Extensions of Schoen–Simon–Yau and Schoen–Simon theorems via iteration à la De Giorgi

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Abstract

We give an alternative proof of the Schoen–Simon–Yau curvature estimates and associated Bernstein-type theorems (1975), and extend the original result by including the case of 6-dimensional (stable minimal) immersions. The key step is an ε-regularity theorem, that assumes smallness of the scale-invariant L^2 norm of the second fundamental form.

Further, we obtain a graph description, in the Lipschitz multi-valued sense, for any stable minimal immersion of dimension $n \geq 2$, that may have a singular set Σ of locally finite $\mathcal{H}^{n-2}$-measure, and that is weakly close to a hyperplane. (In fact, if the $\mathcal{H}^{n-2}$-measure of the singular set vanishes, the conclusion is strengthened to a union of smooth graphs.) This follows directly from an ε-regularity theorem, that assumes smallness of the scale-invariant L^2 tilt-excess (verified when the hypersurface is weakly close to a hyperplane). Specialising the multi-valued decomposition to the case of embeddings, we recover the Schoen–Simon theorem (1981).

In both ε-regularity theorems the relevant quantity (respectively, length of the second fundamental form and tilt function) solves a non-linear PDE on the immersed minimal hypersurface. The proof is carried out intrinsically (without linearising the PDE) by implementing an iteration method à la De Giorgi (from the linear De Giorgi–Nash–Moser theory). Stability implies estimates (intrinsic weak Caccioppoli inequalities) that make the iteration effective despite the non-linear framework. (In both ε-regularity theorems the method gives explicit constants that quantify the required smallness.)

1 Introduction

Part I: curvature estimates. In the renowned 1975 work, Schoen–Simon–Yau proved that any properly immersed two-sided stable minimal hypersurface M in $\mathbb{R}^{n+1}$, with $n \leq 5$, and with Euclidean mass growth at infinity, is necessarily a union of affine hyperplanes. We develop a new and alternative approach to this, that additionally solves the case $n = 6$, which had since remained a well-known open question. More precisely, we prove:

Theorem 1 (Bernstein-type theorem)**.** *Let M be a (smooth) properly immersed two-sided stable minimal hypersurface in $\mathbb{R}^{n+1}$, for $n \in \{2,3,4,5,6\}$, with Euclidean mass growth at infinity, i.e. there exists $\Lambda \in (0,\infty)$ such that $\mathcal{H}^n\big(M \cap B^{n+1}_R(0)\big) \leq \Lambda R^n$ for all $R > 0$. Then M is a union of affine hyperplanes.*

As is well-known, an equivalent formulation of this property is given via a priori (interior) curvature estimates, as follows:

Theorem 2 (Pointwise curvature estimates)**.** *Assume that M is a (smooth) properly immersed two-sided stable minimal hypersurface in $B^{n+1}_{4R}(0)$, with $0 \in M$ and $n \in \{2,3,4,5,6\}$, with $\frac{\mathcal{H}^n\left(M \cap B^{n+1}_{4R}(0)\right)}{(4R)^n} \leq \Lambda \in (0,\infty)$. There exists $\beta > 0$ depending only on Λ and n such that*

$$\sup_{x \in B^{n+1}_{\frac{R}{2}}(0)} |A_M|(x) \leq \frac{\beta}{R},$$

where A_M is the second fundamental form of M.

Remark 1.1. As usual with interior-type estimates, the choice of $\frac{1}{8}$ as ratio between the relevant radii is arbitrary. Any ratio smaller than 1 can be allowed, and the constant β will depend on the chosen ratio.

Remark 1.2. Theorem 1 holds if we replace properness with the fact that the immersion is complete. Indeed, combining the coarea formula and the isoperimetric inequality one can bound from below the n-area of an intrinsic ball of radius r with $c(n)r^n$, for a dimensional constant $c(n)$ (see e.g. [21]). This implies (using completeness and area bounds) that the immersion is proper.

As recalled above, the Schoen–Simon–Yau theory obtained these theorems for $n \leq 5$, see [17, Theorem 3]. (In fact, under the weaker assumption of Euclidean area growth in the intrinsic sense.) For $n = 6$, the validity of these properties under an additional multiplicity-2 condition, more precisely under the restriction $\Lambda < 3\,\omega_6$, was obtained by Wickramasekera, see [22, Theorems 9.1 and 9.2] (the notation ω_n stands for the n-volume of the n-dimensional unit ball). If M is assumed to be properly embedded, rather than immersed, the above results are known to be valid for $n \leq 6$ in view of the fundamental sheeting theorem by Schoen–Simon, see [16, Theorems 1 and 3]. For $n \geq 7$, on the other hand, it has long been known that the situation is drastically different and the above results do not hold, even assuming that M is properly embedded, as in the example of the Hardt–Simon foliation [10].

We recall that the case $n = 2$ can be treated by means of a logarithmic cut-off argument, and, in fact, it has long been known that the above theorems hold for $n = 2$ without any mass hypothesis (see do Carmo–Peng [7], Fischer-Colbrie–Schoen [9], and Pogorelov [14]). Very recently, the Euclidean mass growth assumption has been shown to be redundant for $n = 3$ in the work of Chodosh–Li [3], which resolved a long-standing conjecture of Schoen (see also subsequent alternative proofs by the same authors [4] and by Catino–Mastrolia–Roncoroni [2]). Such Bernstein theorems imply that, for $n = 2, 3$, the constant β in Theorem 2 does not depend on Λ.[1]

We obtain Theorems 1 and 2 as a consequence of an ε-regularity theorem, in which the relevant (small) quantity is the scale-invariant L^2-norm of the second fundamental form:

Theorem 3 (ε-regularity for the second fundamental form)**.** *Let $n \leq 6$. There exists $\epsilon_0 > 0$, depending only n (a sufficiently small ϵ_0 is explicitly given in* (12) *below) with the following significance. Let M be a properly immersed two-sided stable minimal hypersurface in $B^{n+1}_{2R}(0)$, with $0 \in M$ and with*

$$\frac{1}{(2R)^{n-2}} \int_{M \cap B^{n+1}_{2R}(0)} |A_M|^2 \leq \epsilon_0.$$

Then for every $x \in M \cap B^{n+1}_{R/2}(0)$ we have $|A_M|(x) \leq \frac{1}{R}$. More precisely, in the above smallness regime, we have, for a (explicit) dimensional constant $c(n)$,

$$\sup_{M \cap B^{n+1}_{R/2}(0)} |A_M| \leq \frac{c(n)}{R} \left(\frac{1}{(2R)^{n-2}} \int_{M \cap B^{n+1}_{2R}(0)} |A_M|^2 \right)^{\frac{1}{2}}.$$

Theorem 3 is established by PDE methods, working intrinsically on the immersed hypersurface. The relevant (non-linear) PDE is the Simons equation for $|A_M|$. The proof is obtained by implementing an iteration scheme à la De Giorgi, in the style of the linear theory in [6] (the widely known De Giorgi–Nash–Moser theory). The iteration relies on the validity of a weak intrinsic Caccioppoli inequality, valid for level set truncations of $|A_M|$. We establish this inequality in Lemma 2.1, from the associated PDE, making (essential) use of the stability hypothesis to control the terms that escape the linear PDE theory framework.

Theorems 1 and 2 then follow employing soft classical geometric measure theory arguments: Allard's compactness, tangent cone analysis, Federer's dimension reduction. Ultimately as a consequence of the well-known Simons classification of stable cones ([20]), the analysis only needs to address scenarios in which M is close to a cone of the following very specific types: a single hyperplane with multiplicity, a classical cone[2], or a union of hyperplanes with multiplicity. (Closeness is understood in the sense of varifolds.) In all these cases, the smallness condition of Theorem 3 is verified. In the first scenario, this is a consequence of the control of the tilt-excess by the height-excess (recalled in Remark 4.2) and of Schoen's

[1] After the appearance of this paper, Chodosh–Li–Minter–Stryker [5] and Mazet [12] showed the validity of 'stable Bernstein' theorems without any mass growth hypothesis, respectively for $n = 4$ and $n = 5$. Consequently, in these dimensions, β in Theorem 2 does not depend on Λ.

[2] A classical cone is the union of three or more (distinct) closed half-hyperplanes, having for boundary a common $(n-1)$-dimensional subspace of $\mathbb{R}^{n+1}$, and all intersecting at said boundary, each half-hyperplane endowed with an integer multiplicity. The common $(n-1)$-dimensional subspace is also referred to as the spine, a term which in general denotes the maximal subspace along which a cone is translation invariant.

inequality ([15], [16], see also (16) below taken with $k = 0$). In the remaining ones, smallness follows by an inductive argument (on the dimension of the spine of the cone), also using a higher integrability estimate close to the spine.

Remark 1.3. For $n = 3$, the proof of Theorem 3 does not require any smallness assumption, thus it establishes directly Theorems 1 and 2 (see Appendix A). In fact, this argument can be carried out also under an intrinsic area growth hypothesis.

Part II: Towards a compactness theory for branched stable minimal immersions. In the second part of this work, we consider stable minimal immersed hypersurfaces of arbitrary dimension n, and allow a singular set. More precisely, we enlarge the class of two-sided stable minimal immersions by allowing M to have a singular set with vanishing 2-capacity, in particular, we allow it to have locally finite $\mathcal{H}^{n-2}$-measure. Classical branch points show that singularities of this size can indeed arise for the class in question (in contrast with the embedded case, see [16] and Theorem 7 below, in which case branching can be ruled out a posteriori).

We take an intrinsic PDE approach that appears to be new in regularity theory for minimal hypersurfaces. The key step is an ε-regularity result for the scale-invariant tilt excess, Theorem 4 below. The tilt function on M is

$$g = \sqrt{1 - (\nu_M \cdot e_{n+1})^2},$$

where ν_M is a choice of unit normal to M and e_{n+1} is (any fixed unit vector, which we can without loss of generality assume to be) the last coordinate vector. The tilt-function thus varies in $[0, 1]$, with values being higher where the tangent to M tilts more with respect to the reference hyperplane $\{x_{n+1} = 0\}$. The smallness condition is assumed on the scale-invariant L^2-norm of g. More precisely, letting C_r denote the cylinder $B^n_r(0) \times (-r, r)$, for $R > 0$ the scale-invariant L^2 tilt-excess of M on C_R is the quantity $E_M(R)$ defined by

$$E_M(R)^2 = \frac{1}{R^n} \int_{M \cap C_R} (1 - (\nu_M \cdot e_{n+1})^2).$$

Theorem 4 (ε-regularity for the tilt)**.** *Let $n \geq 2$. Let M be a properly immersed, two-sided, stable minimal hypersurface in $C_{2R} \setminus \Sigma$, where Σ is closed in $C_{2R} = B^n_{2R}(0) \times (-2R, 2R)$, and with $cap_2(\Sigma) = 0$ (in particular, $\mathcal{H}^{n-2}\big(\Sigma \cap K\big) < \infty$ for every $K \subset\subset C_{2R}$ is permitted). There exists a positive dimensional constant $k(n)$ (a sufficiently small $k(n)$ is given explicitly in (23) below) with the following significance. Assume that*

$$E_M(R)^2 = \frac{1}{R^n} \int_{M \cap C_R} g^2 \leq k(n).$$

Then

$$\sup_{M \cap C_{\frac{R}{2}}} g \leq \frac{1}{2n}.$$

In fact, there exists a dimensional constant $c(n)$ such that, if $E_M(R)^2 \leq k(n)$, then for every $x \in M \cap C_{\frac{R}{2}}$ we have

$$g(x) \leq c(n) E_M(R).$$

Theorem 4 is applicable when $M \cap C_R$ is sufficiently close to a hyperplane, which we can assume to be $\{x_{n+1} = 0\}$ by a suitable rotation. This follows from the standard control of $E_R(M)$ by means of the L^2 height-excess (as recalled in Remark 4.2 this is an easy consequence of the minimality assumption). The bound on g obtained in Theorem 4 forces a decomposition of $\overline{M} \cap C_{\frac{R}{2}}$ into a union of graphs (over $B^{n+1}_{\frac{R}{2}} \cap \{x_{n+1} = 0\}$), where $\overline{M}$ denotes the closure of M in C_{2R}. In the general case considered in Theorem 4 these graphs are Lipschitz. However, with stronger assumptions we obtain stronger conclusions as well. We illustrate three main instances: the general case of singular immersions, the case of smooth immersions, the case of singular embeddings, respectively Theorems 5, 6 and 7 below. All three follow from the ε-regularity result, Theorem 4, very directly.

Theorem 4 is (as was the case for Theorem 3) established by PDE methods, working intrinsically on the immersed hypersurface M. The relevant (non-linear) PDE, for the tilt-function g, is a direct consequence of the Jacobi field equation for $(\nu \cdot e_{n+1})$. We prove that a weak intrinsic Caccioppoli inequality is valid for level set truncations of g, see Lemma 4.2; this uses the associated PDE and the stability hypothesis (to control the terms that escape the linear PDE theory framework, which involve $|A_M|$). We then implement an iteration à la De Giorgi. In Remarks 2.4 and 4.4 we discuss similarities and differences between the two intrinsic weak Caccioppoli inequalities (Lemmas 2.1 and 4.2), as well as compare them to those in De Giorgi's work [6].

Theorem 5 (sheeting theorem for singular immersions)**.** *In the hypotheses of Theorem 4, with the further assumption that* $\sup_{M \cap C_{2R}} |x_{n+1}| < \frac{R}{2}$*, we have*

$$\overline{M} \cap C_{\frac{R}{2}} = \cup_{j=1}^{q} graph(u_j)$$

for some $q \in \mathbb{N}$, and $u_j : B^n_{\frac{R}{2}} \to \mathbb{R}$ are Lipschitz functions, with Lipschitz constant at most $\frac{1}{2n}$ and $u_j \leq u_{j+1}$ for every $j \in \{1, \ldots, q-1\}$. (We identify $B^n_{\frac{R}{2}}$ with $B^{n+1}_{\frac{R}{2}}(0) \cap \{x_{n+1} = 0\}$ and the target $\mathbb{R}$ with the x_{n+1} coordinate axis.) More precisely, the Lipschitz constant of each u_j is bounded by $c(n)E_M(R)$, with $c(n)$ a dimensional constant.

In fact, $M \cap C_{\frac{R}{2}}$ in Theorem 5 is naturally a smooth q-valued function on $B^n_{\frac{R}{2}} \setminus \pi(\Sigma)$, where $\pi : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is the standard projection. In general, we do not have q smooth graphs on $B_{\frac{R}{2}} \setminus \pi(\Sigma)$: the size of Σ permits classical branching, therefore smoothness only holds for the q-valued function and, in general, there is no "selection" of q smooth functions on $B_{\frac{R}{2}} \setminus \pi(\Sigma)$. On the other hand, one can easily write $\overline{M} \cap C_{\frac{R}{2}}$ as the union of graphs of q Lipschitz functions by ordering the

q values increasingly and extending the Lipschitz functions across $\pi(\Sigma)$, as done in the statement of Theorem 5.

