\!]$ in B_2 by assumption).

Let F be a set with finite perimeter that coincides with E in $B_2 \setminus B_1$. Set

$$F_j = (F \cap B_1) \cup \big(E_j \cap (B_2 \setminus B_1)\big).$$

Then $F_j \to F$; moreover, $\langle [\![F_j]\!], |x| = 1^+\rangle = \langle [\![E_j]\!], |x| = 1^+\rangle$ by definition of F_j, and $\langle [\![F]\!], |x| = 1^+\rangle = \langle [\![E]\!], |x| = 1^+\rangle$ by definition of F.

With notation as in Remark 4.1, we remark that $\theta^+_E = \theta^+_F$, $\theta^-_{F_j} = \theta^-_F$ and $\theta^+_{F_j} = \theta^+_{E_j}$. We use (12) (see Lemma A.1) with E_j, F_j in place of D and thus rewrite the minimising condition $J_\lambda(E_j) \leq J_\lambda(F_j)$ in the form,

$$\mathrm{Per}_{B_1} E_j + \mathbb{M}\big(\langle [\![E_j]\!], |x| = 1^+\rangle - \langle [\![E_j]\!], |x| = 1^-\rangle\big) - \lambda \mathcal{H}^{n+1}\big(E_j\big) \leq$$

$$\mathrm{Per}_{B_1} F + \mathbb{M}\big(\langle [\![F_j]\!], |x| = 1^+\rangle - \langle [\![F_j]\!], |x| = 1^-\rangle\big) - \lambda \mathcal{H}^{n+1}\big(F_j\big).$$

(We have used $\mathrm{Per}_{B_1} F = \mathrm{Per}_{B_1} F_j$ and $\mathrm{Per}_{B_2 \setminus \overline{B_1}} E_j = \mathrm{Per}_{B_2 \setminus \overline{B_1}} F_j$.) The second term on the right-hand-side is written as $\int_{\partial B_1} |\theta^+_{F_j} - \theta^-_{F_j}| = \int_{\partial B_1} |\theta^+_{E_j} - \theta^-_F|$. Since ∂B_1 is compact, $|\theta^+_{E_j} - \theta^-_F| \leq 1$, and $\theta^+_{E_j} \to \theta^+_E = \theta^+_F$ pointwise (by the hypothesis $\langle [\![E_j]\!], |x| = 1^+\rangle \to \langle [\![E]\!], |x| = 1^+\rangle$), we conclude that (by dominated convergence) $\int_{\partial B_1} |\theta^+_{E_j} - \theta^-_F| \to \int_{\partial B_1} |\theta^+_F - \theta^-_F|$. The latter is $\mathbb{M}\big(\langle [\![F]\!], |x| = 1^+\rangle - \langle [\![F]\!], |x| = 1^-\rangle\big)$. Sending $j \to \infty$ and using the lower-semi-continuity of mass and perimeter on the left-hand-side, we find

$$\mathrm{Per}_{B_1} E + \mathbb{M}\big(\langle [\![E]\!], |x| = 1^+\rangle - \langle [\![E]\!], |x| = 1^-\rangle\big) - \lambda \mathcal{H}^{n+1}\big(E\big) \leq$$

$$\mathrm{Per}_{B_1} F + \mathbb{M}\big(\langle [\![F]\!], |x| = 1^+\rangle - \langle [\![F]\!], |x| = 1^-\rangle\big) - \lambda \mathcal{H}^{n+1}\big(F\big).$$

Adding $\mathrm{Per}_{B_2 \setminus \overline{B_1}} E = \mathrm{Per}_{B_2 \setminus \overline{B_1}} F$ to both sides, and using (12) again (with E, F in place of D), the inequality obtained becomes $J_\lambda(E) \leq J_\lambda(F)$. Therefore E minimises J_λ (among sets that coincide with E in $B_2 \setminus \overline{B_1}$).

Repeating the above argument with E in place of F shows that we must have $\mathrm{Per}_{B_1} E = \lim_{j\to\infty} \mathrm{Per}_{B_1} E_j$, therefore $\|\partial^* E_j\| \to \|\partial^* E\|$ as Radon measures in B_1 (and, by Allard's compactness for integral varifolds, $|\partial^* E_j| \to |\partial^* E|$ in B_1). $\square$

Remark 4.2. As a consequence of Lemma 4.1, if E minimises J_λ in an open set U then, given a point in U and a blow up sequence of dilations, any resulting varifold tangent cone to $|\partial^* E|$ agrees with the multiplicity-1 varifold associated to the blow up set (in the sense of sets with finite perimeter); in particular, any varifold tangent has multiplicity 1 on its regular part.

Remark 4.3. If $\lambda < n$ then for a minimiser such as E (or E_j) in Lemma 4.1, one has $\mathcal{H}^n(\partial^* E \cap \partial B_1) = 0$ (see Lemma 2.3). Therefore $\partial[\![E]\!] \llcorner \partial B_1 = 0$ and $\langle [\![E]\!], |x| = 1^+\rangle = \langle [\![E]\!], |x| = 1^-\rangle$ by (11) (and the standard slice $\langle [\![E]\!], |x| = 1\rangle$ exists).

We are now ready to prove the main result of this section, an instance of maximum principle for CMC hypersurfaces with isolated singularities.

Proposition 4.1. *Let E and F be sets with finite perimeter in B_2 that minimise J_λ with respect to their own boundary condition, assumed in $B_2 \setminus B_1$. Assume that $\partial E \cap (B_1 \setminus \{0\})$ is smooth, and that a tangent cone to $|\partial^* E|$ at 0 is regular (which means, it is smooth away from 0 and has multiplicity 1 on its regular part). Assume further that $F \subset E$ and that $0 \in \partial E$ and $0 \in \overline{\partial^* F}$. Then $E \cap B_1 = F \cap B_1$.*

Remark 4.4. Under the assumed condition on a tangent cone, by L. Simon's renowned result [21], $|\partial^* E|$ possesses a unique tangent cone at 0 (which has to be the one about which the regularity and multiplicity hypotheses are made).

