SEMICLASSICAL ANALYSIS

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1. Introduction

These notes are being prepared for a course in Fall 2021 in the London Taught Course Center. The goal of the course is to introduce the basic structure of semiclassical analysis and to give several applications. Although we will spend some time covering the method of stationary phase, we will avoid spending a great deal of time on the technical details for the calculus, instead taking a more axiomatic approach and giving applications of the theory to solutions of partial differential equations. We refer the reader to 'Semiclassical Analysis' by M. Zworski and Appendix E in 'Mathematical Theory of Scattering Resonances' by S. Dyatlov and M. Zworski for the details of the calculus. We have also used some ideas from the notes by S. Dyatlov written for a course on semiclassical analysis during the Summer Northwestern Analysis Program in 2019.

1.1. **Some basic notation.** Throughout these notes we use the following notation.

(1) $D_x := -i\partial_x$
(2) $\langle \xi \rangle := (1+|\xi|^2)^{\frac{1}{2}}$.
(3) Let $\mathcal{B}$ be a Banach space and $f:(0,1)\to(0,\infty)$. We say that $u = O_\epsilon(f(h))_{\mathcal{B}}$ if there are $h_0>0$ and $C>0$ depending on the parameters ϵ such that
$$\|u\|_{\mathcal{B}} \leq Cf(h), \qquad 0<h<h_0.$$
(4) Let $\mathcal{B}$ be a Banach space and $f:(0,1)\to(0,\infty)$. We say that $u=o(f(h))_{\mathcal{B}}$ if
$$\limsup_{h\to 0^+} \frac{\|u\|_{\mathcal{B}}}{f(h)} = 0.$$
(5) We say that $u = O_\epsilon(h^\infty)_{\mathcal{B}}$ if there is $h_0>0$ depending on the parameters ϵ and for all $N>0$ there is $C_N>0$ depending on N and ϵ such that
$$\|u\|_{\mathcal{B}} \leq C_N h^N, \qquad 0<h<h_0.$$
(6) $\mathbb{M}(m\times n)$ - the set of $m\times n$ matrices
(7) $\mathbb{S}(d\times d)$ - the set of $d\times d$ symmetric matrices.

2. Stationary phase and the method of steepest descent

In both harmonic analysis and the study of partial differential equations, methods for understanding asymptotics for various integrals are indispensable. In this section, we study the asymptotics of the integrals

$$\mathbf{R}_h(\phi,a) := \int e^{-\phi(x)/h} a(x)dx, \qquad \mathbf{I}_h(\phi,a) := \int e^{i\phi(x)/h} a(x)dx, \tag{2.1}$$

where $0 < h < 1$ is an asymptotic parameter tending to 0, $a \in C_c^\infty(\mathbb{R}^d)$, and $\phi \in C^\infty(\mathbb{R}^d;\mathbb{R})$. These integrals are ubiquitous in the analysis of partial differential equations, but can also be found in simple asymptotic formulae such as Stirling's formula for $n!$ (see Exercise 2.4). Because they are heuristically simpler, we will start by studying $\mathbf{R}_h$.

We will study asymptotics as $h \to 0$ in the following sense. We write

$$\mathbf{B}_h \sim \sum b_j h^j$$

if for all $N \geq 0$, there is C_N such that for $0 < h < 1$,

$$\Big|\mathbf{B}_h - \sum_{j=0}^{N-1} b_j h^j\Big| \leq C_N h^N.$$

If there are b_j and $M \in \mathbb{R}$ such that $\mathbf{B}_h h^{-M} \sim \sum_j b_j h^j$, we say that $\mathbf{B}_h$ has *a full asymptotic expansion in powers of h.* Note that the sum on the right-hand side need not converge and, in fact, for any sequence $\{b_j\}_{j=0}^\infty$, one can find $\mathbf{B}_h$ such that $\mathbf{B}_h \sim \sum_j b_j h^j$. This result is known as Borel's lemma.

2.1. **The method of steepest descent.** We start by studying the asymptotics for $\mathbf{R}_h$ as in (2.1). We will assume throughout that there is $x_0 \in \operatorname{supp} a$ such that $\phi(x_0) > \phi(x)$ for all $x \neq x_0$ with $x \in \operatorname{supp} a$ and $\partial^2\phi(x_0)$ is non-degenerate. Under these assumptions, it is natural to think that, as $h \to 0$, the main contribution to $\mathbf{R}_h(\phi)$ comes from a neighborhood of x_0. Indeed, one can easily check that for any $\epsilon > 0$, there is $C_\epsilon > 0$ such that

$$\Big|\int_{\mathbb{R}^d\setminus B(x_0,\epsilon)} e^{-\phi(x)/h} a(x)dx\Big| \leq C_\epsilon e^{-1/(C_\epsilon h)}, \qquad 0 < h < 1. \tag{2.2}$$

Because of this, we will be able to assume that a is supported in an arbitrarily small neighborhood of x_0 when studying $\mathbf{R}_h(\phi, a)$. Since we may assume a is supported in an arbitrarily small neighborhood of x_0, it is natural to guess that $\mathbf{R}_h(\phi, a)$ can be (at least heuristically) understood by replacing ϕ by its Taylor polynomial,

$$\phi(x) \approx \phi(x_0) + \langle \phi'(x_0), x - x_0\rangle + \tfrac{1}{2}\langle \partial^2\phi(x_0)(x - x_0), (x - x_0)\rangle + O(|x - x_0|^3).$$

We first notice that $\phi'(x_0)$ is 0 since ϕ is maximal at x_0 and hence

$$\phi(x) \approx \phi(x_0) + \tfrac{1}{2}\langle \partial^2\phi(x_0)(x - x_0), (x - x_0)\rangle + O(|x - x_0|^3).$$

Now, since $\partial^2\phi$ is non-degenerate and ϕ has a maximum at x_0, we have $\partial^2\phi(x_0) > 0$ and hence

$$\phi(x) \geq \phi(x_0) + c|x - x_0|^2 \tag{2.3}$$

Since are interested in asymptotics modulo powers of h, it is natural then to examine where $e^{-\phi/h} \geq \epsilon > 0$. Based on (2.3), one can check that this is only true when $|x - x_0| \leq Ch^{\frac{1}{2}}$, a region with volume $Ch^{\frac{d}{2}}$. Thus, it is natural to expect

(1) The main term in $\mathbf{R}_h(\phi, a)$ is $\sim e^{-\phi(x_0)/h} h^{\frac{d}{2}}$.
(2) The full asymptotic formula for main term involves only the behavior of a and ϕ in an $h^{\frac{1}{2}}$ neighborhood of x_0, and hence can be written in terms of the derivatives of these functions at x_0.

2.2. **Quadratic phase asymptotics.** Although there are many proofs of the asymptotic formula for $\mathbf{R}_h(\phi, a)$, we choose one which adapts easily to the case of $\mathbf{I}_h(\phi, a)$. The Fourier transform will be a useful tool for this proof and we write

$$\hat{u}(\xi) := \int e^{-i\langle x,\xi\rangle} u(x)dx$$

for the Fourier transform of u. Recall also Parseval's formula

$$\int u(x)v(x)dx = \frac{1}{(2\pi)^d}\hat{u}(\xi)\hat{v}(\xi), \tag{2.4}$$

the Fourier inversion formula

$$u(x) = \frac{1}{(2\pi)^d}\int e^{i\langle x,\xi\rangle}\hat{u}(\xi)d\xi. \tag{2.5}$$

and the relationship between derivatives and the Fourier transform

$$\widehat{D_x^\alpha u}(\xi) = \xi^\alpha \hat{u}(\xi), \qquad D_x := -i\partial_x, \tag{2.6}$$

where, for $\alpha \in \mathbb{N}^d$, we use the notation $\xi^\alpha := \prod_{i=1}^d \xi_i^{\alpha_i}$.

We begin by studying the Fourier transform of a Gaussian.