The multi-valued graph structure obtained in Theorem 5 rules out, for example, that (in a branched stable minimal immersion with singular set of locally finite $(n-2)$-measure) there may be an accumulation of necks (connecting different sheets) onto a flat branch point.

Remark 1.4. Working with immersions of the same type as in Theorem 5, and under an additional 'multiplicity 2' assumption, Wickramasekera [22] obtained that when the L^2 height-excess $\int_{C_R} |x_{n+1}|^2$ is sufficiently small then a sheeting description is valid by means of a $C^{1,\alpha}$ 2-valued function. (The strategy in [22] involves a 2-valued Lipschitz approximation of M, followed by a linearisation of the problem, which in particular prevents a quantitative smallness condition.)

Theorem 5 advances towards a compactness theory for branched stable minimal immersions. (This was obtained for a "multiplicity 2 class" in [22].) The natural missing step is the analysis of the situation in which, rather than being close to a hyperplane with multiplicity, M is close to a classical cone. In view of Theorem 5, and of the multiplicity-2 case in [22], it seems natural to expect that:
Conjecture: the class of branched two-sided stable minimal n-dimensional immersions with singular set of locally finite $(n-2)$-measure is compact under varifold convergence.

A further natural aim would then be to obtain a finer structure result for said singular set, in the style of [22, Theorem 1.5]. It may be possible to use an intrinsic approach. This lies outside the scope of this work.

Remark 1.5 (*Unique tangent hyperplanes and Bernstein-type theorem*)**.** An immediate byproduct of Theorem 5, for the class of immersions under study, is that: if $x \in \overline{M}$ is such that one tangent cone (in the sense of varifolds) to M at x is supported on a hyperplane, then that is the unique tangent cone at x (Corollary 1). Similarly, it follows immediately from Theorem 5 that, if M is entire, with Euclidean mass growth, and if one tangent cone at infinity is a hyperplane with multiplicity, then M is a union of hyperplanes (Corollary 2).

If the singular set Σ in Theorem 4 is a priori assumed empty then the graphical decomposition is stronger and prompts a linear PDE behaviour, with a linear (interior) control of $\sup |A_M|$ by $E_M(R)$:

Theorem 6 (sheeting theorem for smooth immersions)**.** *In the hypotheses of Theorem 4, with the further assumptions that* $\sup_{M \cap C_{2R}} |x_{n+1}| < \frac{R}{2}$ *and that* $\Sigma = \emptyset$ *(that is, M is a closed immersed hypersurface in C_{2R}), we have*

$$M \cap C_{\frac{R}{2}} = \cup_{j=1}^{q} graph(v_j),$$

where $v_j : B^n_{\frac{R}{2}}(0) \equiv B^{n+1}_{\frac{R}{2}}(0) \cap \{x_{n+1} = 0\} \to \mathbb{R} \equiv span(e_{n+1})$ *are smooth functions and* $q \in \mathbb{N}$. *(We note that these graphs are not ordered, they may cross.) Moreover,*

$\sup_{j\in\{1,\dots,q\}} \|\nabla v_j\|_{C^{1,\alpha}\left(B^n_{\frac{R}{2}}(0)\right)} \leq c(n)E_M(R)$, *for a dimensional constant* $c(n)$*. In particular,*

$$\sup_{C_{\frac{R}{2}}\cap M} |A_M| \leq c(n)E_M(R).$$

Remark 1.6. Theorem 6 rules out the appearance of a flat branch point when taking a (varifold) limit of smooth stable minimal immersions.

Remark 1.7 (*Sufficiency of* $\mathcal{H}^{n-2}(\Sigma)=0$)**.** The assumption $\Sigma=\emptyset$ in Theorem 6 can be weakened to $\mathcal{H}^{n-2}(\Sigma)=0$. Indeed, under this assumption, $B_{\frac{R}{2}}\setminus\pi(\Sigma)$ is a simply connected open set. This implies (using the conclusions of Theorem 5) that M can be written as the union of graphs (not ordered ones) of smooth functions on $B_{\frac{R}{2}}\setminus\pi(\Sigma)$. Then a removal of singularity for the minimal surface PDE ([18]) shows that in fact $\Sigma=\emptyset$. In view of this (in analogy with the earlier discussion on the branched case), a compactness theory for stable minimal immersions with a codimension-7 singular set could be obtained[3] upon addressing the natural missing step, in which M is close to a classical cone, rather than to a hyperplane with multiplicity. (Again, [22, Theorem 1.3] obtained this for a "multiplicity 2 class".)

If we instead specialise Theorem 5 to hypersurfaces M that are *embedded* away from Σ, then by removal of singularities for the minimal surface PDE, used for each function u_j, we recover (a quantitative version of) the well-known Schoen–Simon sheeting theorem ([16, Theorem 1]):

Theorem 7 (sheeting theorem for singular embeddings)**.** *Let* $n\geq 2$*. Let* M *be a properly embedded, two-sided stable minimal hypersurface in* $C_{2R}\setminus\Sigma$*, where* Σ *is closed in* $C_{2R}=B^n_R(0)\times(-R,R)$*, with locally finite* $\mathcal{H}^{n-2}$*-measure, or, more generally, with* $cap_2(\Sigma)=0$*. Assume that* $\sup_{M\cap C_{2R}}|x_{n+1}|<\frac{R}{2}$ *and*

$$E_M(R)^2=\frac{1}{R^n}\int_{M\cap C_R}(1-\nu\cdot e_{n+1})^2\leq k(n),$$

where $k(n)$ *is the (positive) dimensional constant in* (23) *and* ν *is a choice of unit normal to* M*. Then*

$$\overline{M}\cap C_{\frac{R}{2}}=\cup_{j=1}^q graph(u_j)$$

with $u_j: B_{\frac{R}{2}}\to\mathbb{R}$ *smooth,* $u_j<u_{j+1}$ *for every* j*. In particular,* $\overline{M}\cap C_{\frac{R}{2}}$ *is smoothly embedded (equivalently,* M *extends smoothly across* Σ *in* $C_{\frac{R}{2}}$*), and* $\sup_{C_{\frac{R}{2}}\cap M}|A_M|\leq c(n)E_M(R)$ *for a dimensional constant* $c(n)$*.*

We recall that Theorem 7 leads (by fairly standard arguments) to the renowned compactness and regularity theory [16, Theorems 2 and 3] for stable minimal embedded hypersurfaces that are allowed to possess a singular set of locally finite

[3] After the appearance of this paper, Hong–Li–Wang [11] carried out a modification of the arguments in Sections 2 and 3 below, obtaining [11, Corollary 1.3], which asserts the validity of such compactness statement.

$\mathcal{H}^{n-2}$-measure. A posteriori, the singular set contains no branch points and in fact has dimension at most $n-7$, it is discrete in the case $n=7$, and empty for $n \leq 6$.

We thus obtain an alternative and hopefully more immediate route to (the main component of) the Schoen–Simon theory. (The approach in [16] involves a partial q-valued graph decomposition of the embedding, and a linearisation of the problem, both of which we avoid.)

The impact of Schoen–Simon's compactness, and of Schoen–Simon–Yau's curvature estimates, for developments in analysis and geometry over the last half century, cannot be overstated.

Part I

Curvature estimates

We will denote by M (and by M_ℓ, $\ell \in \mathbb{N}$, when considering a sequence) a smooth two-sided properly immersed stable minimal hypersurface in an open set $U \subset \mathbb{R}^{n+1}$. Typically, the open set U will be a ball $B^{n+1}_R(0)$, or the whole of $\mathbb{R}^{n+1}$, or a cylinder of the form $B^n_R(0) \times (-R, R)$. In other words $M = \iota(S)$, with S an n-dimensional manifold and $\iota : S \to U$ a proper two-sided stable minimal immersion. We recall that the stability condition is the non-negativity of the second variation of the n-area, and that this amounts to the validity of

$$\int_S |A_M|^2 \phi^2 \leq \int_S |\nabla \phi|^2$$

for any $\phi \in C^1_c(S)$, where S is endowed with the pull-back metric from U, ∇ is the metric gradient on S and $|A_M|$ the length of the second fundamental form. We note that whenever $\varphi \in C^1_c(U)$, then $\varphi \circ \iota \in C^1_c(S)$ (since the immersion is proper); with a slight abuse of notation, we will write the integrals directly on M, with $\int_M |\nabla \varphi|^2$ in place of $\int_S |\nabla(\varphi \circ \iota)|^2$ and the inequality taking the form $\int_M |A_M|^2 \varphi^2 \leq \int_M |\nabla \varphi|^2$.

2 Proof of Theorem 3

In this section we prove Theorem 3. We recall the well-known Simons identity ([20]), for the second fundamental form A of a minimal hypersurface:

$$\frac{1}{2}\Delta |A|^2 = |\nabla A|^2 - |A|^4. \tag{1}$$

Clearly, $|\nabla A| \geq \big|\nabla |A|\big|$; in [17, (1.33)] it is shown that the minimality condition implies the following improved inequality, with $c = \frac{2}{n}$:

$$|\nabla A|^2 \geq (1+c)\big|\nabla |A|\big|^2. \tag{2}$$

We will also use the following variant of (1): as $\frac{1}{2}\Delta|A|^2 = |A|\Delta|A| + |\nabla|A||^2$, we find

$$|A|\Delta|A| = |\nabla A|^2 - \left|\nabla|A|\right|^2 - |A|^4. \tag{3}$$

The following lemma contains the relevant weak (intrinsic) Caccioppoli inequality for the level set truncations of $|A|$:

Lemma 2.1. *Let M be a properly immersed smooth two-sided stable minimal hypersurface in $U \subset \mathbb{R}^{n+1}$. For any $k \geq 0$ and any $\eta \in C^1_c(U)$ we have*

$$\int_{\{|A|>k\}} \left(1 - \frac{k}{|A|}\right) \left|\nabla|A|\right|^2 \eta^2 \leq \frac{1}{c} \int ((|A|-k)^+)^2 |\nabla\eta|^2 +$$
$$\frac{k}{c}\int ((|A|-k)^+)^3\,\eta^2 + \frac{2k^2}{c}\int ((|A|-k)^+)^2\,\eta^2 + \frac{k^3}{c}\int (|A|-k)^+\,\eta^2.$$

Remark 2.1. As mentioned above, the integrals are implicitly understood to be on S, with $\eta \circ \iota$ in place of η. For $k = 0$ the inequality is $\int \left|\nabla|A|\right|^2 \eta^2 \leq \frac{1}{c}\int |A|^2|\nabla\eta|^2$, which appears as an intermediate step along the proof of [17, Theorem 1].

Proof. We use the stability inequality with the Lipschitz test function $(|A|-k)^+\eta$, for $k \in [0,\infty)$ and $\eta \in C^1_c(U)$. We note that ${(|A|-k)^+}^2 \in C^1(M) \cap W^{2,\infty}_{\text{loc}}(M)$: indeed, being Lipschitz, its (distributional) gradient is the function $\nabla{(|A|-k)^+}^2 = 2(|A|-k)^+\nabla(|A|-k)^+ = 2(|A|-k)^+\nabla|A|$, which in turn is locally Lipschitz. We find

$$\int |A|^2((|A|-k)^+)^2\eta^2 \leq \int |\nabla((|A|-k)^+\eta)|^2 =$$
$$\int |\nabla(|A|-k)^+|^2\eta^2 + 2\int (|A|-k)^+\,\eta\,\nabla(|A|-k)^+\,\nabla\eta + \int ((|A|-k)^+)^2|\nabla\eta|^2$$
$$= \int |\nabla(|A|-k)^+|^2\eta^2 + \frac{1}{2}\underbrace{\int \nabla((|A|-k)^+)^2\,\nabla\eta^2} + \int ((|A|-k)^+)^2|\nabla\eta|^2$$

and integrating by parts the braced term we can continue the equality chain

$$= \int |\nabla(|A|-k)^+|^2\eta^2 - \frac{1}{2}\int_{\{|A|>k\}} \Delta|A|^2\,\eta^2 + \int_{\{|A|>k\}} \frac{k}{|A|}|A|\Delta|A|\,\eta^2$$
$$+ \int ((|A|-k)^+)^2|\nabla\eta|^2$$

$$\underbrace{=}_{(1),(3)} \int_{\{|A|>k\}} \left|\nabla|A|\right|^2\eta^2 + \int_{\{|A|>k\}} (-|\nabla A|^2 + |A|^4)\,\eta^2 +$$
$$\int_{\{|A|>k\}} \frac{k}{|A|}\left(|\nabla A|^2 - \left|\nabla|A|\right|^2\right)\eta^2 - k\int_{\{|A|>k\}} |A|^3\,\eta^2 + \int ((|A|-k)^+)^2|\nabla\eta|^2.$$

The left-most side (of the above chain of inequalities) is expanded as follows: $\int |A|^2(|A|-k)^+)^2\eta^2 = \int_{\{|A|>k\}}(|A|^4 - 2k|A|^3 + k^2|A|^2)\eta^2$. We thus find

$$\int_{\{|A|>k\}} \left(1 - \frac{k}{|A|}\right) (|\nabla A|^2 - \left|\nabla |A|\right|^2)\eta^2 \leq$$

$$\int ((|A|-k)^+)^2|\nabla\eta|^2 + k\int_{\{|A|>k\}} |A|^3\,\eta^2 - k^2\int_{\{|A|>k\}} |A|^2\,\eta^2$$

and using (2) we conclude

$$\int_{\{|A|>k\}} \left(1 - \frac{k}{|A|}\right) \left|\nabla |A|\right|^2 \eta^2 \leq \tag{4}$$

$$\frac{1}{c}\int \left((|A|-k)^+\right)^2|\nabla\eta|^2 + \frac{k}{c}\int |A|^2(|A|-k)^+\,\eta^2.$$

The desired inequality follows upon rewriting the last term on the right-hand-side of (4), on the set $\{|A| > k\}$:

$$|A|^2(|A|-k) = (|A|-k+k)^2\,(|A|-k) = (|A|-k)^3 + 2k(|A|-k)^2 + k^2(|A|-k). \tag{5}$$

□

Lemma 2.1 provides the weak intrinsic Caccioppoli inequality that will lead, through an iteration scheme à la De Giorgi, to Theorem 3.