Proof. Step 1. We begin by proving that $\overline{\partial^* F}$ is smooth in $B_r \setminus \{0\}$ for some $r > 0$. Let $\Sigma \subset \overline{\partial^* F}$ denote the singular set of $\overline{\partial^* F}$. Arguing by contradiction, assume that $x_i \to 0$, $x_i \in \Sigma$. Letting $\rho_i = |x_i|$, we consider the sequence of dilations $x \to \frac{x}{\rho_i}$ and take a blow up of F at 0 by setting $F_{0,\rho_i} = \frac{F}{\rho_i}$ and taking a subsequential limit F_0 of F_{0,ρ_i}. By the assumption that $F \subset E$ we have that $F_0 \subset E_0$, where E_0 is the blow up of E at 0 obtained by taking the limit for said subsequence of dilations (as remarked above, the blow up for E at 0 is independent of the sequence of dilations). The stationarity property of F with respect to J_λ translates into stationarity of F_{0,ρ_i} with respect to $J_{\rho_i \lambda}$, which implies that F_0 is stationary for the perimeter (equivalently, J_0). Similarly, E_0 is perimeter-stationary, that is, both $|\partial^* E_0|$ and $|\partial^* F_0|$ are stationary varifolds in $\mathbb{R}^{n+1}$. (We remark that both $|\partial^* E_0|$ and $|\partial^* F_0|$ are non-zero, since the origin is in the support of both $|\partial^* E|$ and $|\partial^* F|$ and thus both densities are ≥ 1 by the monotonicity formula.)

More precisely, by Lemma 4.1, E_0 and F_0 are perimeter minimisers in any compact set $K \subset \mathbb{R}^{n+1}$, for their own boundary condition (assumed in the complement of K). Clearly, $0 \in \mathrm{spt}|\partial^* E_0| \cap \mathrm{spt}|\partial^* F_0|$. Then the singular maximum principle [16, Theorem A (iii)] implies that $\mathrm{spt}|\partial^* E_0| = \mathrm{spt}|\partial^* F_0|$, and thus $|\partial^* E_0| = |\partial^* F_0|$. (Alternatively, one may use the maximum principle in the form given in [20].)

Lemma 4.1 (see Remark 4.2) also gives that $|\partial^* F_{0,\rho_i}|$ converge (as varifolds) to $|\partial^* F_0|$. By the choice of dilations, and by Allard's theorem, the points $\frac{x_i}{\rho_i}$ lie in ∂B_1 and have density $\Theta(\|\partial^* F_{0,\rho_i}\|, \frac{x_i}{\rho_i}) \geq 1 + \varepsilon_0$, where $\varepsilon_0 > 0$ is the dimensional constant in Allard's theorem. This contradicts the hypothesis that the density of $|\partial^* E_0| = |\partial^* F_0|$ is 1 at any point distinct from 0. We have therefore established the smoothness of $\overline{\partial^* F}$ in $B_r \setminus \{0\}$ for some $r > 0$.

Step 2. As remarked above, $|\partial^* E_0|$ is the unique tangent cone to $|\partial^* E|$ at 0. This also implies that $|\partial^* F_0|$ is the unique tangent cone for $|\partial^* F|$ at 0 (since, given any blow up sequence, the resulting blow up of F at 0 is contained in E_0, and the maximum principle implies, as above, that the two blow up sets must coincide). In particular (see [21, Section 7]), we are able to write $\partial E \cap (B_\delta \setminus \{0\})$ and $\partial F \cap (B_\delta \setminus \{0\})$, for sufficiently small $\delta > 0$, as graphs of C^2 functions over the common cone $C_\delta = C \cap (B_\delta \setminus \{0\})$, where $C = \overline{\partial^* E_0}$, as follows:

$$\begin{aligned}
&\partial E \cap (B_\delta \setminus \{0\}) = \mathrm{gr}_{C_\delta} u, \text{ with } u \in C^2(C_\delta; \mathbb{R}),\\
&\partial F \cap (B_\delta \setminus \{0\}) = \mathrm{gr}_{C_\delta} v \text{ with } v \in C^2(C_\delta; \mathbb{R}),\\
&\lim_{|x|\to 0} \Big(\frac{|u(x)|}{|x|} + |\nabla u(x)|\Big) = 0\\
&\lim_{|x|\to 0} \Big(\frac{|v(x)|}{|x|} + |\nabla v(x)|\Big) = 0.
\end{aligned} \tag{10}$$

Taking the identification of $\mathbb{R}$ with $(TC)^\perp$ so that the orientation is inward (for E_0), we have, in view of $E \subset F$ and the fact that $|\partial^* E|$ and $|\partial^* F|$ are stationary for J_λ,

$$u \leq v \text{ and } \mathcal{M}_C u = \lambda \det(Id - uA_C),\ \mathcal{M}_C v = \lambda \det(Id - vA_C).$$

Note that due to (10) the PDE for u and v satisfies the estimates (6) in C_δ and from standard elliptic estimates, see also [21, Section 1], we deduce that $|x||\nabla^2 u(x)| + |x||\nabla^2 v(x)| \to 0$ as $|x| \to 0$ hence the radial decay (3) is satisfied. In particular, we may consider $h = v - u \geq 0$ and from (7) we have that h satisfies the linear PDE

$$L_C h = A_1 \cdot \nabla^2 h + \frac{1}{|x|} A_2 \cdot \nabla h + \frac{1}{|x|^2} A_3 h,$$

where $A_1, A_2, A_3 \xrightarrow[|x|\to 0]{} 0$. Thus for any $K \subset\subset C_\delta$ we can apply the Harnack inequality to get that

$$\sup_K h \leq C_K \inf_K h.$$

Hence either $h > 0$ or $h \equiv 0$.