Lemma 2.1. *Let $Q \in \mathbb{M}(d \times d)$ be a positive definite, symmetric matrix with real coefficients. Then,*

$$\widehat{e^{-\frac{1}{2}\langle Qx,x\rangle}}(\xi) = \frac{(2\pi)^{d/2}}{(\det Q)^{1/2}}e^{-\frac{1}{2}\langle Q^{-1}\xi,\xi\rangle}.$$

Proof. We compute

$$\begin{aligned}\widehat{e^{-\frac{1}{2}\langle Qx,x\rangle}}(\xi) &= \int e^{-i\langle x,\xi\rangle-\frac{1}{2}\langle Qx,x\rangle}dx\\ &= \int e^{-\frac{1}{2}\langle Q(x-Q^{-1}i\xi),x-Q^{-1}i\xi\rangle-\frac{1}{2}\langle Q^{-1}\xi,\xi\rangle}dx.\end{aligned}$$

We now deform the contour in x to the contour $\Gamma(y) = y - Q^{-1}i\xi$. Since Q is positive definite, the contributions from ∞ in x vanish and we have

$$\widehat{e^{-\frac{1}{2}\langle Qx,x\rangle}}(\xi) = e^{-\frac{1}{2}\langle Q^{-1}\xi,\xi\rangle}\int e^{-\frac{1}{2}\langle Qy,y\rangle}dy.$$

Now, since Q is symmetric, we can make an orthogonal change of variables in y so that

$$\frac{1}{2}\langle Qy,y\rangle = \frac{1}{2}\sum_i \lambda_i y_i^2,$$

and hence

$$\int e^{-\frac{1}{2}\langle Qy,y\rangle}dy. = \prod_{i=1}^d\int e^{-\frac{1}{2}\lambda_i y_i^2}dy_i = \prod_{i=1}^d\frac{(2\pi)^{1/2}}{\lambda_i^{\frac{1}{2}}} = \frac{(2\pi)^{d/2}}{(\det Q)^{1/2}}.$$

□

Now that we have computed the Fourier transform of a quadratic exponential, we can compute the asymptotics of $\mathbf{R}_h(\frac{1}{2}\langle Qx,x\rangle, a)$.

Lemma 2.2. *Let $Q \in \mathbb{M}(d \times d)$ be a positive definite, symmetric matrix with real coefficients. Then, for $a \in C_c^\infty(\mathbb{R}^d)$,*

$$\mathbf{R}_h(\tfrac{1}{2}\langle Qx, x\rangle, a) \sim \frac{(2\pi h)^{\frac{d}{2}}}{(\det Q)^{\frac{1}{2}}} \sum_j \frac{(-1)^j h^j}{2^j j!} \langle Q^{-1} D_x, D_x\rangle^j a(x)\big|_{x=0}$$

Proof. We start by applying Parseval's formula (2.4) and using Lemma 2.1 with Q replaced by $h^{-1}Q$,

$$\mathbf{R}_h(\tfrac{1}{2}\langle Qx, x\rangle, a) = \frac{h^{\frac{d}{2}}}{(2\pi)^{d/2}} \int e^{-\frac{h}{2}\langle Q^{-1}\xi, \xi\rangle} \hat{a}(\xi) d\xi.$$

Now, by Taylor's formula

$$\Big| e^{-\frac{h}{2}\langle Q^{-1}\xi,\xi\rangle} - \sum_{j=0}^{N-1} \frac{(-1)^j h^j \langle Q^{-1}\xi, \xi\rangle}{2^j j!} \Big| \le C_N h^N |\xi|^N.$$

Therefore,

$$\Big| \mathbf{R}_h(\tfrac{1}{2}\langle Qx, x\rangle, a) - \frac{h^{\frac{d}{2}}}{(2\pi)^{\frac{d}{2}}} \sum_{j=0}^{N-1} \int \frac{(-1)^j h^j \langle Q^{-1}\xi, \xi\rangle}{2^j j!} \hat{a}(\xi) d\xi \Big| \le C_N h^N \int |\xi|^N |\hat{a}(\xi)| d\xi.$$

Next, notice that by the Fourier inversion formula (2.5)

$$\int (\langle Q^{-1}\xi, \xi\rangle)^j \hat{a}(\xi) d\xi = (2\pi)^d (\langle Q^{-1} D_x, D_x\rangle)^j a(x)\big|_{x=0},$$

and thus it remains to estimate

$$\int |\xi|^N |\hat{a}(\xi)| d\xi \le \int \langle \xi\rangle^{-d-1} \langle \xi\rangle^{N+d+1} |\hat{a}(\xi)| d\xi \le C_d \|\langle \xi\rangle^{N+d+1} \hat{a}(\xi)\|_{L^\infty} \le C_d \sum_{|\alpha| \le N+d+1} \|\partial_x^\alpha a\|_{L^1}.$$

(We leave the proof of the last estimate to the exercises, see Exercise 2.3.) □

2.3. **The method of steepest descent.** We now return to the general case of $\mathbf{R}_h(\phi, a)$. We will actually be able to reduce this to the case of a quadratic phase using the following well known lemma.

Lemma 2.3 (Morse Lemma)**.** *Suppose that $\phi \in C^\infty(\mathbb{R}^d; \mathbb{R})$ such that $\phi(0) = \partial_x \phi(0) = 0$ and* $\det \partial^2 \phi(0) \neq 0$. *Then there are neighborhoods, U, V of 0 and $f : U \to V$ a diffeomorphism such that*

$$\phi \circ f(x) = \frac{1}{2} \langle \partial^2 \phi(0) x, x\rangle,$$

and $f(0) = 0$, $\partial_x f(0) = I$.

Proof. Since $\phi(0) = \partial\phi(0) = 0$, Taylors formula shows that

$$\phi(x) = \int_0^1 (1-t)^2 \partial_t^2 \varphi(tx) dt = \frac{1}{2}\langle Qx, x\rangle,$$

with $Q \in C^\infty(\mathbb{R}^d; \mathbb{S}(d \times d))$ and

$$Q(0) = \partial^2 \phi(0).$$

Now, we want to find a map $B \in C^\infty(\mathbb{R}^d; GL(d,\mathbb{R}))$ such that

$$\langle Q(x)x, x\rangle = \langle Q(0)B(x)x, B(x)x\rangle. \tag{2.7}$$

Note that to obtain (2.7), it is sufficient to find $B(x)$ such that

$$B^t(x)Q(0)B(x) = Q(x). \tag{2.8}$$

We define the map $F : \mathbb{M}(d\times d) \to \mathbb{S}(d \times d)$ by

$$F(B) := B^t Q(0)B.$$

It is then necessary to find a right inverse, R for F near $Q(0)$ such that $R(Q(0)) = \mathrm{Id}$ and then to put $f(x) := R(Q(x))x$.

We now use the inverse function theorem to find a right inverse for F near $Q(0)$. Observe that

$$F(I) = Q(0),$$

and hence it is enough to show that $\partial F(I) : \mathbb{M}(d\times d) \to \mathbb{S}(d\times d)$ has a right inverse $\tilde{R} : \mathbb{S}(d\times d) \to \mathbb{M}(d \times d)$.

For this, we compute

$$(\partial F(I))E = E^t Q(0) + Q(0)E$$

Now, putting $E = \frac{1}{2}Q(0)^{-1}D$ for some $D \in \mathbb{S}(d\times d)$, we have

$$(\partial F(I))E = \frac{1}{2}DQ(0)^{-1}Q(0) + Q(0)\frac{1}{2}Q(0)^{-1}D = D.$$

Therefore, the map, $\tilde{R} : D \mapsto \frac{1}{2}Q(0)^{-1}D$ is a right inverse for $\partial F(I)$, and hence, by the inverse function theorem, F has a right inverse near $F(I) = Q(0)$ as claimed.

□

We can now find an asymptotic formula for $\mathbf{R}_h(\phi, a)$.

Theorem 2.1 (The method of steepest descent)**.** *Let $a \in C_c^\infty(\mathbb{R}^d)$ and $\phi \in C^\infty(\mathbb{R}^d;\mathbb{R})$ such that there is $x_0 \in \operatorname{supp} a$ satisfying $\phi(x) < \phi(x_0)$ for all $x \in \operatorname{supp} a$ with $x \neq x_0$ and $\det \partial^2\phi(x_0) \neq 0$. Then for $j = 0, 1, \dots$ there are differential operators L_j of order less than or equal to $2j$ such that*

$$\mathbf{R}_h(\phi, a) \sim e^{-\phi(x_0)/h}\frac{(2\pi h)^{\frac{d}{2}}}{(\det \partial^2\phi(0))^{1/2}}\sum_j h^j L_{2j}(a)\big|_{x=x_0},$$

and $L_0(a) = a$.