Remark 2.2. Note that both assumption and conclusion in Theorem 3 are scale-invariant. The scale-invariant quantity $\frac{1}{(R)^{n-2}}\int_{M\cap B_R(0)} |A|^2$ is uniformly bounded under the Euclidean mass growth hypothesis that we have. Indeed, stability used with a test function $\varphi \in C^1_c(B_{2R}(0))$ with $\varphi \equiv 1$ on $B_R(0)$ and $|\nabla\varphi| \leq \frac{2}{R}$ gives $\int_{B_R(0)} |A|^2 \leq \frac{4}{R^2}\left(\int_{B_{2R}(0)} 1\right) \leq 2^{n+2}R^{n-2}\Lambda$.

Remark 2.3. The standard catenoids for $n \geq 3$, for which $\frac{1}{R^{n-2}}\int_{B_R(0)\cap M} |A|^2 \to 0$ as $R \to \infty$, show that stability is essential for Theorem 3. (On the other hand, the dimensional restriction may not be essential.) Catenoids also show that, given M minimal, the relevant "scale-invariant energy" $\frac{1}{R^{n-2}}\int_{B_R(0)\cap M} |A|^2$ is not an increasing function of R (unlike in several well-known ε-regularity theorems).

Proof of Theorem 3. We will assume $n \geq 3$ (see Remark 2.5) and consider the sequences $k_\ell = d - \frac{d}{2^{\ell-1}}$ and $R_\ell = \frac{R}{2} + \frac{R}{2^\ell}$ for $\ell \in \{1, 2, \dots,\}$ (respectively increasing and decreasing), where $d > 0$ is for the moment left undetermined, and will be quantified in terms of $\frac{1}{(2R)^{n-2}}\int_{B_{2R}} |A|^2$. Here and below, when writing integrals on $B_r = B^{n+1}_r(0)$ we implicitly understand that the integration is on $\iota^{-1}(B^{n+1}_r(0))$.

Using Lemma 2.1 with k_ℓ in place of k, noting the inclusion $\{|A| > k_\ell\} \supset \{|A| > k_{\ell+1}\}$, and that on the set $\{|A| > k_{\ell+1}\}$ we have $\left(1 - \frac{k_\ell}{|A|}\right)^+ \geq 1 - \frac{k_\ell}{k_{\ell+1}} =$

$\frac{d}{k_{\ell+1}2^\ell} \geq \frac{1}{2^\ell}$ we have

$$\frac{1}{2^\ell}\int_{\{|A|>k_{\ell+1}\}} |\nabla|A||^2\,\eta^2 \leq \frac{1}{c}\int((|A|-k_\ell)^+)^2|\nabla\eta|^2 + \frac{k_\ell}{c}\int((|A|-k_\ell)^+)^3\,\eta^2 + \frac{2k_\ell^2}{c}\int((|A|-k_\ell)^+)^2\,\eta^2 + \frac{k_\ell^3}{c}\int(|A|-k_\ell)^+\,\eta^2.$$

Since

$$|\nabla((|A|-k_{\ell+1})^+\eta)|^2 \leq 2\chi_{\{|A|>k_{\ell+1}\}}|\nabla|A||^2\,\eta^2 + 2((|A|-k_{\ell+1})^+)^2|\nabla\eta|^2,$$

and $(|A|-k_{\ell+1})^+ \leq (|A|-k_\ell)^+$, it follows that

$$\int|\nabla((|A|-k_{\ell+1})^+\eta)|^2 \leq \left(\frac{2^{\ell+1}}{c}+2\right)\int((|A|-k_\ell)^+)^2|\nabla\eta|^2 + \frac{k_\ell 2^{\ell+1}}{c}\int((|A|-k_\ell)^+)^3\,\eta^2 + \frac{2k_\ell^2 2^{\ell+1}}{c}\int((|A|-k_\ell)^+)^2\,\eta^2 + \frac{k_\ell^3 2^{\ell+1}}{c}\int(|A|-k_\ell)^+\,\eta^2. \tag{6}$$

We will use the following Michael–Simon inequality ([13, Theorem 2.1], see also [1]), to bound from below the left-hand-side of (6):

$$\left(\int\left|(|A|-k_{\ell+1})^+\eta\right|^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}} \leq C_{MS}\int|\nabla((|A|-k_{\ell+1})^+\eta)|^2, \tag{7}$$

for a dimensional constant C_{MS} explicitly given by $C_{MS} = \left(\frac{2(n-1)4^{n+1}}{(n-2)\omega_n^{1/n}}\right)^2$, where ω_n is the n-volume of the unit ball in $\mathbb{R}^n$.

We define, for each ℓ, the function η_ℓ to be identically 1 on $B_{R_{\ell+1}}$, with $\mathrm{spt}\eta_\ell = B_{R_\ell}$ and $|\nabla\eta_\ell| \leq \frac{2}{|R_\ell-R_{\ell+1}|} = \frac{2^{\ell+2}}{R}$, and with $0 \leq \eta_\ell \leq 1$. From (6) and (7), making the choice $\eta = \eta_\ell$, we find (using $k_\ell \leq d$ on the right-hand-side of (6))

$$\frac{1}{C_{MS}}\left(\int_{B_{R_{\ell+1}}}((|A|-k_{\ell+1})^+)^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}} \leq \left(\frac{2^{\ell+1}}{c}+2\right)\left(\frac{4^{\ell+2}}{R^2}\right)\int_{B_{R_\ell}}((|A|-k_\ell)^+)^2 + \frac{d2^{\ell+1}}{c}\int_{B_{R_\ell}}((|A|-k_\ell)^+)^3 + \frac{2d^2 2^{\ell+1}}{c}\int_{B_{R_\ell}}((|A|-k_\ell)^+)^2 + \frac{d^3 2^{\ell+1}}{c}\int_{B_{R_\ell}}(|A|-k_\ell)^+. \tag{8}$$

For the right-hand-side of (8) we use Hölder's inequality three times, first with $1 = \frac{n-2}{n}+\frac{2}{n}$,

$$\int_{B_{R_\ell}}((|A|-k_\ell)^+)^2 \leq \left(\int_{B_{R_\ell}}((|A|-k_\ell)^+)^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}}\left(\int\chi_{\{|A|>k_\ell\}\cap B_{R_\ell}}\right)^{\frac{2}{n}},$$

then with $\frac{n-2}{2n} + \frac{n+2}{2n} = 1$,

$$\int_{B_{R_\ell}} (|A| - k_\ell)^+ \leq \left(\int_{B_{R_\ell}} ((|A| - k_\ell)^+)^{\frac{2n}{n-2}} \right)^{\frac{n-2}{2n}} \left(\int \chi_{\{|A|>k_\ell\} \cap B_{R_\ell}} \right)^{\frac{n+2}{2n}},$$

and finally (this is possible for $n \leq 6$) with exponents $\frac{2n}{3(n-2)}$ and $\frac{2n}{6-n}$ (with $1 = \frac{3n-6}{2n} + \frac{6-n}{2n}$), where in the case $n = 6$ the two exponents are just 1 and ∞ (hence for $n = 6$ the second factor on the right–hand-side in the following inequality is just 1)

$$\int_{B_{R_\ell}} ((|A| - k_\ell)^+)^3 \leq \left(\int_{B_{R_\ell}} ((|A| - k_\ell)^+)^{\frac{2n}{n-2}} \right)^{\frac{3(n-2)}{2n}} \left(\int \chi_{\{|A|>k_\ell\} \cap B_{R_\ell}} \right)^{\frac{6-n}{2n}}.$$

We will use the notation

$$S_\ell = \int_{B_{R_\ell}} ((|A| - k_\ell)^+)^{\frac{2n}{n-2}},$$

and aim for a superlinear decay estimate for this quantity as $\ell \to \infty$.

We note that $\mathcal{H}^n\left(\{|A| > k_\ell\} \cap B_{R_\ell}\right)$ can be bounded as follows, for $\ell \geq 2$, using Markov's inequality and the fact that on the set $\{|A| > k_\ell\}$ we have $(|A| - k_{\ell-1})^+ \geq k_\ell - k_{\ell-1} = \frac{d}{2^{\ell-1}}$:

$$\{|A| > k_\ell\} \cap B_{R_\ell} \subset \left\{ ((|A| - k_{\ell-1})^+)^{\frac{2n}{n-2}} \geq \frac{d^{\frac{2n}{n-2}}}{(2^{\frac{2n}{n-2}})^{\ell-1}} \right\} \cap B_{R_\ell} \Rightarrow$$

$$\mathcal{H}^n\left(\{|A| > k_\ell\} \cap B_{R_\ell}\right) \leq \frac{(2^{\frac{2n}{n-2}})^{\ell-1}}{d^{\frac{2n}{n-2}}} \int_{B_{R_\ell}} ((|A| - k_{\ell-1})^+)^{\frac{2n}{n-2}} \leq \frac{(2^{\frac{2n}{n-2}})^{\ell-1}}{d^{\frac{2n}{n-2}}} S_{\ell-1}. \tag{9}$$

The above three instances of Hölder's inequality, together with (9), give

$$\int_{B_{R_\ell}} ((|A| - k_\ell)^+)^2 \leq S_\ell^{\frac{n-2}{n}} \left(\frac{(2^{\frac{2n}{n-2}})^{\ell-1}}{d^{\frac{2n}{n-2}}} S_{\ell-1} \right)^{\frac{2}{n}},$$

$$\int_{B_{R_\ell}} (|A| - k_\ell)^+ \leq S_\ell^{\frac{n-2}{2n}} \left(\frac{(2^{\frac{2n}{n-2}})^{\ell-1}}{d^{\frac{2n}{n-2}}} S_{\ell-1} \right)^{\frac{n+2}{2n}},$$

$$\int_{B_{R_\ell}} ((|A| - k_\ell)^+)^3 \leq S_\ell^{\frac{3(n-2)}{2n}} \left(\frac{(2^{\frac{2n}{n-2}})^{\ell-1}}{d^{\frac{2n}{n-2}}} S_{\ell-1} \right)^{\frac{6-n}{2n}}.$$

With these we bound from above the right-hand-side of (8) and obtain

$$\frac{1}{C_{MS}} S_{\ell+1}^{\frac{n-2}{n}} \leq \left(\left(\frac{2^{\ell+1}}{c} + 2 \right) \left(\frac{4^{\ell+2}}{R^2} \right) + \frac{2d^2 2^{\ell+1}}{c} \right) S_\ell^{\frac{n-2}{n}} \left(\frac{(2^{\frac{4}{n-2}})^{\ell-1}}{d^{\frac{4}{n-2}}} \right) S_{\ell-1}^{\frac{2}{n}}$$

$$+ \frac{d 2^{\ell+1}}{c} S_\ell^{\frac{3n-6}{2n}} \left(\frac{(2^{\frac{6-n}{n-2}})^{\ell-1}}{d^{\frac{6-n}{n-2}}} \right) S_{\ell-1}^{\frac{6-n}{2n}} + \frac{d^3 2^{\ell+1}}{c} S_\ell^{\frac{n-2}{2n}} \left(\frac{(2^{\frac{n+2}{n-2}})^{\ell-1}}{d^{\frac{n+2}{n-2}}} \right) S_{\ell-1}^{\frac{n+2}{2n}},$$

for every $\ell \geq 2$. Noting that $S_\ell \leq S_{\ell-1}$ by definition, and that $2 - \frac{4}{n-2} = 1 - \frac{6-n}{n-2} = 3 - \frac{n+2}{n-2} = \frac{2(n-4)}{n-2}$, the last inequality implies (using $n = \frac{2}{c}$)

$$S_{\ell+1}^{\frac{n-2}{n}} \leq (16nC_{MS})\, C^\ell \left(\frac{1}{R^2 d^{\frac{4}{n-2}}} + d^{\frac{2(n-4)}{n-2}} \right) S_{\ell-1},$$

where C is a dimensional constant that can be explicitly taken (using rough estimates for the constants appearing above) to be $C = 2^{\frac{3n-2}{n-2}}$.

Letting $d = \frac{1}{R}$ we find $S_{\ell+1}^{\frac{n-2}{n}} \leq 32nC_{MS}\, C^\ell \frac{1}{R^{\frac{2(n-4)}{n-2}}} S_{\ell-1}$ and hence arrive at the decay relation ($\ell \geq 2$)

$$S_{\ell+1} \leq (32nC_{MS})^{\frac{n}{n-2}} \frac{\tilde{C}^\ell}{R^{\frac{2n(n-4)}{(n-2)^2}}} S_{\ell-1}^{1+\frac{2}{n-2}},$$

where the dimensional constant $\tilde{C} = C^{\frac{n}{n-2}}$ can be explicitly taken to be $\tilde{C} = 2^{\frac{n(3n-2)}{(n-2)^2}}$. This relation forces $S_\ell \to 0$, as long as S_1 is sufficiently small, by Lemma B.1; precisely, if $S_1 \leq \frac{R^{\frac{n(n-4)}{(n-2)}}}{(32nC_{MS})^{\frac{n}{2}} \tilde{C}^{\frac{n(n-2)}{2}}}$. (We use the relation above with ℓ even; the sequence S_ℓ is decreasing by definition, hence it suffices to prove the convergence to 0 for the subsequence of odd indices. Setting $2j = \ell$ and $T_j = S_{2j-1}$ for $j \geq 1$, we obtain the recursive relation $T_{j+1} \leq \frac{(32nC_{MS})^{\frac{n}{n-2}}}{R^{\frac{2n(n-4)}{(n-2)^2}}} \tilde{C}^{2j} T_j^{1+\frac{2}{n-2}}$ and Lemma B.1 gives $T_j \to 0$ if $T_1 = S_1$ is as specified above.) The smallness assumption on S_1 can be written as follows, for the associated scale-invariant quantity (recall that $R_1 = R$ and $k_1 = 0$ in the definition of S_1):

$$\frac{1}{R^{n-\frac{2n}{n-2}}} \int_{B_R} |A|^{\frac{2n}{n-2}} \leq \frac{1}{(32nC_{MS})^{\frac{n}{2}} \tilde{C}^{\frac{n(n-2)}{2}}} = \frac{1}{(32nC_{MS})^{\frac{n}{2}}\, 2^{\frac{n^2(3n-2)}{2(n-2)}}}. \tag{10}$$

Under the condition (10), the convergence $S_\ell \to 0$ obtained implies that $|A| \leq \frac{1}{R}$ a.e. on $M \cap B_{\frac{R}{2}}$ (by our choice $d = 1$), and thus everywhere (by smoothness of M).