Step 3. The minimising property of E_0 implies that C is a stable minimal cone and thus all positive Jacobi fields are of the form (9). To prove that $u \equiv v$, we will construct a non-existent positive Jacobi field on $C \setminus \{0\}$ under the contradiction assumption that $h > 0$. We argue as in [15, Lemma 1.20].

From the property that $h \to 0$ as $|x| \to 0$ we can construct a sequence of ρ'_j such that

$$\sup_{C_{\rho'_{j+1}}} h < \sup_{C_{\rho'_j}} h.$$

Let x_j be the points where $\sup\limits_{C_{\rho'_j}} h$ is achieved and set $r_j = |x_j|$. Then $r_j \searrow 0$ (since $r_j \in (\rho'_{j+1}, \rho'_j))$ and $\sup_{C_{r_j}} h = \sup_{\partial C_{r_j}} h$. We define $h_j(x) = h(r_j x)$, for $x \in C_{\frac{\delta}{r_j}}$ and we have that

$$\sup_{C_1} h_j = \sup_{\partial C_1} h_j.$$

Let $x_j' \in \partial C_1$ where $\sup\limits_{C_1} h_j$ is achieved and set $f_j(x) = \dfrac{h_j(x)}{M_j}$, for $x \in C_{\frac{\delta}{r_j}}$, where $M_j = h_j(x_j')$. From the PDE for h we have that f_j satisfies the following PDE

$$L_C f_j = \tilde{A}_j^{(1)} \cdot \nabla^2 f_j + \frac{1}{|x|}\tilde{A}_j^{(2)} \cdot \nabla f_j + \frac{1}{|x|^2}\tilde{A}_j^{(3)} f_j,$$

where $\tilde{A}_j^{(i)}(x) = A_i(r_j x)$ for $x \in C_{\frac{\delta}{r_j}}$ and $i = 1, 2, 3$.

Fix a set $K \subset\subset C\backslash\{0\}$ and let K' be another set with $K \subset\subset K' \subset\subset C\backslash\{0\}$ and $x_j' \in K'$. Notice that, from the standard regularity theory for CMC hypersurfaces, we have that $u, v \in C^\infty$ thus the coefficients A_i, for $i = 1, 2, 3$ of the PDE are in $C^{0,\alpha}(K')$ and since $\tilde{A}_j^{(i)}$ are rescalings of A_i we have that $[\tilde{A}_j^{(i)}]_{\alpha;K'} \leq M_1 r_j^\alpha$, where M_1 is a constant independent of j and $[\cdot]_{\alpha;K'}$ is the Hölder semi-norm in K' with exponent α. In particular, if we combine with (3), we conclude that $||\tilde{A}_j^{(i)}||_{0,\alpha;K'} \to 0$, as $j \to \infty$ for $i = 1, 2, 3$. Thus from the $C^{2,\alpha}$-Schauder estimates, see Theorem 6.1 of [13], we get that

$$||f_j||_{2,\alpha;K} \leq M_3 ||f_j||_{0;K'},$$

where M_3 is a constant independent of j.

From the Harnack inequality on K' and since $x_j \in K'$ and $f_j(x_j) = 1$ we have that $||f_j||_{0;K'} \leq C_{K'} \inf_{K'} f_j \leq C_{K'}$ where $C_{K'}$ is a constant that depends on K'. Putting together we get that

$$||f_j||_{2,\alpha;K} \leq M_4,$$

where M_4 is a constant independent of j. From Arzelà-Ascoli theorem, after a diagonal argument and passing to a subsequence that we still index with j, we have that $f_j \xrightarrow[C^2_{\text{loc}}(C\backslash 0)]{} f \in C^{2,\alpha}(C \setminus \{0\})$. From the uniform convergence of $\tilde{A}_j^{(i)}$ on compact sets to zero, for $i = 1, 2, 3$, we get that $L_C f = 0$ in $C \setminus \{0\}$. Furthermore, up to a subsequence, we have that $x_j' \to x_0 \in \partial C_1$ and so $f(x_0) = 1$. Thus from Harnack's inequality $f > 0$.

In conclusion, we have constructed a positive solution of $L_C f = 0$ that is defined on the whole $C \setminus \{0\}$ for a stable minimal cone C of $\mathbb{R}^{n+1}$ and satisfies

$$\sup_{C_1} f = \sup_{\partial C_1} f.$$

The latter contradicts (9) and thus proves that $\partial E \cap B_\delta = \partial F \cap B_\delta$.

Step 4. Finally we show that $E \cap B_1 = F \cap B_1$. Let

$$r_0 = \sup\{r : \partial E \cap B_r = \partial F \cap B_r\}$$

and note that the set over which we take the supremum is non-empty due to the existence of δ, from the previous step, and it is in fact a maximum. Assume for the contrary that $r_0 < 1$ and let $x_0 \in \partial B_{r_0} \cap \partial F \cap \partial E$. Then from Remark 4.2 there exists a varifold tangent cone $|\partial^* G|$ at x_0 for $|\partial^* F|$ that is stationary for the perimeter functional in $\mathbb{R}^{n+1}$ and $\mathrm{spt}|\partial^* G|$ lies on the half-space given by the tangent plane of $|\partial^* E|$ at x_0, since $F \subset E$. Then from Theorem 36.5 of [19] we have that $|\partial^* G|$ is a plane hence the regularity theory implies that we can find a neighborhood $B_{\rho'}(x_0)$ where ∂F is smooth and $\partial F, \partial E$ meet tangentially at x_0. Since $F \subset E$ and due to the variational equations satisfied by J_λ the mean curvature vectors point in the same direction at x_0 thus the standard maximum principle implies that $\partial E \cap B_{\rho'}(x_0)$ coincides with $\partial F \cap B_{\rho'}(x_0)$. In particular, since x_0 is arbitrary and $\partial B_r \cap \partial F$ is compact we can find $\epsilon > 0$ such that $\partial E \cap B_{r_0+\epsilon} = \partial F \cap B_{r_0+\epsilon}$ contradicting the choice of r_0. Thus $r_0 = 1$ and we conclude that $E \cap B_1 = F \cap B_1$. □