Proof. Let $\tilde{\phi}(x) = \phi(x + x_0) - \phi(x_0)$. Then $\tilde{\phi}(0) = \partial\tilde{\phi}(0) = 0$. Therefore, by the Morse Lemma (Lemma 2.3), there are neigbhorhoods U, V of 0 and a diffeomorphims $f : U \to V$ such that

$$\tilde{\phi} \circ f = \tfrac{1}{2}\langle \partial^2\phi(x_0)x, x, \rangle.$$

Now, let $\chi \in C_c^\infty(\mathbb{R}^d)$ with $\chi \equiv 1$ near 0 and $\operatorname{supp}\chi \subset V$. Then,

$$\mathbf{R}_h(\phi, a) = \int e^{-\phi(x)/h}a(x)dx = e^{-\phi(x_0)/h}\int e^{\tilde{\phi}(x)}a(x + x_0)dx.$$

By (2.2), there is $C>0$ such that for $0<h<1$

$$\Big|\int e^{\tilde{\phi}(x)}(1-\chi(x))a(x+x_0)dx\Big|\leq Ce^{-1/(Ch)}.$$

Therefore, changing variables, $y=f^{-1}(x)$,

$$\mathbf{R}_h(\phi,a)=e^{-\phi(x_0)/h}\int e^{-\frac{1}{2}\langle\partial^2\phi(x_0)y,y\rangle}\chi(f(y))a(f(y)+x_0)|\det\partial f(y)|dy+O(e^{-\phi(x_0)/h-1/(Ch)})$$

Since $\partial^2\phi(x_0)>0$, we may apply Lemma 2.2 to see that

$$\mathbf{R}_h(\phi,a)\sim e^{-\phi(x_0)/h}\frac{(2\pi h)^{d/2}}{(\det\partial^2\phi(x_0))^{1/2}}\sum_j\frac{h^j}{2^j j!}\langle[\partial^2\phi(x_0)]^{-1}D_y,D_y\rangle^j(a(f(y)+x_0)|\det\partial f(y)|)\big|_{y=0}.$$

Now, the theorem follows from the fact that $f(0)=0$ and $\partial_y f(0)=I$. □

2.4. **Stationary phase asymptotics.** We now turn our attention to $\mathbf{I}_h(\phi,a)$. It turns out that the proof of an asymptotic formula for $\mathbf{I}_h(\phi,a)$ is almost the same as that for $\mathbf{R}_h(\phi,a)$. However, the the reason that critical points of ϕ play such an important role is somewhat less obvious. One of the messages of this section is that rapid oscillation can produce decay of an integral, so one should think that only the regions where the integrand is *not* rapidly oscillating contribute. Many readers may have seen this, for instance, in the fact that $\sin(nx)\underset{L^2([0,2\pi])}{\rightharpoonup}0$ as $n\to\infty$. Another place where this appears is in the Riemann Lebesgue lemma.

We now give a more quantitative version of this idea.

Lemma 2.4 (non-stationary phase)**.** *Suppose that $a\in C_c^\infty(\mathbb{R}^d)$ and $\phi\in C^\infty(\mathbb{R}^d;\mathbb{R})$ such that $|\partial\phi(x)|>0$ for all $x\in\operatorname{supp}a$. Then,*

$$\mathbf{I}_h(\phi,a)=O(h^\infty).$$

Proof. Observe that since $|\partial\phi(x)|>0$ on $\operatorname{supp}a$, there is $c>0$ such that $|\partial\phi(x)|>c$ on $\operatorname{supp}a$. Let $L:=\frac{\langle\partial_x\phi,hD_x\rangle}{|\partial_x\phi(x)|^2}$. Then,

$$L(e^{\frac{i}{h}\phi})=e^{\frac{i}{h}\phi}.$$

Therefore, integration by parts shows that

$$\mathbf{I}_h(\phi,a)=\int e^{\frac{i}{h}\phi(x)}(L^t)^N a(x)dx,\qquad L^t=-\frac{\langle\partial_x\phi,hD_x\rangle}{|\partial_x\phi(x)|^2}-h\frac{\Delta\phi}{|\partial_x\phi|^2}+h\frac{\sum_{ij}\partial^2_{x^ix^j}\phi\partial_{x^j}\phi}{|\partial_x\phi|^4}.$$

In particular, $h^{-1}L^t:C^M\to C^{M-1}$ is uniformly bounded in h has $h\to0$ and the lemma follows. □

One may now wonder, ‘From what region do the main contributions to $\mathbf{I}_h(\phi,a)$ come from?’. To understand this, we again imagine that x_0 is a non-degenerate critical point for ϕ so that

$$\phi(x)=\phi(x_0)+\tfrac{1}{2}\langle\partial^2\phi(x_0)(x-x_0),x-x_0\rangle+O(|x-x_0|^3).$$

It is now much less clear than when in the case of the method of steepest descent, but we will see that, once again, the region for $|x-x_0|\leq Ch^{\frac{1}{2}}$ plays a crucial role.

For simplicity we will assume from now on that ϕ has a single non-degenerate critical point on $\operatorname{supp} a$. The case of finitely many such critical points can be reduced to this situation by a partition of unity. We proceed as in the proof of method of steepest descent. That is, we first study the case of quadratic phases.

Lemma 2.5. *Let $Q \in \mathbb{S}(d \times d)$ be a real, non-singular matrix. Then,*

$$\widehat{e^{\frac{i}{2}\langle Qx,x\rangle}}(\xi) = \frac{e^{\frac{i\pi}{4}\operatorname{sgn}(Q)}(2\pi)^{d/2}}{|\det Q|^{1/2}} e^{-\frac{i}{2}\langle Q^{-1}\xi,\xi\rangle}.$$

Proof. Observe that

$$\begin{aligned}\widehat{e^{\frac{i}{2}\langle Qx,x\rangle}}(\xi) &= \lim_{\epsilon\to 0^+}\int e^{-i\langle x,\xi\rangle+\frac{i}{2}\langle Qx,x\rangle-\epsilon|x-Q^{-1}\xi|^2}dx\\ &= \lim_{\epsilon\to 0^+} e^{-\frac{i}{2}\langle Q^{-1}\xi,\xi\rangle}\int e^{\frac{i}{2}\langle Q(x-Q^{-1}\xi),x-Q^{-1}\xi\rangle-\epsilon|x-Q^{-1}\xi|^2}dx\\ &= e^{-\frac{i}{2}\langle Q^{-1}\xi,\xi\rangle}\lim_{\epsilon\to 0^+}\int e^{\frac{i}{2}\langle Qy,y\rangle-\epsilon|y|^2}dy\end{aligned}$$

So, it is enough to compute

$$\lim_{\epsilon\to 0^+}\int e^{\frac{i}{2}\langle Qy,y\rangle-\epsilon|y|^2}dy = \prod_{j=1}^{d}\lim_{\epsilon\to 0^+}\int e^{(\frac{i}{2}\lambda_j-\epsilon)y_j^2}dy_j,$$

where $\{\lambda_j\}_{j=1}^d$ are the eigenvalues of Q.

We first consider the case $\lambda_j > 0$. In this case

$$\operatorname{Re}(\frac{i}{2}\lambda_j-\epsilon)y_j^2 < 0 \text{ for } 0 \le \operatorname{Arg}(y_j) \le \frac{\pi}{4},$$

and hence we may deform the contour to $\Gamma_+(s) = e^{i\pi/4}s$ to obtain

$$\lim_{\epsilon\to 0^+}\int e^{(\frac{i}{2}\lambda_j-\epsilon)y_j{}^2}dy_j = e^{i\pi/4}\lim_{\epsilon\to 0^+}\int e^{(-\frac{1}{2}\lambda_i-i\epsilon)s^2}ds = e^{i\pi/4}\int e^{-\frac{1}{2}\lambda_i s^2}ds = \frac{e^{i\pi/4}(2\pi)^{1/2}}{\lambda_j^{1/2}}.$$

Next, when $\lambda_j < 0$, we deform the contour to $\Gamma_-(s) = e^{-i\pi/4}s$ to obtain

$$\lim_{\epsilon\to 0^+}\int e^{(\frac{i}{2}\lambda_j-\epsilon)y_j{}^2}dy_j = e^{-i\pi/4}\lim_{\epsilon\to 0^+}\int e^{(-\frac{1}{2}|\lambda_i|+i\epsilon)s^2}ds = e^{-i\pi/4}\int e^{-\frac{1}{2}|\lambda_i|s^2}ds = \frac{e^{-i\pi/4}(2\pi)^{1/2}}{|\lambda_j|^{1/2}}.$$

The claim now follows from the definition of $\operatorname{sgn}(Q)$. □

We are now in a position to prove the main result of stationary phase.

Theorem 2.2. *Let $a \in C_c^\infty(\mathbb{R}^d)$ and $\phi \in C^\infty(\mathbb{R}^d;\mathbb{R})$ such that there is $x_0 \in \operatorname{supp} a$ such that $\partial\phi(x_0) = 0$, $\det\partial^2\phi(x_0) \neq 0$, and $|\partial\phi(x)| > 0$ on $\operatorname{supp} a \setminus x_0$. Then, Then for $j = 0, 1, \ldots$ there are differential operators L_j of order less than or equal to $2j$ such that*

$$\mathbf{I}_h(\phi,a) \sim e^{i\phi(x_0)/h}e^{\frac{i\pi}{4}\operatorname{sgn}(\partial^2\phi(x_0))}\frac{(2\pi h)^{\frac{d}{2}}}{|\det\partial^2\phi(0)|^{1/2}}\sum_j h^j L_{2j}(a)\big|_{x=x_0},$$

and $L_0(a) = a$.

The theorem follows from our next lemma and Lemma 2.4 as in the proof of Theorem 2.1.