We next check that there exists $\epsilon_0 > 0$ such that (10) is implied the hypotheses of Theorem 3. Lemma 2.1, used with $k = 0$, gives $\int_{B_{2R}} |\nabla(|A|\psi)|^2 \leq \left(\frac{2}{c}+2\right) \int_{B_{2R}} |A|^2 |\nabla\psi|^2$ for $\psi \in C_c^1(B_{2R})$. Combining this with the Michael-Sobolev inequality, which gives $\left(\int_{B_{2R}}(|A|\psi)^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}} \leq C_{MS}\int_{B_{2R}}|\nabla(|A|\psi)|^2$, choosing ψ to be identically 1 in B_R and with $|\nabla\psi| \leq \frac{2}{R}$, we obtain (using $c = \frac{2}{n}$)

$$\int_{B_R}|A|^{\frac{2n}{n-2}} \leq C_{MS}^{\frac{n}{n-2}}\left(\frac{4(n+2)}{R^2}\int_{B_{2R}}|A|^2\right)^{\frac{n}{n-2}}, \tag{11}$$

that is, the following inequality holds between the two relevant scale-invariant quantities:

$$\frac{1}{R^{n-\frac{2n}{n-2}}}\int_{B_R}|A|^{\frac{2n}{n-2}} \leq (4(n+2)\,2^{n-2}C_{MS})^{\frac{n}{n-2}}\left(\frac{1}{(2R)^{n-2}}\int_{B_{2R}}|A|^2\right)^{\frac{n}{n-2}}.$$

We have thus established the first conclusion of Theorem 3, that is, there exists a dimensional $\epsilon_0 > 0$ for which $|A| \leq \frac{1}{R}$ on $M \cap B_{\frac{R}{2}}$. Explicitly, this can be quantified from the above by requiring $\left(\epsilon_0\, 4(n+2)\,2^{n-2}C_{MS}\right)^{\frac{n}{n-2}} \leq \frac{1}{(32nC_{MS})^{\frac{n}{2}}\,2^{\frac{n^2(3n-2)}{2(n-2)}}}$, that is, one can take

$$\epsilon_0 = \frac{1}{C_{MS}^{1+\frac{(n-2)}{2}}\,4(n+2)\,2^{n-2}\,(32n)^{\frac{(n-2)}{2}}\,2^{\frac{n(3n-2)}{2}}}. \tag{12}$$

To see the second assertion of Theorem 3, we exploit the freedom on d. We choose $d = \frac{1}{mR}$ for $m \in [1,\infty)$, and the conclusion $S_\ell \to 0$, i.e. $|A| \leq \frac{1}{mR}$ on $B_{\frac{R}{2}}$, follows if S_1 is suitably small. Indeed, the decay relation becomes (for $\ell \geq 2$ and for an explicit dimensional constant C)

$$S_{\ell+1} \leq \left(m^{\frac{4}{n-2}} + \frac{1}{m^{\frac{2(n-4)}{n-2}}}\right)^{\frac{n}{n-2}}\frac{C^\ell}{R^{\frac{2n(n-4)}{(n-2)^2}}}S_{\ell-1}^{1+\frac{2}{n-2}} \leq 2^{\frac{n}{n-2}}m^{\frac{4n}{(n-2)^2}}\frac{C^\ell}{R^{\frac{2n(n-4)}{(n-2)^2}}}S_{\ell-1}^{1+\frac{2}{n-2}}.$$

Hence if $S_1 \leq \frac{R^{\frac{n(n-4)}{(n-2)}}}{m^{\frac{2n}{(n-2)}}\,\overline{C}}$ (for an explicit dimensional constant $\overline{C}$) then $S_\ell \to 0$, that is, $|A| \leq \frac{1}{mR}$ on $M \cap B_{\frac{R}{2}}$. With the same considerations given for the case $m = 1$ above, we have that the smallness requirement on S_1 is implied by our hypotheses, as long as $\frac{1}{(2R)^2}\int_{B_{2R}}|A|^2$ is sufficiently small. The explicit relations that we have obtained show, in fact, that, for a (explicit) dimensional constant $c(n)$,

$$R\sup_{M\cap B_{\frac{R}{2}}}|A| \leq c(n)\Big(\int_{B_{2R}}|A|^2\Big)^{\frac{1}{2}}.$$

$\square$

Remark 2.4. De Giorgi [6] exploits the Caccioppoli inequality $\int_{\{u>k\}\cap B_\rho^n(p)}|Du|^2 \leq \frac{c(n)}{(r-\rho)^2}\int_{\{u>k\}\cap B_r^n(p)}(u-k)^{+2}$ (for any k, with $\rho < r < R$) to prove his theorem, for a weak solution $u : B^n_R(p) \to \mathbb{R}$ of a linear PDE in divergence form with L^∞ strictly elliptic coefficients. In Lemma 2.1 we have an *intrinsic* Caccioppoli inequality on M, when $k = 0$. The multiplicative factor $\big(1 - \frac{k}{|A|}\big)^+$ appears on the left-hand-side for $k > 0$: as shown, this does not disturb the iterative scheme. Extra terms (that weaken the inequality further, when comparing to the classical case) appear on the right-hand-side when $k > 0$. These terms involve L^p-norms of the truncations up to $p = 3$, with multiplicative factors k^{4-p}, which influence the dependence on d in the decay relation. The smallness requirement in Theorem 3 is due[4] to these extra terms (which also force the dimensional bound $n \leq 6$).

While in [6] the classical Caccioppoli inequality for $(u-k)^+$ follows from the linearity of the PDE, in our case stability provides sufficient control on the non-linearity of (1), leading to the weak intrinsic Caccioppoli inequality. (In the absence of stability Theorem 3 fails, as pointed out in Remark 2.3.)

Remark 2.5 ($n = 2$)**.** We proved Theorem 3 for $n \neq 2$. The proof adapts to treat the case $n = 2$ by using the Michael–Simon inequality to embed L^3 into $W^{1,\frac{6}{5}}$ and the Hölder inequality to bound $\int |\nabla\big((|A| - k_\ell)^+\eta_\ell\big)|^{\frac{6}{5}}$ by means of the product $\Big(\int |\nabla\big((|A| - k_\ell)^+\eta_\ell\big)|^2\Big)^{\frac{3}{5}} \mathcal{H}^2\Big(M \cap B_{R_\ell}\Big)^{\frac{2}{5}}$. We do not carry out the iteration explicitly, also in view of the fact that (as mentioned in the introduction) curvature estimates for $n = 2$ admit a simple treatment.

3 Proof of Theorems 1 and 2

The proof of Theorem 2 is reduced to the analysis of scenarios in which Theorem 3 is applicable. The first scenario handles the case in which M (properly immersed smooth two-sided and stable) is weakly close (i.e. as a varifold) to a hyperplane with multiplicity; the second one handles the case in which M is weakly close to a classical cone; the third one handles the case in which M is weakly close to a (finite) union of hyperplanes with multiplicity. In all scenarios the conclusion is that M must be a smooth perturbation (as an immersion) of the cone in question (which is a posteriori always a union of hyperplanes with multiplicity).

3.1 Closeness to a hyperplane

Let M be a properly immersed two-sided stable minimal hypersurface that is weakly close to a hyperplane with multiplicity (as varifolds). Upon rotating coordinates, we assume that the hyperplane is $\{x_{n+1} = 0\}$. We recall the following standard inequality (implied by minimality, via the first variation formula, with a suitable choice of test function, see [19, Section 22] and Remark 4.2 below)

$$\Big(\frac{1}{3R}\Big)^n \int_{B^{n+1}_{3R} \cap M} |\nabla x_{n+1}|^2 \leq C(n) \Big(\frac{2}{7R}\Big)^{n+2} \int_{B^{n+1}_{\frac{7R}{2}} \cap M} |x_{n+1}|^2,$$

[4] As observed explicitly in Appendix A, smallness is only needed for $n \in \{4, 5, 6\}$.

and $|\nabla x_{n+1}| = |\mathrm{proj}_{TM}(e_{n+1})| = \sqrt{1-(\nu\cdot e_{n+1})^2}$. This says that the L^2 height-excess controls the L^2 tilt-excess linearly (both excesses defined in a scale-invariant fashion). Moreover, Schoen's inequality ([15], [16, Lemma 1], see also (16) below taken with $k=0$) gives

$$\frac{1}{(2R)^{n-2}}\int_{B^{n+1}_{2R}\cap M}|A|^2 \leq 2n\Big(\frac{1}{3R}\Big)^n\int_{B^{n+1}_{3R}\cap M}|\nabla x_{n+1}|^2,$$

where the scale-invariant tilt excess appears on the right-hand-side. Here and below, domains of the type $D\cap M$ for D open, are implicitly understood to be the inverse image of D via the immersion that gives M (we will use this with M belonging to a sequence of immersed hypersurfaces).

When $\hat{M}_j$ converge (as varifolds for $j\to\infty$) in B^{n+1}_{4R} to $\{x_{n+1}=0\}$ with multiplicity, by the monotonicity formula we have that $\sup_{\hat{M}_j\cap B^{n+1}_{\frac{7R}{2}}}|x_{n+1}|\to 0$, so we conclude that for all sufficiently large j the quantity $\frac{1}{(2R)^{n-2}}\int_{B^{n+1}_{2R}\cap\hat{M}_j}|A_{\hat{M}_j}|^2$ is at most ϵ_0 and Theorem 3 applies. As immediate consequence, we have:

Lemma 3.1. *Let $\hat{M}_j$ be a sequence of smooth properly immersed two-sided stable minimal hypersurfaces in $B^{n+1}_{4R}(0)$, $n\leq 6$, such that $\hat{M}_j$ converge (as varifolds) to $q|P|$ as $j\to\infty$, where P is a hyperplane and $q\in\mathbb{N}$. Then $\sup_{B^{n+1}_{\frac{R}{2}}(0)\cap\hat{M}_j}|A_{\hat{M}_j}|\to 0$ as $j\to\infty$, and $\hat{M}_j$ converge smoothly to P in $B^{n+1}_{\frac{R}{2}}(0)$ (as immersions, with the limit being an immersion with q connected components, each covering P once).*

3.2 Closeness to a classical cone

Let C be a classical cone, i.e. a sum of half-hyperplanes all intersecting at a given $(n-1)$-dimensional subspace, $C=\sum_{i=1}^N q_i|H_i|$ with $q_i\in\mathbb{N}$, H_i a half-hyperplane whose boundary is the given $(n-1)$-dimensional subspace. Without loss of generality we assume that the $(n-1)$-dimensional subspace is the span of $\{\mathbf{e}_3,\ldots,\mathbf{e}_{n+1}\}$. For $\tau>0$, let C_τ denote the cylinder $\{x_1^2+x_2^2<\tau^2\}$.

Lemma 3.2. *Let $\hat{M}_j$ be a sequence of smooth properly immersed two-sided stable minimal hypersurfaces in B_{4R}. Assume that $\hat{M}_j$ converge (as varifolds in B_{4R}) to C as $j\to\infty$, where C is a classical cone as above. Then $\int_{B_{2R}\cap\hat{M}_j}|A_{\hat{M}_j}|^2\to 0$ as $j\to\infty$.*

Proof. By scale-invariance we may take $4R=2$. By Lemma 3.1, for $\tau>0$ we have that $\hat{M}_j$ converge strongly to C in $B_2\setminus C_{\frac{\tau}{2}}$. In particular, given $\tau>0$, we have $\int_{(B_1\setminus C_\tau)\cap\hat{M}_j}|A_{\hat{M}_j}|^2\to 0$ as $j\to\infty$. Further,

$$\int_{\hat{M}_j\cap C_\tau\cap B_1}|A_{\hat{M}_j}|^2 \leq \left(\int_{\hat{M}_j\cap C_\tau\cap B_1}|A_{\hat{M}_j}|^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}}\left(\int_{\hat{M}_j\cap C_\tau\cap B_1}1\right)^{\frac{2}{n}}$$
$$\leq \left(\int_{\hat{M}_j\cap B_1}|A_{\hat{M}_j}|^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}}\mathcal{H}^n(\hat{M}_j\cap C_\tau\cap B_1)^{\frac{2}{n}}.$$

Lemma 2.1 taken with $k=0$ implies (as argued for (11), for a dimensional $K(n)$)

$$\int_{\hat{M}_j\cap B_1}|A_{\hat{M}_j}|^{\frac{2n}{n-2}}\leq K(n)\left(\int_{\hat{M}_j\cap B_{\frac{3}{2}}}|A_{\hat{M}_j}|^2\right)^{\frac{n}{n-2}}.$$

Letting η be a fixed function that is equal to 1 in $B_{\frac{3}{2}}$, is supported in B_2, and with $|\nabla\eta|\leq 4$, we find $\int_{\hat{M}_j\cap B_{\frac{3}{2}}}|A_{\hat{M}_j}|^2\leq\int_{\hat{M}_j\cap B_2}|\nabla\eta|^2\leq 16\Lambda(C)$, for all sufficiently large ℓ (we used the stability inequality), where we let $\Lambda(C)=\|C\|(B_2)+1$. Therefore, $\left(\int_{B_1}|A_{\hat{M}_j}|^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}}$ is uniformly bounded for all sufficiently large ℓ. Next, observe that $C_\tau\cap B_1$ is contained in the union of $\frac{b(n)}{\tau^{n-1}}$ balls of radius τ, where $b(n)$ is a dimensional constant (a rough cover shows that $b(n)=n^{\frac{n-1}{n}}$ works), and in each such ball the n-area of $\hat{M}_j$ is at most $k(n)\Lambda(C)\tau^n$, by the monotonicity formula for the area ratio, for a dimensional constant $k(n)$. Hence $\mathcal{H}^n(\hat{M}_j\cap C_\tau\cap B_1)^{\frac{2}{n}}\leq\Big(\Lambda(C)b(n)k(n)\Big)^{\frac{2}{n}}\tau^{\frac{2}{n}}$ and

$$\int_{B_1\cap\hat{M}_j}|A_{\hat{M}_j}|^2\leq\left(\int_{(B_1\setminus C_\tau)\cap\hat{M}_j}|A_{\hat{M}_j}|^2\right)+\Lambda(C)b(n)k(n)\tau^{\frac{2}{n}}.$$

As $\tau>0$ can be chosen arbitrarily small, $\int_{B_1\cap\hat{M}_j}|A_{\hat{M}_j}|^2\to 0$ as $j\to\infty$. □

This shows that, for all sufficiently large j, the quantity $\int_{B^{n+1}_{2R}\cap\hat{M}_j}|A_{\hat{M}_j}|^2$ is at most ϵ_0, so Theorem 3 applies. As immediate consequence, we have:

Lemma 3.3. *Let $\hat{M}_j$ be a sequence of smooth properly immersed two-sided stable minimal hypersurfaces that converge (as varifolds) in $B^{n+1}_{4R}(0)$ to a classical cone C, as $j\to\infty$, with $n\leq 6$. Then C is a sum of hyperplanes with multiplicity, which we describe as a smooth immersion; moreover, $\sup_{B^{n+1}_{\frac{R}{2}}(0)\cap\hat{M}_j}|A_{\hat{M}_j}|\to 0$ as $j\to\infty$, and $\hat{M}_j$ converge smoothly to C in $B^{n+1}_{\frac{R}{2}}(0)$ as immersions (with q connected components, with $q=\Theta(\|C\|,0)$).*

3.3 Closeness to a union of hyperplanes

With arguments analogous to those in Section 3.2, we will establish:

Lemma 3.4. *Let $n\leq 6$ and let $\hat{M}_j$ be a sequence of smooth properly immersed two-sided stable minimal hypersurfaces that converge (as varifolds) in $B^{n+1}_{4R}(0)$, as $j\to\infty$, to a cone C given by a (finite) union of hyperplanes passing through the origin, each taken with an integer multiplicity. Then $\sup_{B^{n+1}_{\frac{R}{2}}(0)\cap\hat{M}_j}|A_{\hat{M}_j}|\to 0$ as $j\to\infty$, and (describing C as a smooth immersion) $\hat{M}_j$ converge smoothly to C in $B^{n+1}_{\frac{R}{2}}(0)$ as immersions (with q connected components, with $q=\Theta(\|C\|,0)$).*

Proof. Let $C = \sum_{j=1}^{N} q_j |H_j|$, where $q_j \in \mathbb{N}$ and H_j is a hyperplane (through the origin), $q = \sum_{j=1}^{N} q_j$. For such a cone, we let $S(C) = \cap_{j=1}^{N} H_j$. Note that if $\dim(S(C)) = n$ then C is a single hyperplane (possibly with multiplicity), while if $\dim(S(C)) = n-1$ then C is a classical cone. Therefore in these two cases, Lemmas 3.1 and 3.3 give the conclusion. We prove the general statement by (finite) induction on the dimension. Let $\dim(S(C)) = D \in \{0, 1, \ldots n-2\}$ and assume (inductively) that the conclusion has been established for any cone $\tilde{C}$ (given by a union of hyperplanes with multiplicity) for which $\dim(S(\tilde{C})) \in \{D+1, \ldots, n\}$.