5 Approximation

We assume that $E \subset \mathbb{R}^{n+1}$ satisfies the following properties. The topological boundary $T = \partial E$ contains 0, the hypersurface $(T \setminus \{0\}) \cap B_R$ is smooth for some $R > 0$ (so the origin is an isolated singularity for T), E minimises J_λ in B_R among Caccioppoli sets that coincide with E in $B_{2R} \setminus B_R$, a tangent cone to $|\partial^* E|$ at 0 is regular (which means, it is smooth away from 0 and has multiplicity 1 on its regular part).

Remark 5.1. Under the assumed condition on a tangent cone, [21] gives that $|\partial^* E|$ possesses a unique tangent cone at 0.

Remark 5.2. We note that if $n = 7$ these properties can be fulfilled whenever we have a Caccioppoli set that minimises J_λ locally. To begin with, one chooses a system of coordinates centred at a singular point, and R smaller than the distance of this to any other singular point (which is possible thanks to the interior regularity theory for minimisers). Moreover (again by the regularity theory) any tangent cone must be smooth away from the origin (for otherwise, the radial invariance would give a singular set of dimension at least one). Finally, any tangent cone must have multiplicity 1 on its regular part since the rescaled varifolds $|\partial E_{\rho_i,0}|$ converge as varifolds to $|\partial^* E_0|$ (see Remark 4.2).

It may not be true, in the above situation, that E is the unique minimiser of J_λ, among Caccioppoli sets that coincide with E in $B_{2R} \setminus B_R$. However, by taking a slightly smaller R (which preserves all the assumptions above), we can ensure said uniqueness, thanks to a standard argument that we now recall.

Lemma 5.1. *Let E, T be as above. If $R' < R$ then E is the unique minimiser of J_λ among sets that coincide with E in $B_{2R} \setminus B_{R'}$. (And therefore also among sets that coincide with E in $B_{2R'} \setminus B_{R'}$.)*

Proof. Let $R' < R$. Clearly, E minimises J_λ in $B_{R'}$ among Caccioppoli sets that coincide with E in $B_{2R} \setminus B_{R'}$. Assume that there exists a Caccioppoli set $E' \neq E$

that minimises J_λ in $B_{R'}$ among Caccioppoli sets that coincide with E in $B_{2R} \setminus B_{R'}$. In particular E' coincides with E in $B_{2R} \setminus B_R$, and on E' the energy J_λ attains the same value as it does on E. Therefore E' is a minimiser of J_λ in B_R, among Caccioppoli sets that coincide with E in $B_{2R} \setminus B_R$. As such, its reduced boundary must enjoy the optimal regularity of minimisers, that is, $\overline{\partial^* E'} \cap B_R$ is a smooth hypersurface (with mean curvature λ) away from a set $\Sigma \subset \overline{\partial^* E'} \cap B_R$ with $\dim_{\mathcal{H}} \Sigma \leq n-7$. We aim to prove that $\overline{\partial^* E'}$ coincides with $\overline{\partial^* E}$ (which is in contradiction with $E' \neq E$ and $E' = E$ in $B_{2R} \setminus B_{R'}$).

We define $r \leq R'$ by

$$r = \inf\{t : \overline{\partial^* E'} = \overline{\partial^* E} \text{ in } B_{2R} \setminus B_t.\}$$

and note that this is a minimum. The conclusion will follow upon establishing that $r = 0$. Assume $r > 0$. We remark that for $p \in \partial B_r \cap \overline{\partial^* E}$ we must have that there exists a unique tangent cone to $|\partial^* E'|$ at p, and it must coincide with the hyperplane that is tangent to ∂E at p. (This follows from $\overline{\partial^* E'} = \overline{\partial^* E}$ in $B_{2R} \setminus B_r$ and the smoothness of $\overline{\partial^* E}$ around p.) The regularity theory implies that $\overline{\partial^* E'}$ is smooth in an open ball $B^{n+1}_\rho(p)$ for some $\rho > 0$. Recall however that

$$\overline{\partial^* E'} = \left(\overline{\partial^* E'} \cap B_r\right) \cup \left(\overline{\partial^* E} \cap (B_{2R} \setminus B_r)\right),$$

and we have established that this is smooth in $B_\rho(p)$. Unique continuation implies that $\overline{\partial^* E'} \cap B_r$ coincides with $\overline{\partial^* E} \cap B_r$ in $B_\rho(p)$.

As $p \in \partial B_r \cap \overline{\partial^* E}$ is arbitrary and $\partial B_r \cap \overline{\partial^* E}$ is compact, it follows that $\overline{\partial^* E'}$ coincides with $\overline{\partial^* E}$ in $B_{2R} \setminus B_{r-\delta}$ for some $\delta > 0$, contradicting the choice of r. Hence $r = 0$ and $E' = E$ in B_{2R}. $\square$

Remark 5.3. By taking R' sufficiently small we also ensure that $\lambda < \frac{n}{R'}$. Therefore, upon dilating $B_{2R'}$ to B_2, we have that the working assumptions stated in the next theorem are fulfilled.