Lemma 2.6. *Let $Q \in \mathbb{S}(d \times d)$ be a non-singular, symmetric matrix with real coefficients. Then, for $a \in C_c^\infty(\mathbb{R}^d)$,*

$$\mathbf{I}_h(\tfrac{1}{2}\langle Qx, x\rangle, a) \sim \frac{e^{\frac{i\pi}{4}\operatorname{sgn}(Q)}(2\pi h)^{\frac{d}{2}}}{|\det Q|^{\frac{1}{2}}} \sum_j \frac{h^j}{i^j 2^j j!} \langle Q^{-1} D_x, D_x\rangle^j a(x)\big|_{x=0}$$

Proof. The proof follows as in Lemma 2.2. □

2.5. Exercises.

Exercise 2.1. Show that for any smooth function $f : \mathbb{R} \to \mathbb{R}$, $f(h) \sim \sum_j \frac{f^{(j)}(0)}{j!} h^j$.

Exercise 2.2. Suppose that $\phi \in C^\infty(\mathbb{R}^d; \mathbb{R}$, $a \in C_c^\infty(\mathbb{R}^d)$, and there is $x_0 \in \operatorname{supp} a$ such that $\phi(x) < \phi(x_0)$ for all $x \neq x_0$ with $x \in \operatorname{supp} a$. Show that $\mathbf{R}_h(\phi, a)$ satisfies (2.2)

Exercise 2.3. Let $M \geq 0$. Show that there is $C > 0$ such that if $\partial_x^\alpha a \in L^1$ for $|\alpha| \leq M$, then there is

$$\|\langle \xi\rangle^M \hat{a}(\xi)\|_{L^\infty} \leq C \sum_{|\alpha| \leq M} \|\partial_x^\alpha a\|_{L^1}.$$

Exercise 2.4 (Stirling's formula). Recall that the gamma function $\Gamma : (0, \infty) \to \mathbb{R}$ is given by

$$\Gamma(s) = \int_0^\infty t^{s-1} e^{-t} dt,$$

and satisfies $\Gamma(n) = (n-1)!$. Use the method of steepest descent to show that $\frac{n! e^n}{n^n}$ has a full asymptotic expansion in powers of n and find the first two terms of Stirling's formula.

Exercise 2.5. Prove Theorem 2.2 and Lemma 2.6

Exercise 2.6. Let $\delta_{S^{d-1}}$ denote the arc length measure on the unit sphere in $\mathbb{R}^d$. Show that there are $a_{j,\pm} \in \mathbb{C}$, $j = 0, 1, \dots$ such that

$$\widehat{\delta_{S^d}}(\xi) \sim e^{i|\xi|} |\xi|^{-\frac{d-1}{2}} \sum_j a_{j,+} |\xi|^{-j} + e^{-i|\xi|} |\xi|^{-\frac{d-1}{2}} \sum_j a_{j,-} |\xi|^{-j}.$$

Exercise 2.7. Let $\chi \in C_c^\infty(\mathbb{R})$ and define $\chi_\epsilon(\xi) = \epsilon^{-1}\chi(\epsilon^{-1}\xi)$. Show that for all $u \in L^\infty(\mathbb{R})$, $\|\chi_\epsilon * \hat{u}\|_{L^\infty} \leq C\epsilon^{-\frac{1}{2}}$. Fix $\delta > 0$. Find $u \in L^\infty$ such that $\|\chi_\epsilon * \hat{u}\|_{L^\infty} \geq c\epsilon^{-\frac{1}{2}+\delta}$. (Hint: Take the Fourier transform and consider functions oscillating *very* rapidly near 0.

3. Definitions and an axiomatic approach to the basic pseudodifferential calculus

In this section we introduce the basic notions of psuedodifferential operators that we will use throughout these notes. We do not intend to give the details of various technical proofs and instead treat many of the technical lemmas as axioms.

3.1. **Motivation for pseudodifferential operators.** Before moving on, it is worth asking the questions, 'Why do we need something other than differential operators and the Fourier transform?' There are, as usual, many possible answers to this question which range from: 'They are a natural mathematical extension of differential operators.' to 'The layer potential operators turn out to be pseudodifferential.' That is, from a purely abstract answer for the sake of mathematical to completeness, to the fact that some specific operators which people care about turn out to be pseudodifferential operators.

We will take an approach somewhere in the middle for our first motivation.

3.1.1. *Motivation 1: inversion.* Consider for the moment, the Laplace equation posed on $L^2(\mathbb{R}^d)$

$$(-\Delta + 1)u = f.$$

How can we understand solutions to this equation? One option is to apply the Fourier transform to both side

$$\widehat{(-\Delta + 1)u}(\xi) = \hat{f}(\xi).$$

Now, by (2.6), this is

$$(|\xi|^2 + 1)\hat{u}(\xi) = \hat{f}(\xi) \qquad\qquad \hat{u}(\xi) = \frac{\hat{f}(\xi)}{|\xi|^2 + 1}.$$

Therefore, we can go on our merry way, knowing that

$$u(x) = \frac{1}{(2\pi)^d} \int e^{i\langle x-y,\xi\rangle} \frac{f(y)}{|\xi|^2 + 1} dy d\xi,$$

and that the inverse of $-\Delta + 1$ is a Fourier multiplier.

Now, it is natural to consider a slightly more general problem. Let $A(x) \in C^\infty(\mathbb{R}^d; \mathbb{S}(d \times d; \mathbb{R}))$ such that $\langle A(x)v, v\rangle \geq c|v|^2$ for some $c > 0$ and all $x \in \mathbb{R}^d$. Then, the following equation

$$(-\nabla A(x)\nabla + 1)u = f$$

is a natural extension to non-trivial metrics of the Laplace equation. We may even assume that $A(x) = \mathrm{Id}$ outside of a compact set. However, as soon as we try to solve this problem using the Fourier transform, we realize there is a problem. Namely, we cannot write the Fourier transform of $(-\nabla A(x)\nabla + 1)u$ as a multiple of that of u. This is simply not true since the operator is no longer translation invariant. Therefore, if we want to invert this equation, we are forced into new territory. It will turn out that the new operators one develops for this purpose are pseudodifferential operators.

Thus, if one wants a class of operators which include elliptic differential operators and is closed under inversion, then one immediately arrives at something resembling pseudodifferential operators.

3.1.2. *Motivation 2: propagation.* Consider the Schrödinger equation in 1-dimension

$$ih\partial_t u - h^2\partial_x^2 u = 0, \qquad u(0) = u_0 \in L^2(\mathbb{R}).$$

Suppose that we put

$$u_0(x) = h^{-\frac{1}{4}} e^{-|x|^2/(2h)} e^{2ix/h} + h^{-\frac{1}{4}} e^{-|x|^2/(2h)} e^{-ix/h}.$$

It will be convenient now to introduce the *semiclassical Fourier transform*

$$\mathcal{F}(u)(\xi) = \hat{u}(\xi/h).$$

Then, it is easy to see that

$$\mathcal{F}(u)(t,\xi) = e^{it|\xi|^2/h}\mathcal{F}(u_0)(\xi).$$

In particular, with our choice of u_0, we have by Lemma 2.5

$$\mathcal{F}(u_0) = (2\pi)^{1/2}h^{1/4}[e^{-\frac{(\xi-2)^2}{2h}} + e^{-\frac{(\xi+1)^2}{2h}}],$$

and hence, completing the square and deforming the contour appropriately,

$$\begin{aligned} u(t,x) &= \frac{1}{\sqrt{2\pi h}}h^{\frac{1}{4}}\int(e^{i(x\xi+t|\xi|^2)/h-|\xi-2|^2/(2h)} + e^{i(x\xi+t|\xi|^2)/h-|\xi+1|^2/(2h)})d\xi \\ &= \frac{h^{-1/4}}{\sqrt{1+2it}}(e^{\frac{-(x+4t)^2}{2(1+4t^2)h}+i\frac{t(4-x^2)+2x}{(1+4t^2)h}} + e^{\frac{-(x-2t)^2}{2(1+4t^2)h}+i\frac{t(1-x^2)-x}{(1+4t^2)h}}). \end{aligned}$$

Notice that $u(t,x)$ consists of a wave packet traveling to the left at speed 4 and one traveling to the right at speed 2.

It is reasonable to ask, 'Why does a single bump at 0 split into two packets traveling at different speeds?'. The reader will of course notice the different phases chosen, one given by $-x/h$ and the other by $2x/h$ which you could guess are responsible for this behavior. However, there are two problems with this analysis:

1. We have no explanation for why different phases result in splitting.
2. If we change the operator even slightly, e.g. replacing ∂_x^2 by $\partial_x^2 + V$, for some smooth function V we lose the ability to do this analysis using explicit formulae.

How can we understand this phenomenon in a more robust way? Semiclassical analysis will turn out to be key to this.