Let $p \in \mathrm{spt}\|C\| \setminus \mathrm{S}(C)$. Upon relabeling, we let H_j for $j \in \{1, \ldots, N_1\}$ be the hyperplanes that pass through p, and H_j for $j \in \{N_1+1, \ldots N\}$ be those that do not. Since $p \in \mathrm{spt}\|C\| \setminus \mathrm{S}(C)$, necessarily $N_1 < N$.

Let η_p be the translation $\eta_p(x) = x - p$ in $\mathbb{R}^{n+1}$. Let $C_1 = \eta_{p\sharp}(\sum_{j=1}^{N_1} q_j |H_j|) = \sum_{j=1}^{N_1} q_j \eta_{p\sharp} |H_j|$. Since the line through 0 and p is contained in H_j for all $j \in \{1, \ldots, N_1\}$, we have $C_1 = \sum_{j=1}^{N_1} q_j |H_j|$. This gives immediately $S(C_1) \supset S(C)$. This inclusion is strict: in fact, $p \in S(C_1)$, while $p \notin S(C)$. In particular, $\dim(S(C_1)) > \dim(S(C))$.

For any $p \in \mathrm{spt}\|C\| \setminus \mathrm{S}(C)$ we consider a ball $B_r^{n+1}(p)$ sufficiently small to ensure that, for each $j \in \{1, \ldots, N\}$, we have $H_j \cap B_r^{n+1}(p) \neq \emptyset \Rightarrow p \in H_j$. Then the inductive hypothesis can be applied to the sequence $\eta_p(\hat{M}_j \cap B_r^{n+1}(p))$, since it converges in $B_r^{n+1}(0)$ to a cone C_1, given by a union of hyperplanes with multiplicity, and with $\dim(S(C_1)) > \dim(S(C))$ (in view of the argument just given). This yields $\sup_{B^{n+1}_{\frac{r}{8}}(p) \cap \hat{M}_j} |A_{\hat{M}_j}| \to 0$ as $j \to \infty$.

Finally, letting $C_{\tau R} = \{x \in \mathbb{R}^{n+1} : \mathrm{dist}_{\mathbb{R}^{n+1}}\big(x, S(C) \cap B_{2R}^{n+1}(0)\big) < \tau R\}$ for $\tau > 0$ given, we note that $C_{\tau R}$ can be covered by $\frac{b(D)}{\tau^D}$ balls of radius τR, where $b(D)$ is a constant that depends on D only. Then we argue as in Lemma 3.2. By Hölder's inequality

$$\frac{1}{R^{n-2}} \int_{\hat{M}_j \cap C_{\tau R} \cap B_{2R}^{n+1}} |A_{\hat{M}_j}|^2$$

$$\leq \left(\frac{1}{R^{n-\frac{2n}{n-2}}} \int_{\hat{M}_j \cap B_{2R}^{n+1}} |A_{\hat{M}_j}|^{\frac{2n}{n-2}} \right)^{\frac{n-2}{n}} \left(\frac{\mathcal{H}^n(\hat{M}_j \cap C_{\tau R} \cap B_{2R}^{n+1})}{R^n} \right)^{\frac{2}{n}}.$$

Using (11) and stability

$$\frac{1}{R^{n-\frac{2n}{n-2}}} \int_{\hat{M}_j \cap B_{2R}^{n+1}} |A_{\hat{M}_j}|^{\frac{2n}{n-2}} \leq K(n) \left(\frac{1}{R^{n-2}} \int_{\hat{M}_j \cap B_{3R}^{n+1}} |A_{\hat{M}_j}|^2 \right)^{\frac{n}{n-2}} \leq \Lambda(C, n),$$

for a constant $\Lambda(C, n)$ determined by n and by the cone C. Moreover, by the monotonicity formula, with $\Lambda(C)$ denoting a constant determined by the cone, for all sufficiently large j we have

$$\mathcal{H}^n(\hat{M}_j \cap C_{\tau R} \cap B_{2R}^{n+1}) \leq b(D) \Lambda(C) \tau^{n-D} R^n.$$

Hence

$$\frac{1}{R^{n-2}}\int_{\hat{M}_j\cap C_{\tau R}\cap B^{n+1}_{2R}}|A_{\hat{M}_j}|^2\leq c(n,D,C)\tau^{\frac{2(n-D)}{n}},$$

which, for τ sufficiently small, and together with the conclusion obtained above around any $p\in \operatorname{spt}\|C\|\setminus \mathrm{S}(C)$, implies that $\frac{1}{R^{n-2}}\int_{\hat{M}_j\cap B^{n+1}_{2R}(0)}|A_{\hat{M}_j}|^2\leq \epsilon_0$ for all sufficiently large j. Then for such j Theorem 3 applies and the lemma follows. □

3.4 Tangent cone analysis and conclusion

Lemma 3.5. *Let M_j be a sequence of smooth properly immersed two-sided stable minimal hypersurfaces in an open set $U\subset\mathbb{R}^{n+1}$, with $n\leq 6$, converging (as varifolds) to a (stationary integral) varifold V. Let $x\in U$ be such that (at least) one tangent cone to V at x is either a (finite) union of hyperplanes with multiplicity (possibly a single hyperplane), or a classical cone. There exists $\rho>0$ such that V is the varifold associated to a smooth immersion in $B^{n+1}_\rho(x)$, and M_j converge smoothly (as immersions) to V in $B^{n+1}_\rho(x)$.*

Proof. It suffices to find $\rho>0$ such that $\limsup_{j\to\infty}\sup_{M_j\cap B^{n+1}_{2\rho}(x)}|A_{M_j}|<\infty$ and $2\rho<\operatorname{dist}(x,\partial U)$ (after which, standard compactness under L^∞ curvature bounds gives the result). Arguing by contradiction, we assume that this fails for every such ρ, hence there exist a subsequence (not relabeled) M_j and associated points $x_j\in M_j$ with $|A_{M_j}(x_j)|\to\infty$ and $x_j\to x$. Upon passing to a further subsequence (determined by the chosen blow up of V at x), we find, for each j, rescalings $\tilde{M}_j$ of M_j around x that converge to the chosen tangent to V at x (a hyperplane with multiplicity, a classical cone, or a union of hyperplanes with multiplicity) and such that $|A_{\tilde{M}_j}(\tilde{x}_j)|\to\infty$, where $\tilde{x}_j\in\tilde{M}_j$ is the image of x_j via the dilation associated to j. For each type of cone (applying respectively Lemma 3.1, Lemma 3.3, or Lemma 3.4) we reach a contradiction. □

The argument that we recall next is a classical procedure (see e.g. [16, Section 6], [22, Section 8]) that involves tangent cone analysis (in the sense of varifolds), Federer's dimension reducing principle (see e.g. [19, Appendix A]), and Simons' classification of stable cones ([20], see also [19, Appendix B] and [17, Section 3]), together with Lemma 3.5 itself, to show that:

Proposition 3.1. *Let U, M_j, V be as in the hypotheses of Lemma 3.5. Then any tangent cone to V is a (finite) union of hyperplanes, each with an integer multiplicity. In particular, the conclusion of Lemma 3.5 applies at every $x\in$ spt$\|V\|$.*

It turns out that it is natural to prove the stronger result that any iterated tangent to V is a finite union of hyperplanes (each with an integer multiplicity). We first recall the relevant notions and facts.

If $M_\ell\to V$ and C_x is a tangent cone to V at x, there exist $r_j\to 0$ and a subsequence $\ell(j)$ such that $\hat{M}_{\ell(j)}=\lambda_{\left(x,\frac{1}{r_j}\right)}M_{\ell(j)}$ converge (as varifolds) to C_x, where

$\lambda_{\left(x,\frac{1}{r_j}\right)}$ is the dilation of factor $\frac{1}{r_j}$ centred at x, combined with the translation that sends x to 0, that is $\lambda_{\left(x,\frac{1}{r_j}\right)}(z) = \frac{z-x}{r_j}$.

The spine $S(C)$ of a cone C is the maximal subspace of translation invariance (and coincides with the set of points of maximal density). (It is immediate that when the cone is a union of hyperplanes, the case addressed in Lemma 3.4, the spine is the intersection of these hyperplanes, justifying the notation used there.) We further recall the notion of iterated tangents (to V at x), by which we mean the collection of cones C for which there exist cones $C_1, \ldots, C_N$, with $C_N = C$ and $N \in \mathbb{N}$, $N \geq 1$, and points $p_1 \in C_1 \setminus S(C_1), \ldots p_N \in C_N \setminus S(C_N)$ such that C_m is a tangent cone to C_{m-1} at p_{m-1} for $m \geq 2$ and $C_1 = C_x$ is a tangent cone to V at x. For every C in the space of iterated tangents to V at x, we can find $r_j \to 0$ a subsequence $\ell(j)$ and points $z_{\ell(j)} \to x$ (not necessarily lying on $M_{\ell(j)}$) such that $\tilde{M}_{\ell(j)} = \lambda_{\left(z_{\ell(j)},\frac{1}{r_j}\right)} M_{\ell(j)}$ converge (as varifolds) to C, where $\lambda_{\left(z_{\ell(j)},\frac{1}{r_j}\right)}$ is the dilation of factor $\frac{1}{r_j}$ centred at $z_{\ell(j)}$, combined with the translation that sends $z_{\ell(j)}$ to 0, that is $\lambda_{\left(z_{\ell(j)},\frac{1}{r_j}\right)}(z) = \frac{z-z_{\ell(j)}}{r_j}$. (In the case $N = 1$ one can take $z_{\ell(j)} = x$, as seen above.)

Proof of Proposition 3.1. We first establish that *for any iterated tangent C the smoothly immersed part of C is stable.* Indeed, in a sufficiently small ambient ball $B^{n+1}_{\rho_y}(y)$, centred at any given $y \in C$ around which C is smoothly immersed, the (dilated) hypersurfaces $\tilde{M}_{\ell(j)}$ converge smoothly (as immersions) to C. This is a consequence of Lemma 3.5, as the (unique, in this case) tangent to C at y is of the type prescribed there. The arbitrariness of y leads to smooth convergence of $\tilde{M}_{\ell(j)}$ to C on the smoothly immersed part of C, and thus the stability condition is inherited by the smoothly immersed part of C.

By Simons' classification, any cone C of dimension $n \in \{2, \ldots, 6\}$ (with vertex at the origin) in $\mathbb{R}^{n+1}$, smoothly immersed away from the origin, which is stable on the smoothly immersed part, must be a union of hyperplanes. From this we will deduce that *the only iterated tangent cones that are smoothly immersed away from the spine are (finite) unions of hyperplanes with multiplicity and classical cones.* Indeed, if the spine dimension is n or $n-1$ then the cone is respectively a single hyperplane with multiplicity or a classical cone. So we assume that $C \setminus S(C)$ is smoothly immersed and stable, with $\dim(S(C)) \leq n-2$. We slice C with affine planes of dimension complementary to the spine $S(C)$, and orthogonal to it. Any such slice is a cone, of dimension at least 2, since $\dim(S(C)) \leq (n-2)$, and at most n, since it is a slice of the n-dimensional cone C. This slice (in $\mathbb{R}^{n+1-\dim S(C)}$) is a smoothly immersed cone except possibly for an isolated singularity at the vertex; moreover, its regular part inherits stability. By Simons' result the slice is a union of hyperplanes in $\mathbb{R}^{n+1-\dim S(C)}$. Then also C is a union of hyperplanes.

A key fact, underlying Federer's dimension reducing principle, is that the spine dimension strictly increases when we take iterated tangents, $\dim(S(C_1)) < \ldots <$

$\dim(S(C_N))$ with the above notation. (This is due to the fact that, choosing a point y away from the spine $S(C)$, the linear subspace spanned by y and $S(C)$ becomes translation invariant for the tangent to C at y.) This is used in the following way, to prove that *any iterated tangent must be smoothly immersed away from its spine.*

Assume that a given cone C in the collection of iterated tangents is not smoothly immersed away from its spine $S(C)$, whose dimension we denote by s. As C is neither a hyperplane with multiplicity nor a classical cone, we must have $s \in \{0, \ldots, n-2\}$. We consider a tangent cone to C at a non-immersed point in $C \setminus S(C)$, and iterate this step until we find a cone $\hat{C}$ that is smoothly immersed away from its spine. This is achieved after at most $n-s-1$ iterations (thanks to the strict increase in spine dimension, after $n-s-1$ iterations we must have a classical cone or a hyperplane with multiplicity). We let $\tilde{C}$ be the iterated tangent cone for which $\hat{C}$ is a tangent cone at a non-immersed point $y \in \tilde{C} \setminus S(\tilde{C})$.