Theorem 5. *Let E a set of finite perimeter in B_2. Assume that the topological boundary $T = \partial E$ contains 0, the hypersurface $T \cap (B_2 \setminus \{0\})$ is smooth, E is the unique minimiser for J_λ in B_2 among Caccioppoli sets that coincide with E in $B_2 \setminus B_1$, $\lambda < n$.*

Given $r \in (0,1)$ there exists a sequence of sets E_j that have finite perimeter in B_2, such that: $\partial E_j \cap B_r$ is smooth for each j, it has constant mean curvature $\lambda \nu_{E_j}$, where ν_{E_j} is the inward unit normal to E_j, $E_j \subset E$, $E_j \to E$ and ∂E_j converge to ∂E smoothly on any $\Omega \subset\subset B_r \setminus \{0\}$.

Proof. Step 1. The first step is to perturb the boundary condition E inwards, and then use this new boundary condition to define E_j. The vector field ν_E is smooth in $(B_2 \setminus \{0\}) \cap \partial E$. Consider a tubular neighbourhood $\mathcal{N}_\rho$ of size $\rho > 0$ around $\partial E \cap (B_{\frac{3}{2}} \setminus B_{\frac{1}{2}})$, then the gradient of the signed distance function to ∂E (taken positive in E and negative in its complement) is a smooth extension of ν_E to $\mathcal{N}_\rho$. Let χ be a smooth function on B_2 that is equal to 1 in $(B_{\frac{5}{4}} \setminus B_{\frac{3}{4}}) \cap \mathcal{N}_{\frac{\rho}{2}}$ and with support contained in $(B_{\frac{3}{2}} \setminus B_{\frac{1}{2}}) \cap \{|d| < \frac{3}{4}\rho\}$. Let $X = \chi\nabla d$, then X extends ν_E and we may consider the flow $\Phi(t,x)$ of X. (We view X as a vector field in B_2.) For any $t \in [0,\delta)$, with $\delta > 0$ sufficiently small, $\phi_t(E) \subset E$. By construction $\phi_t(\partial E \cap \partial B_1)$ is disjoint from $\partial E \cap \partial B_1$ for all $t \in (0,\delta)$, and $\phi_t(E) \cap (B_{\frac{5}{4}} \setminus B_{\frac{3}{4}})$ is strictly contained in $E \cap (B_{\frac{5}{4}} \setminus B_{\frac{3}{4}})$.

The sequence E_j in the statement is built with the boundary condition $E_j = \phi_{t_j}(E)$ in $B_2 \setminus B_1$, for a sequence $t_j \to 0$. Namely, we define E_j to be a minimiser of J_λ for said boundary condition. The results proved in Section 2 guarantee existence of E_j and interior regularity for ∂E_j in B_1.

Step 2. We have $E_j \to E$ as $j \to \infty$. (Therefore $\partial[\![E_j]\!] \to \partial[\![E]\!]$ as well.) This follows from the uniqueness property of E, as we now show. To begin with, we have $J_\lambda(E_j) \le J_\lambda(\phi_{t_j}(E))$ (by the minimising property of E_j). By smoothness of X, using the area formula we find that $J_\lambda(\phi_{t_j}(E)) \to J_\lambda(E)$ as $j \to \infty$. In particular, there exists a uniform upper bound for $J_\lambda(\phi_{t_j}(E))$, and thus (since $|E_j| \le |B_2|$) a uniform upper bound for $\mathrm{Per}_{B_2}(E_j)$. Standard BV-compactness then gives the existence of a subsequential limit $E_j \to D$ with $|E_j| \to |D|$ and (by lower semi-continuity of perimeter) $J_\lambda(D) \le \liminf_{j\to\infty} J_\lambda(E_j)$. Recalling the previous considerations, $J_\lambda(D) \le \liminf_{j\to\infty} J_\lambda(E_j) \le J_\lambda(E)$. Finally, noting that $E_j \cap (B_2 \setminus \overline{B_1}) = \phi_{t_j}(E) \cap (B_2 \setminus \overline{B_1}) \to E \cap (B_2 \setminus \overline{B_1})$, we obtain that $D = E$ in $B_2 \setminus \overline{B_1}$ and therefore D is a minimiser (among sets with finite perimeter that coincide with E in $B_2 \setminus \overline{B_1}$). The uniqueness hypothesis on E gives $E = D$.

Next we will prove that $E_j \subset E$, for each given j. Considering the sets with finite perimeter $E_j \cap E$ and $E_j \cup E$, we have $[\![E_j \cap E]\!] + [\![E_j \cup E]\!] = [\![E_j]\!] + [\![E]\!]$, so that $\partial[\![E_j \cap E]\!] + \partial[\![E_j \cup E]\!] = \partial[\![E_j]\!] + \partial[\![E]\!]$. Clearly we also have $E_j \cap E \subset E_j \cup E$. This implies that $\mathcal{H}^n$-a.e. $x \in \partial^*(E_j \cap E) \cap \partial^*(E_j \cup E)$ one must obtain the same half-space as blow up for both sets $E_j \cap E$ and $E_j \cup E$, and therefore the measure-theoretic outer normals are the same at x for both sets. The common orientation $\mathcal{H}^n$-a.e. gives the equality

$$\mathbb{M}(\partial[\![E_j \cap E]\!]) + \mathbb{M}(\partial[\![E_j \cup E]\!]) = \mathbb{M}(\partial[\![E_j \cap E]\!] + \partial[\![E_j \cup E]\!]),$$

and therefore

$$\mathbb{M}(\partial[\![E_j \cap E]\!]) + \mathbb{M}(\partial[\![E_j \cup E]\!]) = \mathbb{M}(\partial[\![E_j]\!] + \partial[\![E]\!]) \le \mathbb{M}(\partial[\![E_j]\!]) + \mathbb{M}(\partial[\![E]\!]).$$

Noting that $|E_j \cap E| + |E_j \cup E| = |E_j| + |E|$, we conclude that

$$J_\lambda(E_j \cap E) + J_\lambda(E_j \cup E) \le J_\lambda(E_j) + J_\lambda(E).$$