3.2. **Basic definitions: Symbol classes, Sobolev spaces, and pseudodifferential operators.** In order to define pseudodifferential operators, we first need to define symbol classes. Typically, we allow functions and operators to implicitly depend on the small parameter h. We say that $a \in C^\infty(T^*\mathbb{R}^d)$ is a symbol of order $m \in \mathbb{R}$ if for all $\alpha, \beta \in \mathbb{N}^d$, there is $C_{\alpha\beta} > 0$ such that

$$|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \le C_{\alpha\beta}\langle\xi\rangle^{m-|\beta|}.$$

In this case, we write $a \in S^m(T^*\mathbb{R}^d)$. Throughout these notes we actually work with a somewhat for restrictive class of symbols. We say that $a \in S^m_{\text{phg}}(T^*\mathbb{R}^d)$ if $a \in S^m$ and there are $a_j \in S^{m-j}$ independent of h such that

$$a - \sum_{j=0}^{N-1} h^j a_j \in h^N S^{m-N}(T^*\mathbb{R}^d). \tag{3.1}$$

When $a \in S^m$ satisfies (3.1), we say that $a \sim \sum_j h^j a_j$.

We will often write simply $a \in S^m_{\text{phg}}$ when the space is clear from context. We also define $S^\infty_{\text{phg}} = \bigcup_m S^m_{\text{phg}}$, $S^{-\infty}_{\text{phg}} = \bigcap_m S^m_{\text{phg}}$, and we define $S^{\text{comp}}_{\text{phg}}$ to be the set of $a \in S^{-\infty}_{\text{phg}}$ which are supported in some h-independent compact set.

It will also be be convenient to have a notion of semiclassical Sobolev spaces. The elements of these spaces are the same as for the standard Sobolev spaces, but the norm is scaled in a way depending on h. In particular,

$$H_h^s(\mathbb{R}^d) := \{u \in \mathcal{S}'(\mathbb{R}^d) \,:\, \langle \xi\rangle^s \mathcal{F}(u) \in L^2\}, \qquad \|u\|_{H_h^s}^2 := (2\pi h)^{-d} \|\langle \xi\rangle^s \mathcal{F}(u)\|_{L^2}^2.$$

For $A : \mathcal{S} \to \mathcal{S}$, we say that $A = O(h^\infty)_{\Psi^{-\infty}}$ if for all N, there is $C_N > 0$ such that

$$\|A\|_{H_h^{-N} \to H_h^N} \leq C_N h^N.$$

We can now introduce the class of pseudodifferential operators on $\mathbb{R}^d$. For $m \in \mathbb{R}$, we say that A is a *pseudodifferential operator of order* m and write $A \in \Psi^m(\mathbb{R}^d)$ if there is $a \in S^m_{\text{phg}}$ such that

$$A = \text{Op}_\text{h}(a) + O(h^\infty)_{\Psi^{-\infty}}, \qquad [\text{Op}_\text{h}(a)u](x) := \frac{1}{(2\pi h)^d} \int e^{\frac{i}{h}\langle x-y,\xi\rangle} a(x,\xi) u(y) dy d\xi.$$

Here, the integral in $\text{Op}_\text{h}(a)u$, can be understood as an iterated integral when $u \in \mathcal{S}$ and it is not hard to check (see exercise 3.1) that operators in Ψ^m are bounded on $\mathcal{S}$ and $\mathcal{S}'$. As with symbols, we sometimes omit the space $\mathbb{R}^d$ from the notation and define $\Psi^\infty = \bigcup_m \Psi^m$, $\Psi^{-\infty} = \bigcap_m \Psi^m$, and we define Ψ^{comp} to be those $A \in \Psi^{-\infty}$ such that

$$A = \text{Op}_\text{h}(a) + O(h^\infty)_{\Psi^{-\infty}}$$

for some $a \in S^{\text{comp}}_{\text{phg}}$.

3.3. **Basic properties of pseudodifferential operators.** Before proceeding to recall the most important properties of pseudodifferential operators, we define the symbol map $\sigma_m : \Psi^m \to S^m$ using the following procedure. Let $A \in \Psi^m$. Then, there is $a \in S^m_{\text{phg}}$ such that $A = \text{Op}_\text{h}(a) + O(h^\infty)_{\Psi^{-\infty}}$. In particular,

$$a \sim \sum_j a_j h^j$$

with $a_j \in S^{m-j}$ independent of h. We then define $\sigma_m(A) = a_0$. We will often abuse notation slightly and write instead $\sigma(A) = a_0$.

We now collect the most important properties of the pseudodifferential calculus in the following theorem.

Theorem 3.1. *(1) Suppose that $A \in \Psi^m$ and $\sigma_m(A) = 0$. Then $A \in h\Psi^{m-1}$.*

(2) Suppose that $A \in \Psi^m$. Then, $A^* \in \Psi^m$ and $\sigma_m(A^*) = \overline{\sigma_m(A)}$.

(3) Let $A \in \Psi^{m_1}$ and $B \in \Psi^{m_2}$. Then $AB \in \Psi^{m_1+m_2}$ and

$$\sigma(AB) = \sigma(A)\sigma(B).$$

Moreover, if $A = \text{Op}_\text{h}(a)$ and $B = \text{Op}_\text{h}(b)$, then $AB = \text{Op}_\text{h}(e)$, with e satisfying

(4) Let $A \in \Psi^{m_1}$ and $B \in \Psi^{m_2}$. Then $[A, B] \in \Psi^{m_1+m_2-1}$ and

$$\sigma([A, B]) = -ih\{\sigma(A), \sigma(B)\}, \qquad \{a, b\} = \sum_{j=1}^d \partial_{\xi_j} a \partial_{x_j} b - \partial_{\xi_j} b \partial_{x_j} a.$$

We will occasionally need the following slightly more precise estimate on the composition

Theorem 3.2. *For all $\alpha, \beta \in \mathbb{N}^d$, there are $C_{\alpha\beta}$ and $M_{\alpha\beta}$ such that for all $a \in S^{m_1}_{\mathrm{phg}}$ and $b \in S^{m_2}_{\mathrm{phg}}$,*

$$\mathrm{Op_h}(a)\,\mathrm{Op_h}(b) = \mathrm{Op_h}(e),$$

for some $e \in S^{m_1+m_2}_{\mathrm{phg}}$ satisfying

$$\sup |\langle \xi \rangle^{-m_1-m_2+|\beta|} \partial_x^\alpha \partial_\xi^\beta e(x,\xi)| \leq C_{\alpha\beta} \sum_{|\gamma_1|+|\nu_1| \leq M_{\alpha\beta}} \sup |\langle \xi \rangle^{-m_1+|\gamma_1|} \partial_x^{\nu_1} \partial_\xi^{\gamma_1} a(x,\xi)| \sum_{|\gamma_2|+|\nu_2| \leq M_{\alpha\beta}} \sup |\langle \xi \rangle^{-m_2+|\gamma_2|} \partial_x^{\nu_2} \partial_\xi^{\gamma_2} b(x,\xi)|.$$

3.4. **The compactified cotangent bundle and wavefront set.** Before proceeding to properties of pseudodifferential operators such as ellipticity, we introduce the wavefront set of a pseudodifferential operator. Before doing so, it will be convenient to introduce the *fiber radially compactified and radially compactifiied cotangent bundle*, $\overline{T}^*\overline{\mathbb{R}^d}$. This is a manfold with interior given by $T^*\mathbb{R}^d$ and

$$\partial \overline{T}^*\mathbb{R}^d \cong S^*\mathbb{R}^d \sqcup T^*\mathbb{R}^d_{S^{d-1}} \sqcup S^*\mathbb{R}^d_{S^{d-1}}.$$

We call $S^*\mathbb{R}^d$, *fiber infinity*, and $T^*\mathbb{R}^d_{S^{d-1}}$, *physical infinity.*

We now describe a neighborhood basis for each point $(x_0, \xi_0) \in \overline{T}^*\overline{\mathbb{R}^d}$ can be described as follows. If $(x_0, \xi_0) \in T^*\mathbb{R}^d$, then, the neighborhoods of (x_0, ξ_0) are the usual neighborhoods in $T^*\mathbb{R}^d$. On the other hand, if $(x_0, \xi_0) \in S^*\mathbb{R}^d \subset \partial \overline{T}^*\overline{\mathbb{R}^d}$, then a neighborhood basis is given as follows,

$$U_\epsilon := \Big\{ (x,\xi) \in \overline{T}^*M \,:\, |x - x_0| < \epsilon,\, |\tfrac{\xi}{|\xi|} - \xi_0| < \epsilon,\, |\xi| \geq \epsilon^{-1} \Big\},$$

if $(x_0, \xi_0) \in T^*_{S^{d-1}}\mathbb{R}^d \subset \partial \overline{T}^*\overline{\mathbb{R}^d}$, then a neighborhood basis is given as follows,

$$U_\epsilon := \Big\{ (x,\xi) \in \overline{T}^*M \,:\, |\frac{x}{|x|} - x_0| < \epsilon,\, |\xi - \xi_0| < \epsilon,\, |x| \geq \epsilon^{-1} \Big\},$$

and if $(x_0, \xi_0) \in S^*_{S^{d-1}}\mathbb{R}^d \subset \partial \overline{T}^*\overline{\mathbb{R}^d}$, then a neighborhood basis is given as follows,

$$U_\epsilon := \Big\{ (x,\xi) \in \overline{T}^*M \,:\, |\frac{x}{|x|} - x_0| < \epsilon,\, |\frac{\xi}{|\xi|} - \xi_0| < \epsilon,\, |\xi| > \epsilon^{-1},\, |x| \geq \epsilon^{-1} \Big\}.$$

We can now define the essential support of a symbol and the wavefront set of a pseudodifferential operator.