As shown above, $\hat{C}$ is either a (finite) union of hyperplanes with integer multiplicity or a classical cone. Lemma 3.5 applies to the sequence $\widetilde{M}_{\ell(j)}$ that converges to $\tilde{C}$ in a suitably small ball, contradicting that y is a non-immersed point. This concludes the proof of Proposition 3.1. □

Remark 3.1. The conclusion of Proposition 3.1 also says that any tangent to V that is a classical cone is in fact a union of hyperplanes (this also follows from Lemma 3.5). We remark that this could be established a priori, employing the framework of integral curvature varifolds (to obtain that if the limit, as varifolds, of a sequence of smooth stable minimal immersions is a classical cone, then it is a union of hyperplanes). If we did that, there would be no need to treat classical cones separately, and Lemma 3.3 could be subsumed under Lemma 3.4.

Theorem 2 follows from Proposition 3.1 by a contradiction argument. Using standard compactness arguments (which require the given mass bounds), we assume the existence of a sequence M_ℓ of hypersurfaces in $B_4^{n+1}(0)$ (that satisfy the same assumptions as M in the theorem) and, arguing by contradiction, assume that there exists $x_\ell \in M_\ell \cap B_{\frac{1}{2}}$ with $\limsup_{\ell \to \infty} |A_{M_\ell}(x_\ell)| = \infty$. Allard's compactness gives a (subsequential) stationary limit V for M_ℓ (without relabelling the subsequence), in the sense of varifolds. By extracting a further subsequence (without relabelling) we also assume $x_\ell \to x \in \mathrm{spt}\|V\|$. Proposition 3.1 (and Lemma 3.5) applied at x contradicts $\limsup_{\ell \to \infty} |A_{M_\ell}(x_\ell)| = \infty$.

Theorem 1 follows by considering $M \cap B_{4R}^{n+1}(p)$ for any chosen $p \in M$, and translating (sending p to 0). As $R \to \infty$, the estimate in Theorem 2 remains valid with the same β. This forces $A_M(p) = 0$. Hence $A \equiv 0$ on M and the result follows.

Part II

Towards a compactness theory for branched stable minimal immersions

We are interested, in this second part, in a wider class of immersed hypersurfaces M: we allow a singular set Sing_M with locally finite $\mathcal{H}^{n-2}$-measure, or, more generally, vanishing 2-capacity. Explicitly, for $U \subset \mathbb{R}^{n+1}$ open, and $\Sigma \subset U$ closed (in U) with $\mathrm{cap}_2(\Sigma) = 0$ (in particular[5], we allow Σ to have locally finite $\mathcal{H}^{n-2}$-measure, that is, $\mathcal{H}^{n-2}(\Sigma \cap K) < \infty$ for every $K \subset\subset U$), we let $\iota : S \to U \setminus \Sigma$ be a (smooth) proper immersion, that we assume to be two-sided minimal and stable, with continuous unit normal ν. Denoting by $\overline{M}$ the closure of $M = \iota(S)$ in U, we say that $x \in \mathrm{Sing}_M$ if, for every $r > 0$, $B^{n+1}_r(x) \cap \overline{M}$ is not the image of a smooth immersion. (In other words, a point in Σ is genuinely singular if M cannot be smoothly extended across it, as an immersion.)

As proved in [16, (1.18) and Section 5], the stationarity condition (with respect to the area functional) is valid for ambient deformations in U, that is, the integral varifold $|\iota_\sharp S|$ is stationary in U (not only in $U \setminus \Sigma$). This follows from a suitable extension of the monotonicity formula, obtained at points in Σ, giving Euclidean area growth around all points in $\overline{M} \cap U$, combined with a standard capacity argument. (In fact, [16] shows that $\mathcal{H}^{n-1}(\Sigma) = 0$ would be sufficient for this.)

4 Proof of Theorem 4

The tilt function and the relevant PDE. For a given fixed unit vector, that we assume without loss of generality to be the last coordinate vector e_{n+1}, consider the function $g = \left(1 - (\nu \cdot e_{n+1})^2\right)^{1/2}$, well-defined on S. Clearly, $0 \leq g \leq 1$. Letting ∇ denote the metric gradient on S, it is immediate that $|\nabla g| \leq \sqrt{1-g^2}|A|$. This follows by direct computation, since

$$|\nabla(\nu \cdot e_{n+1})| = |(D\nu)(e^T_{n+1})| \leq |A||e_{n+1} - (\nu \cdot e_{n+1})\nu| = |A|g,$$

where e^T_{n+1} denotes the tangential part of e_{n+1} and $D\nu$ is the shape operator, and

$$\nabla g = \frac{-(\nu \cdot e_{n+1})\nabla(\nu \cdot e_{n+1})}{\sqrt{1-(\nu \cdot e_{n+1})^2}}, \text{ that is, } g^2|\nabla g|^2 = (1-g^2)|\nabla(\nu \cdot e_{n+1})|^2. \quad (13)$$

We recall the standard Jacobi field equation $\Delta(\nu \cdot e_{n+1}) = -|A|^2(\nu \cdot e_{n+1})$, or, equivalently,

$$-\Delta(\nu \cdot e_{n+1})^2 = -2|\nabla(\nu \cdot e_{n+1})|^2 + 2|A|^2(\nu \cdot e_{n+1})^2,$$

[5]We refer to [8] for details on capacity. In our context, the implication $\mathcal{H}^{n-2}(\Sigma) < \infty \Rightarrow \mathrm{cap}_2(\Sigma) = 0$ is implicitly proved in [16] when $\mathcal{H}^{n-2}(\Sigma) = 0$ and refined in [22] for the case $\mathcal{H}^{n-2}(\Sigma) < \infty$ using a Federer-Ziemer argument.

where Δ is the Laplace-Beltrami operator on S. This implies a PDE for g on S (using the relation (13)):

$$(1-g^2)(2g\Delta g + 2|\nabla g|^2) = (1-g^2)\big(\Delta g^2\big) = -2g^2|\nabla g|^2 + 2|A|^2(1-g^2)^2,$$

and therefore (in view of $|\nabla g|^2 \le (1-g^2)|A|^2$ the following is well-defined)

$$g\Delta g = -\frac{|\nabla g|^2}{1-g^2} + |A|^2(1-g^2). \tag{14}$$

We recall that the following improved inequality (see [16, (2.7)]) is implied by the minimality condition:

$$\frac{|\nabla(\nu\cdot e_{n+1})|^2}{(1-(\nu\cdot e_{n+1})^2)} \le \left(1-\frac{1}{n}\right)|A|^2 \Leftrightarrow \frac{|\nabla g|^2}{1-g^2} \le \left(1-\frac{1}{n}\right)|A|^2. \tag{15}$$

Remark 4.1. The quantity $E_M(R)^2 = \frac{1}{R^n}\int_{C_R} g^2$ (appearing in Theorem 4) is the square of the scale-invariant tilt-excess of $|M|$ in $C_R = B^n_R(0)\times(-R,R)$, with respect to the hyperplane $\mathbb{R}^n\times\{0\}$ (orthogonal to e_{n+1}). As in Part I, with slight notational abuse we will write domains D, or $M\cap D$, to mean $\iota^{-1}(D)$, where $\iota : S \to C_{2R}$ is the immersion with image M.

Remark 4.2 (*height and tilt excess*)**.** We recall that the (scale-invariant) L^2 height-excess $\hat{E}_M(r)$, defined by $\hat{E}_M(r)^2 = \frac{1}{r^{n+2}}\int_{M\cap C_r}|x_{n+1}|^2$, bounds (linearly) $E_M(\frac{r}{2})^2$. Indeed, stationarity implies, using the first variation formula with a vector field $x_{n+1}\varphi^2 e_{n+1}$, for $\varphi\in C^1_c(C_{2R})$ taken to be identically 1 in C_R and with $|\nabla\varphi|\le\frac{1}{R}$, the validity of the inequality (see e.g. [19, Section 22])

$$\frac{1}{R^n}\int_{M\cap C_R}|\nabla x_{n+1}|^2 \le \frac{2^{n+4}}{(2R)^{n+2}}\int_{M\cap C_{2R}}|x_{n+1}|^2,$$

and $|\nabla x_{n+1}| = |\mathrm{proj}_{TM}(e_{n+1})| = \sqrt{1-(\nu\cdot e_{n+1})^2} = g$.

The proof of Theorem 4 will be carried out by means of an iteration à la De Giorgi, for which the fundamental lemma is an intrinsic weak Caccioppoli inequality, for level set truncations of g (Lemmas 4.1 and 4.2 below).

Lemma 4.1. *Let M be as above. Then for any $k\in[0,\frac{1}{2n}]$ and $\phi\in C^{0,1}_c(S)$ we have*

$$\frac{1}{2n}\int_{\{g>k\}}|\nabla g|^2\left(1-\frac{k}{g}\right)\phi^2 \le \int (g-k)^{+2}|\nabla\phi|^2,$$

where $(g-k)^+$ denotes the function $(g-k)^+ = \begin{cases} g-k & \textit{when } g>k \\ 0 & \textit{when } g\le k\end{cases}$.

Proof. We use the stability condition, whose analytic form is the validity of

$$\int|A|^2\eta^2 \le \int|\nabla\eta|^2$$

for all $\eta \in C^1_c(S)$. A standard approximation argument implies that $\eta \in C^{0,1}_c(S)$ is also allowed and we choose $\eta = (g-k)^+\phi$, where $\phi \in C^{0,1}_c(S)$ (as in the statement). We compute (on the right-hand-side of the stability inequality)

$$\int |\nabla((g-k)^+\phi)|^2 =$$

$$\int |\nabla(g-k)^+|^2\phi^2 + \underbrace{2\int (g-k)^+\,\phi\,\nabla(g-k)^+\,\nabla\phi}_{\frac{1}{2}\int \nabla((g-k)^{+2})\,\nabla(\phi^2)} + \int (g-k)^{+2}|\nabla\phi|^2.$$

We note that the function $(g-k)^{+2}$ is in $C^1 \cap W^{2,\infty}(S)$. Indeed, $\nabla((g-k)^{+2}) = 2(g-k)^+\nabla g$ and this function is locally Lipschitz. In particular, we have that $\Delta((g-k)^{+2})$ is the L^∞ function that vanishes in the complement of $\{g \geq k\}$ and is equal to $2(g-k)^+\Delta(g-k)^+ + 2|\nabla((g-k)^+|^2$ on $\{g < k\}$. Hence we can integrate by parts and the braced term becomes

$$-\frac{1}{2}\int \Delta((g-k)^{+2})\,\phi^2 = -\int |\nabla(g-k)^+|^2\,\phi^2 - \int_{\{g>k\}} (g-k)^+\Delta(g-k)^+\,\phi^2.$$

The right-hand-side of the stability inequality is therefore

$$-\int_{\{g>k\}} (g-k)\Delta g\,\phi^2 + \int (g-k)^{+2}|\nabla\phi|^2 \overset{\text{by (14)}}{=}$$

$$\int_{\{g>k\}}\left(1-\frac{k}{g}\right)\frac{|\nabla g|^2}{1-g^2}\,\phi^2 - \int_{\{g>k\}}\left(1-\frac{k}{g}\right)|A|^2(1-g^2)\,\phi^2 + \int (g-k)^{+2}|\nabla\phi|^2.$$

(When $k=0$ we do not need to multiply the PDE (14) by $\frac{g-k}{g} = 1-\frac{k}{g}$.) We now use the improved inequality (15) (for the first integrand in the last expression) and find, from the stability inequality,

$$\int_{\{g>k\}} |A|^2(g-k)^2\phi^2 \leq \int_{\{g>k\}}\left(1-\frac{1}{n}\right)\left(1-\frac{k}{g}\right)|A|^2\,\phi^2$$
$$-\int_{\{g>k\}}\left(1-\frac{k}{g}\right)|A|^2(1-g^2)\,\phi^2 + \int (g-k)^{+2}|\nabla\phi|^2.$$

Moving all terms containing $|A|^2$ to the left-hand-side we compute

$$(g-k)^2 - \left(1-\frac{1}{n}\right)\left(1-\frac{k}{g}\right) + \left(1-\frac{k}{g}\right)(1-g^2) =$$

$$(g-k)\left(g-k-\frac{1}{g}\Big(1-\frac{1}{n}-1+g^2\Big)\right) = \frac{g-k}{g}\left(g^2-kg+\frac{1}{n}-g^2\right),$$

which gives

$$\int_{\{g>k\}} |A|^2 \frac{(g-k)}{g} \left(\frac{1}{n} - kg\right) \phi^2 \le \int (g-k)^{+2} |\nabla \phi|^2.$$

As $g \in [0,1]$, if we restrict $k \in [0, \frac{1}{2n}]$ as in the hypotheses we get $\frac{1}{n} - kg \ge \frac{1}{2n}$, hence

$$\frac{1}{2n} \int_{\{g>k\}} |A|^2 \left(1 - \frac{k}{g}\right) \phi^2 \le \int (g-k)^{+2} |\nabla \phi|^2. \tag{16}$$

Using $|\nabla g| \le |A|$ we reach

$$\frac{1}{2n} \int_{\{g>k\}} |\nabla g|^2 \left(1 - \frac{k}{g}\right) \phi^2 \le \int (g-k)^{+2} |\nabla \phi|^2.$$

□

Lemma 4.2. *Let M be as above. Then for any $k \in [0, \frac{1}{2n}]$ and $\varphi \in C_c^{0,1}(U)$ we have*

$$\frac{1}{2n} \int_{M \cap \{g>k\}} |\nabla g|^2 \left(1 - \frac{k}{g}\right) \varphi^2 \le \int_M (g-k)^{+2} |\nabla \varphi|^2,$$

where $(g-k)^+$ denotes the function $(g-k)^+ = \begin{cases} g-k & \text{when } g>k \\ 0 & \text{when } g \le k \end{cases}$.

Proof. The statement is just Lemma 4.1 when $\varphi \in C_c^{0,1}(U \setminus \Sigma)$. (We are implicitly choosing $\phi = \varphi \circ \iota$; the immersion is proper so $\varphi \circ \iota \in C_c^1(S)$.) Taking this as starting point, the extension of the inequality to $\varphi \in C_c^{0,1}(U)$ relies on the Euclidean area growth of n-area (valid at all points in $\overline{M}$, as recalled above) and on the assumption that $\mathrm{cap}_2(\Sigma) = 0$. The (now standard) 2-capacity argument is carried out in [16] for the case $\mathcal{H}^{n-2}(\Sigma) = 0$ and in [22] for the case $\mathcal{H}^{n-2}(\Sigma) < \infty$. □

Remark 4.3. In the case $k = 0$, (16) is the well-known Schoen inequality, [15], [16, Lemma 1]. The instance $k = 0$ of the lemma gives the intrinsic Caccioppoli inequality $\frac{1}{2n} \int |\nabla g|^2 \varphi^2 \le \int g^2 |\nabla \varphi|^2$.