On the other hand, the minimising properties of E_j and E imply respectively that

$$J_\lambda(E_j \cap E) \ge J_\lambda(E_j),\ J_\lambda(E_j \cup E) \ge J_\lambda(E),$$

since $E_j = E_j \cap E$ and $E_j \cup E = E$ are valid in $B_2 \setminus \overline{B_1}$. Combining the inequalities we find that equalities hold throughout and therefore $E_j \cup E$ is also a minimiser of J_λ (among sets with finite perimeter that coincide with E in $B_2 \setminus \overline{B_1}$), so that the uniqueness of E gives $E_j \cup E = E$, that is, $E_j \subset E$. [3]

Step 3. As a consequence of Allard's theorem, and of the smoothness of ∂E away from the origin, we must then have that, for any $r < 1$ and $\sigma \in (0, r)$, there is $C^{1,\alpha}$ convergence of ∂E_j to ∂E in $B_r \setminus B_\sigma$. By elliptic regularity, the convergence is in fact smooth, and $\partial E_j \cap (B_r \setminus B_\sigma)$ is smooth for all sufficiently large j, depending on the choice of σ, r.

Let Σ_j denote the singular set of $\overline{\partial^* E_j}$ in B_1 (which is of dimension at most $n-7$). Let $r_0 < 1$ be fixed and let $p_j \in \Sigma_j \cap B_{r_0}$. In view of the previous conclusion, we must have $p_j \to 0$ as $j \to \infty$. Also remark that, by Proposition 4.1, we must have $0 \notin \overline{\partial^* E_j}$ for all j, so $p_j \neq 0$ for all sufficiently large j. We will dilate E_j around 0 by the homothethy $\eta_j(x) = \frac{x}{|p_j|}$. Then $\tilde{E}_j = \eta_j(E_j)$ is a Caccioppoli set in $B_{\frac{1}{|p_j|}}$, in particular in B_2 for all sufficiently large j; moreover, the point $\tilde{p}_j = \frac{p_j}{|p_j|}$ is singular for $\overline{\partial^* \tilde{E}_j}$ and lies on ∂B_1. Upon extracting a subsequence that we do not relabel, we can assume that $\tilde{E}_j \to \Omega$ and $|\partial^* \tilde{E}_j|$ converge to the (stationary) integral varifold $\partial^* \Omega$ in B_2. The minimising property of E_j with respect to J_λ implies that Ω minimises perimeter in any compact set. Moreover, as $E_j \subset E$, we have $\Omega \subset E_0$, where E_0 is the blow up of E at 0 obtained from η_j. Then [15, Theorem 2.1] (specifically, its final assertion) implies that either $\Omega = E_0$, or Ω belongs to the "Hardt-Simon family" of sets $G_s = \eta_{0,s}(G)$, where $\eta_{0,s}(\cdot) = \frac{\cdot}{s}$, $s > 0$, and $G \subsetneq E_0$ has smooth minimising boundary. On the other hand, the presence of a sequence of singular points $p_j \in \partial B_1$ implies, by Allard's theorem, that a subsequential limit $p \in \partial B_1$ of p_j must occur with density $\geq 1 + \epsilon_0$ in $|\partial^* \Omega|$, contradicting the smoothness and unit density of ∂E_0 and of ∂G_s (regardless of s) in a tubular neighbourhood of ∂B_1. The contradiction shows that $\Sigma_j \cap B_{r_0} = \emptyset$ for all sufficiently large j, so that $\partial E_j \cap B_{r_0}$ is a smooth hypersurface (for all sufficiently large j). $\square$

Remark 5.4. In fact, $\partial E_j \cap B_1$ is smooth. Assume for the contrary that there exist $y_j \in \operatorname{sing}|\partial^* E_j| \cap B_1$ then since $|\partial^* E_j| \to |\partial^* E|$ we have $\operatorname{spt}|\partial^* E_j| \xrightarrow[d_{\mathcal{H}}]{} \operatorname{spt}|\partial^* E|$, where $d_{\mathcal{H}}$ denotes the Hausdorff distance. Thus, up to a subsequence, $y_j \to x_0 \in \partial E \cap \overline{B_1}$. Since the boundary $\Gamma_j = \overline{\partial^* E_j} \cap \partial B_1$ is smooth there exists, from Allard's boundary regularity [1], a neighborhood V of Γ_j, uniform in j, such that $\overline{\partial^* E_j} \cap B_1$ is a smooth hypersurface-with-boundary Γ_j thus $d(y_j, \Gamma_j) \geq C$, where C is a constant independent of j. Since $\Gamma_j \xrightarrow[C^2]{} \partial E \cap \partial B_1$ we conclude that $x_0 \in \partial E \cap B_1$. Note that $\Theta^n(|\partial^* E_j|, y_j) \geq 1 + \epsilon_0$, where ϵ_0 is the constant of Allard's regularity theorem, thus $x_0 = 0$ since else we would have a contradiction to the fact that away from the origin $\partial E \cap B_1$ has unit density. Thus we must have $y_j \to 0$ and we can now proceed exactly as in the proof of Theorem 5.

[3] We point out that the conclusion $E_j \subset E$ would follow also without the uniqueness assumption on E, by exploiting interior regularity for the minimiser $E_j \cup E$ in B_1 to conclude that $\partial^* E_j$ and $\partial^* E$ cannot intersect transversely on their regular parts, and by then applying the maximum principle and unique continuation to exclude tangential intersections.

A Auxiliary results

We give a proof of the following general property.