Definition 3.1. Let $a \in S^m$. For $(x_0, \xi_0) \in \overline{T}^*\overline{\mathbb{R}^d}$, we say that $(x_0, \xi_0) \notin \operatorname{ess\,supp}(a)$ if there is a neighborhood, U of (x_0, ξ_0) such that for all $\alpha, \beta \in \mathbb{N}^d$, and $N \in \mathbb{R}$, there is $C_{\alpha\beta N} > 0$ such that for $0 < h < 1$,

$$|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \leq C_{\alpha\beta N} h^N \langle \xi \rangle^{-N}, \qquad (x,\xi) \in U \cap T^*\mathbb{R}^d.$$

Definition 3.2. Let $A \in \Psi^m$. For $(x_0, \xi_0) \in \overline{T}^*\overline{\mathbb{R}^d}$, we say that $(x_0, \xi_0) \notin \mathrm{WF}(A)$ if there is $a \in S^m$ such that $(x_0, \xi_0) \notin \operatorname{ess\,supp}(a)$ and

$$A = \mathrm{Op_h}(a) + O(h^\infty)_{\Psi^{-\infty}}.$$

Remark 1. It is easy to see from the definition that for any $A \in \Psi^m$, $\mathrm{WF}(A) \subset \overline{T}^*\overline{\mathbb{R}^d}$ is closed.

The crucial feature of the wavefront set is contained in the following lemma.

Lemma 3.1. *Suppose that* $A \in \Psi^{m_1}$, $B \in \Psi^{m_2}$. *Then,*

$$\mathrm{WF}(AB) \subset \mathrm{WF}(A) \cap \mathrm{WF}(B).$$

As with Theorem 3.1, we do not give the details for the proof of Lemma 3.1. We will instead treat it as one of the axioms of our operators.

3.5. **Boundedness.** With Theorem 3.1, it becomes fairly easy to obtain boundedness on L^2 for pseudodffiferential operators of order 0.

Lemma 3.2. *Let* $m < -d$ *and* $A \in \Psi^m$. *Then,* $A : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ *is uniformly bounded in* h.

Proof. It is enough to check that for $a \in S^m$, $\mathrm{Op_h}(a) : L^2 \to L^2$ is bounded. Note that $\mathrm{Op_h}(a)$ has kernel

$$K(x, y) = \frac{1}{(2\pi h)^d} \int e^{\frac{i}{h}\langle x-y,\xi\rangle} a(x, \xi) d\xi.$$

Put $L^t = \frac{1-\langle D_\xi, \frac{x-y}{|x-y|}\rangle}{1+h^{-1}|x-y|}$ so that for any $N > 0$,

$$K(x, y) = \frac{1}{(2\pi h)^d} \int e^{\frac{i}{h}\langle x-y,\xi\rangle} (L^t)^N a(x, \xi) d\xi.$$

Now, it is easy to check that for $a \in S^m$,

$$|(L^t)^N a(x, y, \xi)| \leq C_N \langle h^{-1}|x - y|\rangle^{-N} \langle \xi\rangle^m \leq \langle \xi\rangle^m \langle h^{-1}|x - y|\rangle^{-N}$$

In particular, taking $N > d$, since $m < -d$,

$$\int |K(x, y)|dy \leq Ch^{-d} \int \langle h^{-1}|x-y|\rangle^{-N} dy \leq C, \quad \int |K(x, y)|dx \leq Ch^{-d} \int \langle h^{-1}|x-y|\rangle^{-N} dx \leq C,$$

we may apply the Schur test for L^2 boundedness (see Lemma A.1) to see that $\mathrm{Op_h}(a)$ is bounded on L^2. □

Lemma 3.3. *Suppose that* $A \in \Psi^0$. *Then,* $A : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ *is uniformly bounded in* h. *Moreover, for all* $\delta > 0$ *there is* $C > 0$ *such that*

$$\|A\|_{L^2\to L^2} \leq (1 + \delta) \sup |\sigma(A)| + Ch^{\frac{1}{2}}. \tag{3.2}$$

Proof. We first claim that for all $m < 0$, any $A \in \Psi^m$ is bounded on L^2 uniformly as $h \to 0$. Indeed, by Lemma 3.2, if $m < -d$, then A is uniformly bounded on L^2.

Now, let $m_1 < 0$ and suppose the claim hold for $m \leq m_1$. Let $m \leq m_1/2$ and $A \in \Psi^m$. Then, by Theorem 3.1, $A^* \in \Psi^m$ and $A^*A \in \Psi^{2m}$. In particular, since $m \leq m_1/2$, $2m \leq m_1$, A^*A is bounded on L^2 uniformly as $h \to 0$ which implies that $A : L^2 \to L^2$ is also bounded uniformly as $h \to 0$. By induction, we then have the claim.

We now need to show that if $A \in \Psi^0$, then $A : L^2 \to L^2$ is uniformly bounded. Let $M = (1 + \delta) \sup_{T^*\mathbb{R}^d} |\sigma_0(A)|$, and put

$$b(x, \xi) = (M - |\sigma_0(A)|^2(x, \xi))^{\frac{1}{2}}.$$

One can check that since $M^2 - |\sigma_0(A)|^2 \geq c > 0$, $b \in S^0_{\mathrm{phg}}$. (See, exercise 3.4.)

Now, since

$$\sigma_0(\mathrm{Op_h}(b)^* \mathrm{Op_h}(b) - (M^2 - A^*A))) = 0,$$

by the first part of Theorem 3.1, we have

$$\mathrm{Op_h}(b)^* \mathrm{Op_h}(b) = M^2 - A^*A + hE$$

for some $E \in \Psi^{-1}$. Therefore, since $\mathrm{Op_h}(b)^* \mathrm{Op_h}(b) \geq 0$,

$$\|Au\|_{L^2}^2 \leq M^2 \|u\|_{L^2}^2 + h\langle Eu, u\rangle.$$

But, since $E \in \Psi^{-1}$, $E : L^2 \to L^2$ is uniformly bounded in h, and hence

$$\|Au\|_{L^2}^2 \leq (M^2 + Ch)\|u\|_{L^2}^2,$$

which completes the proof of the lemma after recalling that $\sqrt{a+b} \leq \sqrt{a} + \sqrt{b}$ □

3.6. **Ellipticity and inverses.** We now define the notion of ellipticity for pseudodifferential operators.

Definition 3.3. Let $A \in \Psi^m$. For $(x_0, \xi_0) \in \overline{T}^*\overline{\mathbb{R}^d}$, we say that A *is elliptic at* (x_0, ξ_0), and write $(x_0, \xi_0) \in \mathrm{ell}(A)$, if there is a neighborhood, $U \subset \overline{T}^*\overline{\mathbb{R}^d}$ of (x_0, ξ_0) and $c > 0$ such that

$$|\sigma(A)(x,\xi)| \geq c\langle \xi\rangle^m, \qquad (x,\xi) \in U \cap T^*\mathbb{R}^d.$$

Remark 2. It is easy to see from the definition that for any $A \in \Psi^m$, $\mathrm{ell}(A) \subset \overline{T}^*\overline{\mathbb{R}^d}$ is open.

Ellipticity gives an appropriate conditions which guarantee that A is invertible on a subset of $\overline{T}^*\mathbb{R}^d$ in the following sense.

Lemma 3.4 (Elliptic parametrix)**.** *Suppose that* $A \in \Psi^{m_1}$ *and* $B \in \Psi^{m_2}$ *with* $\mathrm{WF}(B) \subset \mathrm{ell}(A)$. *Then there are* $E_L, E_R \in \Psi^{m_2 - m_1}$ *such that*

$$B = E_L A + O(h^\infty)_{\Psi^{-\infty}}, \qquad B = AE_R + O(h^\infty)_{\Psi^{-\infty}}.$$

As with many constructions in semiclassical analysis, this lemma is proved by an iterative construction. The nonlinear part of the construction is done by solving a top order equation, and then each successive iteration involves only the solution of a linear equation. In the case of the elliptic parametrix construction, this is particularly simple since the equations involved are algebraic.

Proof. Let $e = \sigma(B)/\sigma(A)$. Then, since $\mathrm{WF}(B) \subset \mathrm{ell}(A)$, $|\sigma(A)| > c > 0$ on $\mathrm{supp}\,\sigma(B)$, and hence, by Exercise 3.4, $e \in S^{m_2 - m_1}_{\mathrm{phg}}$. Putting $E_{L,0} := \mathrm{Op_h}(e)$, we have

$$\sigma_{m_2}(E_{L,0}A - B) = 0,$$

and therefore,

$$E_{L,0}A = B + hR_1,$$

with $R_1 \in \Psi^{m_1 - 1}$.