We will employ Lemma 4.2, with suitable choices of φ. We will obtain a superlinear rate of decay for the L^2-norm of $(g-k)^+$ in C_r as $k \in \mathbb{R}$ grows from 0 to $\frac{1}{2n}$, and r decreases from the initial scale R to $\frac{R}{2}$. Via the elementary Lemma B.1, such a decay forces $(g - \frac{1}{2n})^+$ to vanish in $C_{\frac{R}{2}}$, as long as the L^2-norm of g is sufficiently small in C_R. This will establish Theorem 4.

proof of Theorem 4 for $n \ge 3$. We consider the dyadic sequences (respectively increasing and decreasing) $k_\ell = \frac{d}{2n}\left(1 - \frac{1}{2^{\ell-1}}\right)$ for $d \in (0,1]$ (for the moment arbitrary), and $R_\ell = \frac{R}{2} + \frac{R}{2^\ell}$ for $\ell \in \{1, 2, \ldots\}$.

We take the inequality of Lemma 4.2 with k_ℓ in place of k, and use the inclusion $\{g > k_\ell\} \supset \{g > k_{\ell+1}\}$:

$$\frac{1}{2n} \int_{\{g>k_{\ell+1}\}} |\nabla g|^2 \left(1 - \frac{k_\ell}{g}\right) \varphi^2 \le \int (g-k_\ell)^{+2} |\nabla \varphi|^2;$$

on the relevant domain on the left-hand-side, $\{g > k_{\ell+1}\}$, we have $1 - \frac{k_\ell}{g} \geq 1 - \frac{k_\ell}{k_{\ell+1}} \geq \frac{1}{2^\ell}$, therefore

$$\int_{\{g>k_{\ell+1}\}} |\nabla g|^2 \varphi^2 \leq (2n)2^\ell \int (g-k_\ell)^{+2} |\nabla \varphi|^2.$$

Since $|\nabla\big((g-k_{\ell+1})^+\varphi\big)|^2 \leq 2\chi_{\{g>k_{\ell+1}\}}|\nabla g|^2\varphi^2 + 2(g-k_{\ell+1})^{+2}|\nabla\varphi|^2$, and by definition $(g-k_{\ell+1})^+ \leq (g-k_\ell)^+$, we find

$$\int_{\{g>k_{\ell+1}\}} \left|\nabla\big((g-k_{\ell+1})^+\varphi\big)\right|^2 \leq 2((2n)2^\ell+1)\int (g-k_\ell)^{+2}|\nabla\varphi|^2. \tag{17}$$

Next (from now we will use $n \geq 3$), we will use (for the left-hand-side of (17)) the following Michael-Simon inequality ([1, 13]) on the minimally immersed hypersurface M, for the function $\varphi(g-k_{\ell+1})^+$:

$$\left(\int \left|\varphi(g-k_{\ell+1})^+\right|^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}} \leq C_{MS}\int \left|\nabla\left(\varphi(g-k_{\ell+1})^+\right)\right|^2, \tag{18}$$

with C_{MS} the dimensional constant given after (7).

Simultaneously, we choose φ, as follows. For $r > \rho$ chosen in $(0, R]$ we will consider φ of the type $\varphi(x, x_{n+1}) = \tilde{\varphi}(x)\tilde{\psi}(x_{n+1})$, with $\tilde{\varphi} : \mathbb{R}^n \to \mathbb{R}$ identically equal to 1 on $B^n_\rho(0)$, vanishing in the complement of $B^n_r(0)$ and with $|D\tilde{\varphi}| \leq \frac{\sqrt{2}}{r-\rho}$; with $\tilde{\psi} \in C^\infty_c(\mathbb{R})$ identically equal to 1 on $[-\rho, \rho]$, vanishing in the complement of $(-r, r)$, with $|\tilde{\psi}'| \leq \frac{\sqrt{2}}{r-\rho}$. Then, for each ℓ, we choose $\tilde{\varphi}_\ell$ and $\tilde{\psi}_\ell$ with $\rho = R_{\ell+1}$, $r = R_\ell$, so that $r - \rho = R_\ell - R_{\ell+1} = \frac{R}{2^{\ell+1}}$, and $\varphi_\ell = \tilde{\varphi}_\ell\tilde{\psi}_\ell$. Note that $|\nabla\varphi_\ell| \leq \frac{2}{R_\ell - R_{\ell+1}}$, $\varphi_\ell \equiv 1$ on $C_{R_{\ell+1}}$ and $\varphi_\ell \equiv 0$ in the complement of C_{R_ℓ}. Combining (17) and (18), with the chosen φ_ℓ in place of φ, we find

$$\begin{aligned}\left(\int_{M\cap C_{R_{\ell+1}}} \left|(g-k_{\ell+1})^+\right|^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}} &\leq \left(\int \left|\varphi_\ell(g-k_{\ell+1})^+\right|^{\frac{2n}{n-2}}\right)^{\frac{n-2}{n}} \leq \\ &\leq C_{MS}(n)\frac{2((2n)2^\ell+1)4^{\ell+2}}{R^2}\int_{M\cap C_{R_\ell}} (g-k_\ell)^{+2}.\end{aligned} \tag{19}$$

Hölder's inequality further gives

$$\begin{aligned}\int_{M\cap C_{R_{\ell+1}}} (g-k_{\ell+1})^{+2} \leq \\ \left(\int_{M\cap C_{R_{\ell+1}}} (g-k_{\ell+1})^{+\frac{2n}{n-2}}\right)^{\frac{n-2}{n}} \mathcal{H}^n\Big(M\cap\{g>k_{\ell+1}\}\cap C_{R_{\ell+1}}\Big)^{\frac{2}{n}}.\end{aligned} \tag{20}$$

Noting that on the set $\{g > k_{\ell+1}\}$ we have $(g-k_\ell)^+ > \frac{d}{n\,2^{\ell+1}}$, and using the inclusion $C_{R_{\ell+1}} \subset C_{R_\ell}$, the last factor in (20) is bounded above (thanks to the standard Markov's inequality) by

$$\mathcal{H}^n\left(M\cap\left\{(g-k_\ell)^{+2}>\frac{d^2}{(n\,2^{\ell+1})^2}\right\}\cap C_{R_\ell}\right)^{2/n}\leq \left(\frac{n^2\,4^{\ell+1}}{d^2}\int_{M\cap C_{R_\ell}}(g-k_\ell)^{+2}\right)^{2/n}. \tag{21}$$

From (20), using (21) for the second factor on the right-hand-side, and using (19) for the first factor on the right-hand-side, we have

$$\int_{M\cap C_{R_{\ell+1}}}(g-k_{\ell+1})^{+2}\leq C_{MS}(n)\frac{8((2n)2^\ell+1)4^{\ell+1}}{d^{\frac{4}{n}}R^2}n^{\frac{4}{n}}(4^{\frac{2}{n}})^{\ell+1}\Big(\int_{M\cap C_{R_\ell}}(g-k_\ell)^{+2}\Big)^{1+\frac{2}{n}}.$$

Writing $G_\ell=\int_{C_{R_\ell}}(g-k_\ell)^{+2}$, this implies

$$G_{\ell+1}\leq \underbrace{C_{MS}(n)\frac{32n}{d^{\frac{4}{n}}R^2}n^{4/n}(4\cdot 4^{2/n})}_{c(n,R,d)}(8\cdot 4^{2/n})^\ell\,G_\ell^{1+\frac{2}{n}}. \tag{22}$$

The superlinear decay estimate (22), $G_{\ell+1}\leq c(n,R,d)C^\ell G_\ell^{1+\frac{2}{n}}$ with $C=2\cdot 4^{1+\frac{2}{n}}$, forces $G_\ell\to 0$ as $\ell\to\infty$, as long as G_1 is sufficiently small, in a quantified fashion determined by $c(n,R,d)$ and C. We make it now explicit, using Lemma B.1.

With the initial choice $d=1$, the smallness condition on G_1 is written, for the scale-invariant tilt-excess ($R_1=R$ so $E_M(R)^2=\frac{1}{R^n}G_1$), as

$$E_M(R)^2\leq\frac{1}{\Big(R^2c(n,R,1)C^{\frac{n+2}{2}}\Big)^{n/2}}=\left(\frac{1}{C_{MS}(n)32\,n^{1+\frac{4}{n}}4^{1+\frac{2}{n}}2^{\frac{n+2}{2}}4^{\frac{(n+2)^2}{2n}}}\right)^{n/2}, \tag{23}$$

where the last term makes explicit the dimensional constant $k(n)$ in Theorem 4.

The convergence $G_\ell\to 0$ implies $\int_{C_{\frac{R}{2}}}(g-\frac{1}{2n})^{+2}=0$, that is, $g\leq\frac{1}{2n}$ a.e. on $M\cap C_{\frac{R}{2}}$. By smoothness of ι, then $g\leq\frac{1}{2n}$ on $M\cap C_{\frac{R}{2}}$.

More generally, with $d\in(0,1]$, we find that, if

$$E_M(R)^2\leq\frac{1}{\Big(R^2c(n,R,d)C^{\frac{n+2}{2}}\Big)^{n/2}}=\frac{d^2}{\Big(C_{MS}(n)32\,n^{1+\frac{4}{n}}4^{1+\frac{2}{n}}2^{\frac{n+2}{2}}4^{\frac{(n+2)^2}{2n}}\Big)^{n/2}},$$

then $g\leq\frac{d}{2n}$ on $M\cap C_{\frac{R}{2}}$. In other words, we have proved that, in the regime $E_M(R)^2\leq k(n)$, we have (for an explicit dimensional constant $c(n)$)

$$\sup_{M\cap C_{\frac{R}{2}}} g\leq c(n)E_M(R).$$

□

Remark 4.4. For $k = 0$, the inequality in Lemma 4.2 is an *intrinsic* Caccioppoli inequality (we have the intrinsic gradient on M, rather than the gradient D in $\mathbb{R}^n$ as in the case of De Giorgi [6], see also Remark 2.4). For $k \in (0, \frac{1}{2n}]$, on the other hand, we only have a weak intrinsic Caccioppoli inequality, due to the multiplicative factor $\left(1 - \frac{k}{g}\right)^+$. As seen also for Lemma 2.1, this weaker inequality is sufficient to implement the iterative scheme. While in [6] it is the linearity of the PDE that permits to obtain the classical Caccioppoli inequality for $(u-k)^+$, in our case the PDE for g escapes the De Giorgi–Nash–Moser framework: in fact, the PDE is a consequence of the minimality of M alone, which would permit e.g. catenoidal necks, with g reaching the value 1 under any smallness assumption on the L^2 height- or tilt-excess. The stability condition provides sufficient control on the non-linearity of the PDE (14) to obtain the weak intrinsic Caccioppoli inequality. We note explicitly that Lemma 4.2 is only valid for truncations at sufficiently small level sets (hence the smallness requirement in Theorem 4).

proof of Theorem 4 for $n = 2$. The case $n = 2$ requires a modification, as the exponent $\frac{2n}{n-2}$ is not well-defined in that case. The choices of k_ℓ, R_ℓ, φ_ℓ remain the same. We start from (17) (only after which we used $n \geq 3$), choosing φ in (17) to be φ_ℓ (recall that $|\nabla\varphi_\ell| \leq \frac{2^{\ell+1}}{R}$ and that $\text{spt}\varphi_\ell \subset C_{R_\ell}$). In what follows, $\sigma, \sigma', \sigma''$ denote explicitly determinable constants. We have

$$\int \left|\nabla\big((g - k_{\ell+1})^+\varphi_\ell\big)\right|^2 \leq \frac{4^\ell\sigma}{R^2}\int_{C_{R_\ell}\cap M} (g - k_\ell)^{+2}.$$

We use Hölder's inequality

$$\Big(\int \left|\nabla\big((g-k_{\ell+1})^+\varphi_\ell\big)\right|\Big)^2 \leq \Big(\int \left|\nabla\big((g-k_{\ell+1})^+\varphi_\ell\big)\right|^2\Big)\mathcal{H}^2\big(M\cap\{g > k_{\ell+1}\}\cap C_{R_\ell}\big),$$

the Michael–Simon inequality

$$\int \big((g - k_{\ell+1})^+\varphi\big)^2 \leq C_{MS}^2\Big(\int \left|\nabla\big((g - k_{\ell+1})^+\varphi\big)\right|\Big)^2$$

and the following consequence of Markov's inequality (as justified earlier)

$$\mathcal{H}^2\big(M \cap \{g > k_{\ell+1}\} \cap C_{R_\ell}\big) \leq \frac{\sigma' 4^\ell}{d^2}\int_{M\cap C_{R_\ell}} (g - k_\ell)^{+2}.$$

Writing $G_\ell = \int_{M\cap C_{R_\ell}} (g - k_\ell)^{+2}$, combining these inequalities we find

$$G_{\ell+1} \leq \frac{16^\ell\sigma''}{R^2d^2}G_\ell^2.$$

At this stage, Lemma B.1 gives that, if $G_1 \leq \frac{R^2 d^2}{16\sigma''}$ then $G_\ell \to 0$ as $\ell \to \infty$. In other words, given $d \in (0,1]$, if $E_M^2(R) = \frac{1}{R^2}\int_{M \cap C_R} g^2 \leq \frac{d^2}{16\sigma''}$, then $\sup_{M \cap C_{\frac{R}{2}}} g \leq \frac{d}{2n}$. The conclusion of Theorem 4 is thus proved for $n = 2$: in the smallness regime $E_M^2(R) \leq \frac{1}{16\sigma''}$ we have the control $\sup_{M \cap C_{\frac{R}{2}}} g \leq c(n) E_M(R)$. □

5 Proof of Theorems 5, 6, 7

Proof of Theorem 6. The pointwise bound $g \leq \frac{1}{2n}$ obtained in Theorem 4 implies the decomposition result by elementary arguments. Being an immersion, ι is locally a diffeomorphism with its image, that is, for every $X \in S$ there exists a neighbourhood D_X such that $\iota|_{D_X}$ is an embedded disk. The bound on g implies that that there exists a choice of continuous unit normal ν such that $\nu \cdot e_{n+1} \geq \frac{\sqrt{(2n)^2-1}}{2n}$. Denote by π the projection $\mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$. Then (for D_X sufficiently small) the disk $\iota(D_X)$ is a smooth graph over its projection. We thus have that $\iota|_{\iota^{-1}(C_{R/2})}$ is a local diffeomorphism with $B^n_{R/2}(0)$. Fix a connected component of $\iota^{-1}(C_{R/2})$, which we denote S_0. Then $\iota|_{S_0} : S_0 \to B^n_{R/2}(0)$ is a local diffeomorphism.