Lemma A.1. *Let D be a set with finite perimeter in B_2, then*

$$\partial[\![D]\!] \llcorner \partial B_1 = \langle [\![D]\!], |x| = 1^- \rangle - \langle [\![D]\!], |x| = 1^+ \rangle \tag{11}$$

and

$$\mathrm{Per}_{B_2} D = \mathrm{Per}_{B_1} D + \mathrm{Per}_{B_2 \setminus \overline{B_1}} D + \mathbb{M}\big(\langle [\![D]\!], |x| = 1^+ \rangle - \langle [\![D]\!], |x| = 1^- \rangle\big). \tag{12}$$

Proof. To check this, we begin by recalling that for an open set $U \subset B_2$, one has $\mathrm{Per}_U D = \mathbb{M}\big(\partial[\![D]\!] \llcorner U\big)$, and $\mathbb{M}\big(\partial[\![D]\!]\big) = \mathbb{M}\big(\partial[\![D]\!] \llcorner B_1\big) + \mathbb{M}\big(\partial[\![D]\!] \llcorner (B_2 \setminus \overline{B_1})\big) + \mathbb{M}\big(\partial[\![D]\!] \llcorner \partial B_1\big)$. Therefore (12) follows from (11).

We recall that the restriction of $\partial[\![D]\!]$ to ∂B_1 is well-defined (since the current is normal) via the limit, for any n-form ω with compact support in B_2,

$$\big(\partial[\![D]\!] \llcorner \partial B_1\big)(\omega) = \lim_{h \to 0} \big(\partial[\![D]\!]\big)\big(\gamma_h(|x| - 1)\, \omega\big),$$

where $\gamma_h : (-\infty, \infty) \to \mathbb{R}$ is C^1, is identically 1 on $(-h, h)$, vanishes on $(-\infty, -2h) \cup (2h, \infty)$, and $\gamma' \in \big[-\frac{2}{h}, 0\big]$ on $(0, \infty)$ and $\gamma' \in \big[0, \frac{2}{h}\big]$ on $(-\infty, 0)$. Then

$$\begin{aligned}\big(\partial[\![D]\!] \llcorner \partial B_1\big)(\omega) &= \lim_{h \to 0} \big([\![D]\!]\big)\big(\gamma_h'(|x| - 1) d|x| \wedge \omega\big) + \lim_{h \to 0} \big([\![D]\!]\big)\big(\underbrace{\gamma_h(|x| - 1)\, d\omega}_{\to 0 \text{ as } h \to 0}\big) \\ &= \lim_{h \to 0} \big([\![D]\!]\big)\big(\gamma_h'(|x| - 1) d|x| \wedge \omega\big).\end{aligned} \tag{13}$$

On the other hand, let $\gamma_h^+ : (-\infty, \infty) \to \mathbb{R}$ be C^1, identically 0 on $(-\infty, 0)$, and equal to $1 - \gamma_h$ on $[0, \infty)$. Let $\gamma_h^- : (-\infty, \infty) \to \mathbb{R}$ be defined by $\gamma_h^+(s) = \gamma_h^-(-s)$. Note that $\gamma_h^+ + \gamma_h^- + \gamma_h = 1$. Then

$$\langle [\![D]\!], |x| = 1^+ \rangle(\omega) = -\partial\Big([\![D]\!] \llcorner \{|x| > 1\}\Big)(\omega) + \Big(\partial[\![D]\!] \llcorner \{|x| > 1\}\Big)(\omega) =$$

$$-\lim_{h \to 0} [\![D]\!](\gamma_h^+(|x| - 1) d\omega) + \lim_{h \to 0} [\![D]\!](d(\gamma_h^+(|x| - 1)\omega)) = \lim_{h \to 0} [\![D]\!]\big((\gamma_h^+)'(|x| - 1) d|x| \wedge \omega)\big)$$

and similarly

$$\langle [\![D]\!], |x| = 1^- \rangle(\omega) = -\lim_{h \to 0} [\![D]\!]\big((\gamma_h^-)'(|x| - 1) d|x| \wedge \omega)\big).$$

Therefore

$$\Big(\langle [\![D]\!], |x| = 1^+ \rangle - \langle [\![D]\!], |x| = 1^- \rangle\Big)(\omega) = -\lim_{h \to 0} [\![D]\!]\big((\gamma_h)'(|x| - 1) d|x| \wedge \omega)\big),$$

which, jointly with (13), gives (11). $\square$

We provide the details regarding the positive solutions to the linear elliptic PDE $L_C f = 0$, which is crucial in the proof of Proposition 4.1.

Lemma A.2. *Let C be a regular stable minimal n-cone in $\mathbb{R}^{n+1}$. Then every positive solution of $L_C f = 0$ is of the form*

$$f(r\omega) = \left(\frac{c_1}{r^{\gamma^+}} + \frac{c_2}{r^{\gamma^-}}\right)\phi_1(\omega),$$

where $\phi_1 > 0$ is the first eigenfunction of L_Σ, and c_1, c_2 are non-negative constants.

Proof. Consider the eigenvalues of the operator $-L_\Sigma$,

$$\lambda_1 < \lambda_2 \leq \lambda_3 \cdots \to \infty$$

and let (ϕ_j) be an orthonormal basis of $L^2(\Sigma)$ such that ϕ_j is an eigenfunction of λ_j. Recall that $\phi_1 > 0$ and λ_1 is a simple eigenvalue.