Suppose we have found e_i, $i = 0, 1, \dots, N-1$, $e_i \in S^{m_2 - m_1 - i}_{\mathrm{phg}}$ such that $\mathrm{supp}\, e \subset \mathrm{WF}(B)$, and, with $E_{L,N-1} := \sum_{j=0}^{N-1} h^j \mathrm{Op_h}(e_j)$, we have

$$E_{L,N-1}A = B + h^N R_N, \tag{3.3}$$

for some $R_N \in \Psi^{m_2-N}$. Now, since $\operatorname{supp} e_i \subset \operatorname{WF}(B)$, $\operatorname{WF}(E_{L,N-1}) \subset \operatorname{WF}(B)$ and hence,

$$\operatorname{WF}(R_N) = \operatorname{WF}(h^{-N}(B - E_{L,N-1}A)) \subset \operatorname{WF}(B).$$

Therefore $\operatorname{WF}(R_N) \subset \operatorname{ell}(A)$ and hence $e_N := -\sigma(R_N)/\sigma(A) \in S^{m_2-N-m_1}_{\mathrm{phg}}$ and

$$(E_{L,N-1} + h^N \operatorname{Op_h}(e_N))A - B = h^N(R_N + \operatorname{Op_h}(e_N)A) \in h^N \Psi^{m_2-N},$$

and

$$\sigma(R_N + \operatorname{Op_h}(e_N)A) = 0.$$

Therefore,

$$(E_{L,N-1} + h^N \operatorname{Op_h}(e_N))A - B = h^{N+1} R_{N+1},$$

for some $R_{N+1} \in \Psi^{m_2-N-1}$. In particular, putting $E_{L,N} = \sum_{j=0}^N h^j \operatorname{Op_h}(e_j)$, we have (3.3) with $N-1$ replaced by N. In particular, there are $e_j \in \Psi^{m_2-m_2-j}$ for $j = 0, 1, \dots$ such that (3.3) holds for any N. Setting $E_L \sim \sum_j h^j \operatorname{Op_h}(e_j)$, completes the proof of the first equality.

The proof of the second equality is nearly identical and we leave the details to the reader. □

We now record a few corollaries. of Lemmas 3.3 and 3.4.

Corollary 3.1. *Let $m \in \mathbb{R}$ and $A \in \Psi^m$. Then, for all $s \in \mathbb{R}$, $A : H^s_h \to H^{s-m}_h$ is uniformly bounded as $h \to 0$.*

Proof. Recall that

$$\|u\|^2_{H^s_h} = (2\pi h)^{-d} \|\langle \xi \rangle^s \mathcal{F} u\|^2_{L^2} = \|\operatorname{Op_h}(\langle \xi \rangle^s) u\|^2_{L^2}.$$

We start by showing that for any m, any $A \in \Psi^m$ is uniformly bounded $L^2 \to H^{-m}_h$ and from $H^m_h \to L^2$. For this, we observe that

$$\|Au\|_{H^{-m}_h} = \|\operatorname{Op_h}(\langle \xi \rangle^{-m}) Au\|_{L^2}.$$

Therefore, since $\langle \xi \rangle^{-m} \in S^{-m}_{\mathrm{phg}}$, $\operatorname{Op_h}(\langle \xi \rangle^{-m})A \in \Psi^0$ and hence, there is $C > 0$ such that

$$\|Au\|_{H^{-m}_h} \leq C \|u\|_{L^2}.$$

In particular $A : L^2 \to H^{-m}_h$ is uniformly bounded. Now, since $A^* \in \Psi^m$, $A^* : L^2 \to H^{-m}_h$ is uniformly bounded, and hence $A : H^m_h \to L^2$ is uniformly bounded as claimed.

Now, let $s \in \mathbb{R}$. Then, by Lemma 3.4, there is $E \in \Psi^{-s}$ such that

$$E_s \operatorname{Op_h}(\langle \xi \rangle^s) = I + R,$$

with $R = O(h^\infty)_{\Psi^{-\infty}}$. Therefore,

$$\|Au\|_{H^{s-m}_h} = \|\operatorname{Op_h}(\langle \xi \rangle^{s-m}) A E_s \operatorname{Op_h}(\langle \xi \rangle^s) u\|_{L^2} + \|\operatorname{Op_h}(\langle \xi \rangle^{s-m}) ARu\|_{L^2}$$

Now, since $E_s \in \Psi^{-s}$, $\operatorname{Op_h}(\langle \xi \rangle^{s-m}) A E_s \in \Psi^0$, and hence

$$\|\operatorname{Op_h}(\langle \xi \rangle^{s-m}) A E_s \operatorname{Op_h}(\langle \xi \rangle^s) u\|_{L^2} \leq C \|\operatorname{Op_h}(\langle \xi \rangle^s) u\|_{L^2} = C \|u\|_{H^s_h}.$$

Next, observe that $\operatorname{Op_h}(\langle \xi \rangle^{s-m}) A \in \Psi^s$, and hence, is uniformly bounded $H^s_h \to L^2$. Therefore, for any $N \leq s$

$$\|\operatorname{Op_h}(\langle \xi \rangle^{s-m}) ARu\|_{L^2} \leq \|Ru\|_{H^s_h} \leq C_N h^N \|u\|_{H^{-N}_h} \leq C_N h^N \|u\|_{H^s_h}.$$

All together, we have that there is $C>0$ such that

$$\|Au\|_{H_h^{s-m}} \leq C\|u\|_{H_h^s},$$

and hence the corollary is proved. □

Corollary 3.2. *Suppose that $A \in \Psi^m$ and $\mathrm{ell}(A) = \overline{T}^*\mathbb{R}^d$. Then, there is $h_0>0$ such that for $0<h<h_0$, $A : H_h^s \to H_h^{s-m}$ is invertible and $A^{-1} \in \Psi^{-m}$.*

Proof. Since $\mathrm{ell}(A) = \overline{T}^*\mathbb{R}^d$, by Lemma 3.4, there is $E_L \in \Psi^{-m}$ such that

$$\tilde{E}_L A = I + R,$$

with $R = O(h^\infty)_{\Psi^{-\infty}}$.

Let $s \in \mathbb{R}$. Then, since $\|R\|_{L^2\to L^2} \leq Ch$, there is h_0 such that for $0<h<h_0$, $\|R\|_{L^2\to L^2} \leq \frac{1}{2}$. In particular, $I + R : L^2 \to L^2$ is invertible for $0<h<h_0$. Now,

$$(I+R)^{-1} = I - R + (I+R)^{-1}R^2.$$

Therefore, since $R = O(h^\infty)_{\Psi^{-\infty}}$, and $(I+R)^{-1}R = R(I+R)^{-1}$, $(I+R)^{-1} = I + O(h^\infty)_{\Psi^{-\infty}}$. In particular, $(I+R)^{-1} \in \Psi^0$ and $(I+R)$ is invertible on H_h^s for every s.

Put $E_L = (I+R)^{-1}\tilde{E}_L$. Then, $E \in \Psi^{-m}$, and

$$E_L A = I.$$

A similar argument shows that there is $E_R \in \Psi^{-m}$ such that $AE_R = I$ and hence that A is invertible and $E_R = E_L = A^{-1}$.

□

3.7. **Gårding inequalities.** We now record the easy Gårding inequality.

Lemma 3.5 (Easy Gårding inequality)**.** *Let $\gamma \in \mathbb{R}$ and suppose that $A \in \Psi^m$ and $\mathrm{Re}\,\sigma(A)(x,\xi) \geq \gamma\langle\xi\rangle^m$ for all (x,ξ). Then, for all $\epsilon>0$, there is $h_0>0$ such that for $0<h<h_0$,*

$$\mathrm{Re}\langle Au, u\rangle \geq (\gamma-\epsilon)\|u\|_{H_h^{m/2}}$$

Proof. Let $s \in \mathbb{R}$ and observe that

$$2\,\mathrm{Re}\langle Au, u\rangle = \langle C\,\mathrm{Op_h}(\langle\xi\rangle^{m/2})u, \mathrm{Op_h}(\langle\xi\rangle^{m/2})u\rangle, \tag{3.4}$$

where

$$C = E_m^*(A + A^*)E_m,$$

and $E_m = \mathrm{Op_h}(\langle\xi\rangle^{m/2})^{-1} \in \Psi^{-m/2}$. In particular, $C \in \Psi^0$ self adjoint and $\sigma(C) \geq 2\gamma$. In particular, by Corollary 3.2, for every $s < 2\gamma$, there is h_0 such that $C - s$ is invertible for $0<h<h_0$. Moreover, one can check that this h_0 can be taken uniform in $s < 2\gamma - \epsilon$ (see Exercise 3.6). Therefore, for any $\epsilon>0$, there is $h_0>0$ such that for $0<h<h_0$, the spectrum of C is contained in $[2\gamma-\epsilon, \infty)$ and hence,

$$\langle Cv, v\rangle_{L^2} \geq 2(\gamma-\epsilon)\|v\|_{L^2}^2,$$

and using (3.4), the proof is complete. □

3.8. **Pseudodifferential operators on manifolds.** Everything we have written above has a generalization to a compact manifold M, with $T^*\mathbb{R}^d$ replaced by T^*M, and $\overline{T}^*\mathbb{R}^d$ replaced by $\overline{T}^*M$. The crucial fact used to define these operators is that the class of pseudodifferential operators is invariant under coordinate changes. We will not spend time on the technical difficulties inherent in defining pseudodifferential operators on manifolds, and instead use the previous sections as though they were written for pseudodifferential operators on manifolds.