The condition on $\nu \cdot e_{n+1}$ guarantees that $\iota|_{S_0}$ is transverse to any line of the form $\{q\} \times \mathbb{R}$ (oriented by e_{n+1}) and the intersection is always positive. Moreover, the intersection index of M with such lines is constant (since $\sup_{M \cap C_{\frac{R}{2}}} |x_{n+1}| < \frac{R}{2}$, $M \cap C_{\frac{R}{2}}$ has no boundary in $B^n_{\frac{R}{2}}(0) \times \mathbb{R}$). Therefore $\iota^{-1}(\{q\} \times \mathbb{R})$ is a subset of S_0 with fixed cardinality $N \in \mathbb{N}$, regardless of q. (The immersion is proper, therefore there can only be finitely many points of intersection.)

The above observations imply that $\iota|_{S_0}$ is a N-cover of $B^n_{R/2}(0)$. On the other hand, the ball $B^n_{R/2}(0)$ is its own universal cover (and S_0 is connected), so $N = 1$. We have proved that each connected component of $\iota^{-1}(C_{R/2})$ is mapped (by ι) to a (smooth) graph over $B^n_{R/2}(0)$, which provides the smooth functions v_j in the conclusion of Theorem 6 (where j ranges over the set of connected components, which are finitely many because ι is proper).

At this stage, one can follow the arguments of De Giorgi [6], or directly invoke the De Giorgi–Nash–Moser theory, to conclude that g is Hölder continuous on every graph(v_j), and that each v_j is in $C^{1,\alpha}(B^n_{R/2}(0))$, with the estimate $\|\nabla v_j\|_{C^{0,\alpha}(B^n_{R/2}(0))} \leq C(n) E_0$. Higher regularity (and the analogous estimate for the $C^{k,\alpha}$-norms) follow from Schauder theory (using the Schoen inequality to control the L^2 norm of A by the tilt excess). □

Proof of Theorem 5. The arguments given for the graph decomposition for Theorem 6 lead to the conclusion that ι restricted to any connected component S_0 of S is a local diffeomorphism and an N-cover of $B^n_{R/2}(0) \setminus \pi(\Sigma)$. This relies on the observation that $B^n_{R/2}(0) \setminus \pi(\Sigma)$ is open and (path) connected (a consequence of the fact that Σ is closed with $\mathcal{H}^{n-1}(\Sigma) = 0$, which follows from $\mathrm{cap}_2(\Sigma) = 0$). This guarantees the possibility to choose a normal that has positive intersections with lines $\{q\} \times \mathbb{R}$ and the constancy of the intersection index of M with such lines for $q \in B^n_{R/2}(0) \setminus \pi(\Sigma)$.

At this stage, we have a description of $M \cap C_{R/2}$ as graph of a smooth q-valued function on $B^n_{R/2}(0) \setminus \pi(\Sigma)$. For any $q \in B^n_{R/2}(0) \setminus \pi(\Sigma)$, by ordering the values $\Pi\Big(M \cap (\{q\} \times \mathbb{R})\Big) \subset \mathbb{R}$ increasingly, where Π is the projection onto the second factor of $B^n_{R/2}(0) \times \mathbb{R}$, we obtain q Lipschitz functions $u_j : B^n_{R/2}(0) \setminus \pi(\Sigma) \to \mathbb{R}$, with Lipschitz constant $\frac{1}{2n}$, $u_j \leq u_{j+1}$ for all $j \in \{1, \ldots, Q-1\}$, which we can extend (preserving the Lipschitz constant) to $u_j : B^n_{R/2}(0) \to \mathbb{R}$. □

Theorem 5 gives a sheeting theorem for immersions, that are allowed to possess a singular set of locally finite $\mathcal{H}^{n-2}$-measure (or vanishing 2-capacity), and that are assumed to be "close" to a hyperplane. For such immersions, genuine branch points may arise, hence the singular set cannot be ruled out in the conclusions.

Proof of Theorem 7. Specialising Theorem 5 to embeddings, that is, if $\iota(M)$ is properly embedded in $C_R \setminus \Sigma$, then the Lipschitz functions $u_j : B^n_{R/2}(0) \setminus \pi(\Sigma) \to \mathbb{R}$ must be such that, for every $j \in \{1, \ldots, Q-1\}$, $u_j < u_{j+1}$. Thanks to the strict inequality, each u_j is a Lipschitz solution of the weak minimal surface PDE on $B^n_{R/2}(0) \setminus \pi(\Sigma)$, hence a smooth strong solution. Simon's well-known singularity removal [18], which only requires $\pi(\Sigma)$ closed in $B^n_{R/2}(0)$ and $\mathcal{H}^{n-1}(\pi(\Sigma)) = 0$ (a consequence of $\text{cap}_2(\Sigma) = 0$), yields a smooth extension $u_j : B^n_{R/2}(0) \to \mathbb{R}$ for each j, so that $\text{sing}_M \cap C_{\frac{R}{2}} = \emptyset$. □

Remark 5.1. As shown in [16], Theorem 7 leads rather quickly to the renowned Schoen–Simon regularity and compactness theory for stable minimal embedded hypersurfaces, see [16, Theorems 2 and 3].

The extra step required for this is a fairly simple slicing argument, see [16, pp. 785–787], which proves that "closeness" to a classical cone cannot arise for embeddings; after that, standard tangent cone analysis and dimension reduction complete the proof. For contrast, in the immersed case, closeness to classical cones can arise (and one would naturally aim for a sheeting result, over the several hyperplanes constituting the classical cone, which for $n \leq 6$ and in the absence of singular set follows from Lemma 3.3 of Part I).

With the multi-valued description of M in Theorem 5, natural questions are a more precise characterisation of the q-valued function obtained (plausibly, one can establish $C^{1,\alpha}$ regularity in the sense of q-valued functions), and a finer structure result for the singular set. While we do not pursue this here, we observe:

Corollary 1 (uniqueness of tangent hyperplanes)**.** *Let M be as in the beginning of Part II, and let $x \in \overline{M}$ be such that there exists a tangent cone (in the sense of varifolds) to M at x that is a hyperplane with multiplicity. Then that is the unique tangent cone at x.*

Proof. We take a blow up that gives rise to a hyperplane with multiplicity, which we assume to be $\{x_{n+1} = 0\}$ by rotating coordinates. For the blow up sequence M_ℓ (obtained by dilations of M) we have $E_{M_\ell}(1) \to 0$ (this follows from the monotonicity formula, using also Remark 4.2). Denoting by g_ℓ the tilt function on M_ℓ, using the estimate $\sup_{M_\ell \cap C_{\frac{R}{2}}} g_\ell \leq c(n) E_{M_\ell}(R)$ obtained in Theorem 5, it follows that $\sup_{M_\ell \cap C_{\frac{1}{2}}} g_\ell \to 0$. If any other blow up gave rise to a different cone, we would have the existence of $y_\ell \in M_\ell \cap C_{\frac{1}{2}}$ with $y_\ell \to 0$ and $\limsup_{\ell\to\infty} g_\ell(y_\ell) > 0$, contradiction. □

If $U = \mathbb{R}^{n+1}$ and the mass growth of M at infinity is Euclidean, then tangents at infinity exist and are cones. The same argument shows:

Corollary 2 (Bernstein-type theorem)**.** *Let M be as in the beginning of Part II with $U = \mathbb{R}^{n+1}$, and assume that one tangent cone to M at infinity (in the sense of varifolds) is a hyperplane with multiplicity. Then M is a union of hyperplanes, parallel to the given tangent. (In particular, the tangent at infinity is unique.)*

Proof. Assume without loss of generality that a tangent at infinity is the hyperplane $\{x_{n+1} = 0\}$ with multiplicity $q \in \mathbb{N}$. Then, by the monotonicity formula and by Remark 4.2 there exists a sequence $R_\ell \to \infty$ such that $E_M(R_\ell) = \frac{1}{R_\ell^n} \int_{M \cap B^{n+1}_{R_\ell}(0)} |\nabla x_{n+1}|^2 \to 0$ as $\ell \to \infty$ (where ν is a chosen unit normal to the immersed hypersurface and ∇ denotes the intrinsic gradient). For all sufficiently large ℓ we can therefore apply Theorem 4 to conclude that $\sup_{M \cap B^{n+1}_{\frac{R_\ell}{2}}(0)} |\nabla x_{n+1}| \leq c(n) E_M(R_\ell)$. For any given $r > 0$ (since $B^{n+1}_r(0) \subset B_{\frac{R_\ell}{2}}(0)$ for all sufficiently large ℓ) one must thus have $\sup_{M \cap B^{n+1}_r(0)} |\nabla x_{n+1}| = 0$. As r is arbitrary, $\nabla x_{n+1} \equiv 0$ on M and the conclusion follows. □

Remark 5.2. If the multiplicity of the hyperplane is at most 2, then the two corollaries follow from [22].

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A The case $n = 3$ of Theorem 3

While not essential for our arguments, we note explicitly that when $n = 3$ a stronger conclusion in Theorem 3 can be obtained from the proof given:

Corollary 3 ($n = 3$)**.** *Let M be a properly immersed two-sided stable minimal hypersurface in $B_{2R}(0)$, with $0 \in M$. There exists an (explicit) increasing continuous function $y : [0, \infty) \to [0, \infty)$ with $y(0) = 0$, such that for every $x \in M \cap B_{R/2}(0)$ we have*

$$|A|(x) \leq \frac{y\left(\frac{1}{2R} \int_{B_{2R}} |A|^2\right)}{R}.$$

Remark A.1. The proof gives $y(a) \sim a$ for a large and $y(a) \sim \sqrt{a}$ for a small.

Proof. Repeating the proof of Theorem 3 with $n = 3$ until the choice of d, and noting that $\frac{1}{R^2 d^{\frac{4}{n-2}}} + d^{\frac{2(n-4)}{n-2}} = \frac{1}{R^2 d^4} + \frac{1}{d^2}$, if we let $d = \frac{x}{R}$ the decay relation becomes

$$S_{\ell+1} \leq c\Big(\frac{1}{x^4} + \frac{1}{x^2}\Big)^3 R^6 \tilde{C}^\ell S_{\ell-1}^3,$$

with $\tilde{C} = 2^{21}$ and $c = 3^3 \cdot 2^{48}$ (using rough estimates, among which $C_{MS}^3 \leq 4^{28}$). A sufficient smallness condition on S_1 (to have $S_\ell \to 0$) is then

$$R^3 S_1 = R^3 \int_{M \cap B_R} |A|^6 \leq \Big(\frac{x^4}{1+x^2}\Big)^{\frac{3}{2}} \frac{1}{c^{\frac{1}{2}} \tilde{C}^{\frac{3}{2}}}.$$

This is in turn implied[6], writing $K = c^{\frac{1}{3}} \tilde{C} (40\, C_{MS})^2$, by

$$K\left(\frac{1}{2R}\int_{M\cap B_{2R}} |A|^2\right)^2 \leq \frac{x^4}{1+x^2}.$$

As $\frac{x^4}{1+x^2}$ is monotonically (strictly) increasing with value 0 at 0, we let f denote its inverse function and set $y(a) = f(Ka^2)$. Then by choosing $d = \frac{y(a)}{R}$, with $a = \frac{1}{2R}\int_{B_{2R}} |A|^2$, we find $|A| \leq \frac{y(a)}{R}$ on $B_{\frac{R}{2}}$. ☐

Remark A.2. In other words, for $n = 3$ curvature estimates of Theorem 2 completely follow from Corollary 3 (without appealing to tangent cone analysis and dimension reducing). Indeed, $\frac{1}{(2R)^{n-2}}\int_{B_{2R}} |A|^2 \leq \omega_n 2^n \Lambda$ (by the stability inequality in B_{4R}), hence $|A|(x) \leq \frac{y(8\omega_3\Lambda)}{R}$ for every $x \in B_{\frac{R}{2}}$. More precisely, as y above is explicit, for $n = 3$ we find, in Theorem 2, $\beta = \sqrt{\frac{\sigma + \sqrt{\sigma^2 + 4\sigma}}{2}}$ with $\sigma = K(8\omega_3\Lambda)^2$ and K as above.

We remark that (always for $n = 3$) [4] establishes the existence of a constant, explicitly determinable in terms of the first Betti number of M and of the number of boundary components of M, that bounds the curvature in $B_{\frac{R}{2}}$ of Theorem 2 .

B An elementary lemma

Lemma B.1. *Let $\tilde{C}, C > 0, \alpha > 0$ be given constants, and let x_ℓ be a sequence of positive real numbers that satisfies the following recursive relation for all $\ell \in \mathbb{N}\backslash\{0\}$:*

$$x_{\ell+1} \leq \tilde{C}\, C^\ell\, x_\ell^{1+\alpha}.$$

Assume that $x_1 \leq \frac{1}{(\tilde{C}\, C^{1+\frac{1}{\alpha}})^{\frac{1}{\alpha}}}$ if $C > 1$, $x_1 < \frac{1}{\tilde{C}^{\frac{1}{\alpha}}}$ if $C \leq 1$. Then $x_\ell \to 0$ as $\ell \to \infty$.

[6] We use $R^3 \int_{B_R} |A|^6 \leq (40\, C_{MS})^3 \big(\frac{1}{2R}\int_{B_{2R}\cap M} |A|^2\big)^3$, obtained in (11).

Proof. Assume that $C > 1$. We show that there exists $a \in (0,1)$ such that $\tilde{C}\, C^{\ell}\, x_{\ell}^{\alpha} \leq a$ for all $\ell \in \mathbb{N}$, from which $x_{\ell+1} \leq a x_{\ell}$ follows (hence the conclusion). For $\ell = 1$ we have

$$\tilde{C} C x_1^{\alpha} \leq \frac{C\,\tilde{C}}{\tilde{C}\, C^{1+\frac{1}{\alpha}}} = \frac{1}{C^{\frac{1}{\alpha}}}$$

and we set $a = \frac{1}{C^{\frac{1}{\alpha}}}$. Now we check inductively, for arbitrary $(\ell+1) \geq 2$, that

$$\tilde{C}\, C^{\ell+1}\, x_{\ell+1}^{\alpha} \leq \tilde{C}\, C^{\ell+1} (\tilde{C}\, C^{\ell}\, x_{\ell}^{1+\alpha})^{\alpha} = C(\tilde{C} C^{\ell} x_{\ell}^{\alpha})^{1+\alpha} \leq C a^{1+\alpha} = \frac{C}{C^{\frac{1+\alpha}{\alpha}}} = a.$$

If $C \leq 1$ then the recursive relation implies $x_{\ell+1} \leq \tilde{C}\, x_{\ell}^{1+\alpha} \leq (\tilde{C}\, x_{\ell}^{\alpha}) x_{\ell}$, in which case the smallness assumption $x_1 < \frac{1}{\tilde{C}^{\frac{1}{\alpha}}}$ implies the conclusion. □

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