For any $r > 0$ the function $f(r, \cdot)$ (on Σ) is of the form $\sum_{j=1}^{\infty} a_j(r)\phi_j(\omega)$. Thus in order to solve $L_C f = 0$ we write L_C in spherical coordinates and from (8) we get, after solving the corresponding ODE for a_j, that $a_j(r) = c_j^+ r^{-\gamma_j^+} + c_j^- r^{-\gamma_j^-}$, where $\gamma_j^{\pm} = \dfrac{n-2}{2} \pm \sqrt{\dfrac{(n-2)^2}{4} + \lambda_j}$ and $c_j^{\pm}$ are constants. Thus

$$f(r\omega) = \sum_{j=1}^{\infty} c_j^{\pm} r^{-\gamma_j^{\pm}} \phi_j(\omega).$$

We want to prove that $c_j^{\pm} = 0$ for all $j \geq 2$. Since $L_C f = 0$ and $f > 0$ from Harnack's inequality on $K_1 = C \cap (\overline{B_2} \setminus B_{\frac{1}{2}})$, Corollary 8.21 of [13], we have that $\sup_{K_1} f \leq C_{K_1} \inf_{K_1} f$, where C_{K_1} is a constant that depends on K_1 and the operator L_C. Let now $K_s = C \cap (\overline{B_{2s}} \setminus B_{s/2})$, for some $s > 0$ to be fixed later. Notice that if we rescale $f_s(x) = f(sx)$ then the scale invariance of the operator L_C implies that

$$\sup_{K_s} f \leq C_{K_1} \inf_{K_s} f.$$

We want to evaluate the L^2-norm of f on K_s with respect to the cone metric $g_C = dr^2 + r^2 g_\Sigma$. First note that

$$||f||_{L^2(K_s)} \leq \left(\mathcal{H}^n(C \cap K_1)s^n\right)^{1/2} \sup_{K_s} f = C_{(K_1,n,\Sigma)} s^{n/2} \sup_{K_s} f,$$

where $C_{(K_1,n,\Sigma)}$ denotes a constant that depends on K_1, n, Σ that may vary from line to line. On the other hand, since ϕ_j is an orthonormal basis of $L^2(\Sigma)$, we have

$$\begin{aligned} ||f||_{L^2(K_s)} &= \left(\int_{s/2}^{2s} \sum_{j=1}^{\infty} (c_j^{\pm})^2 r^{-2\gamma_j^{\pm}} r^{n-1} dr\right)^{1/2} \\ &= \left(\sum_{j=1}^{\infty} (c_j^{\pm})^2 s^{n-2\gamma_j^{\pm}} \left(\frac{2^{n-2\gamma_j^{\pm}} - 2^{2\gamma_j^{\pm}-n}}{n - 2\gamma_j^{\pm}}\right)\right)^{1/2}, \end{aligned}$$

and since $\dfrac{2^x - 2^{-x}}{x} \geq 1$ for any $x \in \mathbb{R} \setminus \{0\}$ we conclude that

$$||f||_{L^2(K_s)} \geq s^{n/2} \bigg(\sum_{j=1}^{\infty} (c_j^{\pm})^2 s^{-2\gamma_j^{\pm}} \bigg)^{1/2}.$$

The three inequalities thus give

$$C_{(K_1,n,\Sigma)} \bigg(\sum_{j=1}^{\infty} (c_j^{\pm})^2 s^{-2\gamma_j^{\pm}} \bigg)^{1/2} \leq \inf_{K_s} f \leq f(r,\omega),$$

for all $r \in [s/2, s]$ and $\omega \in \Sigma$. Multiplying the latter with ϕ_1, and integrating over Σ, we get

$$C_{(K_1,n,\Sigma)} \bigg(\sum_{j=1}^{\infty} (c_j^{\pm})^2 s^{-2\gamma_j^{\pm}} \bigg)^{1/2} \leq c_1^+ r^{-\gamma_1^+} + c_1^- r^{-\gamma_1^-},$$

for all $r \in [\frac{s}{2}, 2s]$. Thus we may take $r = s$ and get that

$$C_{(K_1,n,\Sigma)} \bigg(\sum_{j=1}^{\infty} (c_j^{\pm})^2 s^{-2\gamma_j^{\pm}} \bigg)^{1/2} \leq c_1^+ s^{-\gamma_1^+} + c_1^- s^{-\gamma_1^-}. \tag{14}$$

Multiplying now (14) by $s^{\gamma_1^+}$ we have that

$$C_{(K_1,n,\Sigma)} \bigg(\sum_{j=1}^{\infty} (c_j^{\pm})^2 s^{2\gamma_1^+ - 2\gamma_j^{\pm}} \bigg)^{1/2} \leq c_1^+ + c_1^- s^{\gamma_1^+ - \gamma_1^-}.$$

In order to prove that $c_j^+ = 0$ for all $j \geq 2$ first note that $\sum_{j=2}^{\infty} (c_j^+)^2 < \infty$ (by Parseval's identity it is bounded by $||f||_{L^2(\Sigma)}$), and recall that $\gamma_j^- \leq \gamma_2^- < \gamma_1^- < \gamma_1^+ < \gamma_2^+ \leq \gamma_j^+$ for all $j \geq 2$. Thus for any $E > 0$ there exists $s_0 > 0$ such that $s \leq s_0 \Rightarrow s^{2\gamma_1^+ - 2\gamma_j^+} > E^2$ for every j, and moreover $s^{\gamma_1^+ - \gamma_1^-} < \frac{1}{|c_1^-|}$ thus we obtain

$$E^2 \sum_{j=2}^{\infty} (c_j^+)^2 \leq C_{(K_1,n,\Sigma)} (c_1^+ + 1)^2,$$

which gives a contradiction for sufficiently large E unless $c_j^+ = 0$ for all $j \geq 2$. If we instead multiply (14) by $s^{\gamma_1^-}$ and choose s sufficiently large a similar argument leads to a contradiction unless $c_j^- = 0$ for all $j \geq 2$.

It remains to show that $c_1, c_2 \geq 0$. Assume for the contrary that $c_1 < 0$ then

$$r^{\gamma^+} f = c_1 \phi_1 + c_2 r^{\gamma^+ - \gamma^-} \phi_1,$$

and letting $r \to 0$ we get a contradiction. A similar argument gives $c_2 \geq 0$. $\square$

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