3.9. **Exercises.**

Exercise 3.1. Show that for all $m \in \mathbb{R}$ and $A \in \Psi^m$, we have $A : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d)$, and $A : \mathcal{S}'(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$. (Hint: For the second part, use duality.)

Exercise 3.2. Suppose that $a, b \in C_c^\infty(T^*\mathbb{R}^d)$. Show that $\mathrm{Op_h}(a)\,\mathrm{Op_h}(b) = \mathrm{Op_h}(c)$, and

$$c \sim \sum_{j=0}^{\infty} \frac{h^j i^j}{j!} \langle D_x, D_\eta\rangle^j a(x,\xi) b(y,\eta)|_{\substack{y=x \\ \eta=\xi}}.$$

Exercise 3.3. This exercise proves what is known as Borel's Theorem: Suppose that $a_j \in S^{m-j}$ for $j = 0, 1, \dots$. Then there exists a symbol $a \in S^m$ such that

$$a \sim \sum_{j=0}^{\infty} h^j a_j.$$

(1) Let $\chi \in C_c^\infty(\mathbb{R})$ with $\chi \equiv 1$ on $[-1, 1]$. Show that if $\{\lambda_j\}_{j=0}^\infty \subset \mathbb{R}$ with $\lambda_j \to \infty$, the sum

$$a(x,\xi) := \sum_{j=0}^{\infty} h^j \chi(\lambda_j h \langle\xi\rangle^{-1}) a_j$$

converges.

(2) Show that there is λ_j increasing with $\lambda_j \to \infty$ such that for any multiindeces $\alpha, \beta \in \mathbb{N}^d$ with $|\alpha| + |\beta| \le j$, we have

$$h^j |\partial_x^\alpha \partial_\xi^\beta \chi(\lambda_j \langle\xi\rangle^{-1} h) a_j| \le 2^{-j} h^{j-1} \langle\xi\rangle^{m-j-|\beta|+1}.$$

(3) With the choice of λ_j from part (2), show that for any $\alpha, \beta \in \mathbb{N}^d$ with $|\alpha| + |\beta| \le N$,

$$\Big|\partial_x^\alpha \partial_\xi^\beta \Big(a - \sum_{j=0}^{N} a_j\Big)\Big| \le C_{\alpha\beta N} h^N \langle\xi\rangle^{m-|\beta|-N},$$

and conclude that

$$a \sim \sum_j h^j a_j.$$

Exercise 3.4. Show that if $b \in S^0_{\mathrm{phg}}$ is real valued, and $b \ge c > 0$, then $\sqrt{b} \in S^0_{\mathrm{phg}}$. Show that if $b \in S^{m_1}_{\mathrm{phg}}$ and $a \in S^{m_2}_{\mathrm{phg}}$ with $|a| > c\langle\xi\rangle^{m_2}$ on $\operatorname{supp} b$, then, $b/a \in S^{m_1-m_2}_{\mathrm{phg}}$.

Exercise 3.5. Show that if $P \in \Psi^m$ and $u \in L^2$ is tempered and satisfies

$$Pu = 0.$$

Then, for any $E \in \Psi^0$ with $\mathrm{WF}(E) \subset \mathrm{ell}(P)$,

$$\|Eu\|_{H_h^s} = O(h^\infty).$$

Exercise 3.6. Suppose that $A \in \Psi^0$ and $\sigma(A) \geq c > 0$. Follow the construction in Lemma 3.4, using Theorem 3.2, to see that for any $\gamma < c$, there is $E_\gamma \in \Psi^0$ such that

$$E_\gamma A = I + O(h^\infty)_{\Psi^{-\infty}},$$

and that for any $\epsilon > 0$ $\{E_\gamma\}_{\gamma \leq c-\epsilon}$ is uniformly bounded in Ψ^0.

4. Damped wave equation

In this section, we study decay for the damped wave equation

$$(\partial_t^2 - \Delta_g + a(x)\partial_t)u = 0, \qquad u|_{t=0} = u_0 \in H^1, \quad \partial_t u|_{t=0} = u_1 \in L^2(M), \tag{4.1}$$

where Δ_g is the (negative definite) Laplace operator on a compact, Riemannian manifold (M, g), and $a \in C^\infty(M; [0, \infty))$. We define the energy of a solution, u to (4.1) by

$$E(t) := \frac{1}{2}\int_M |\partial_t u|^2 + |\nabla_g u|_g^2 dx. \tag{4.2}$$

We then have the following elementary energy estimate.

Lemma 4.1. *The map $t \mapsto E(t)$ is non-increasing. In particular,*

$$E'(t) = -\operatorname{Re}\int_M a|\partial_t u|^2 dx \tag{4.3}$$

Proof. Observe that

$$\begin{aligned} E'(t) &= \operatorname{Re}\int_M (\partial_t u\overline{\partial_t^2 u} + \langle \nabla_g \partial_t u, \nabla_g \overline{u}\rangle_g dx \\ &= \operatorname{Re}\int_M (\partial_t u\overline{(\partial_t^2 u - \Delta_g u)} dx \\ &= \operatorname{Re}\int_M -a|\partial_t u|^2 dx \leq 0. \end{aligned}$$

□

Because of (4.3), it is natural to try to understand when solutions the the damped wave equation can be observed by a. In particular, when

$$\int_0^T \int a|\partial_t u|^2 dx > cE(0).$$

If this were the case, it is not hard to check that there is exponential decay of the energy.

We will use semiclassical tools to find conditions on a which guarantee exponential decay of u. In order to do this, we first introduce a tool that has found many applications in microlocal analysis.

4.1. **Defect measures.** The concept of a defect measure gives a precise notion of where a sequence of functions $\{u_h\}_{0<h<h_0}$ lives in the limit $h \to 0$. More precisely, we consider a family sequence of functions $\{u_{h_n}\}_{n=1}^{\infty}$ with $h_n \to 0$ such that

$$\sup_n \|u_{h_n}\|_{L^2} \leq C < \infty. \tag{4.4}$$

Let μ be a radon measure on T^*M. We say that the sequence $\{u_{h_n}\}$ satisfying (4.4) has *defect measure* μ if

$$\lim_{n\to\infty} \langle Au_{h_n}, u_{h_n}\rangle_{L^2} = \int \sigma(A)d\mu$$

for all $A \in \Psi^{\text{comp}}$.

Theorem 4.1 (Existence of defect measures)**.** *Suppose that $\{u_{h_n}\}_{n=1}^{\infty}$ satisfies* (4.4)*, then there is a subsequence $\{h_{n_k}\}_{k=1}^{\infty}$ and a radon measure μ such that $\{u_{h_{n_k}}\}_{k=1}^{\infty}$ has defect measure μ.*

Proof. Let $\{a_m\}_{m=1}^{\infty} \subset C_c^{\infty}(T^*M)$ be dense in $C_c^0(T^*M)$. Then, observe that by (3.2),

$$\sup_n |\langle \mathrm{Op_h}(a_m)u_{h_n}, u_{h_n}\rangle| \leq C_m$$

In particular, there is a subsequence, $n_{k,1}$ such that

$$\lim_{k\to\infty} \langle \mathrm{Op_h}(a_m)u_{h_{n_{k,1}}}, u_{h_{n_{k,1})}}\rangle =: L(a_1).$$

Then, for each $m \geq 2$, we find a $n_{k,m}$ a subsequence of $n_{k,m-1}$ such that

$$\lim_{k\to\infty} \langle \mathrm{Op_h}(a_m)u_{h_{n_{k,m}}}, u_{h_{n_{k,m})}}\rangle =: L(a_m).$$

Taking $n_k := n_{k,k}$, we then have that

$$\lim_{k\to\infty} \langle \mathrm{Op_h}(a_m)u_{h_{n_k}}, u_{h_{n_k})}\rangle = L(a_m)$$

for all m. Moreover, by (3.2), the map

$$L : \mathrm{span}\{a_m\}_{m=1}^{\infty} \to \mathbb{C}$$

is a bounded functional on subspace of $C_c^0(T^*M)$. In particular, by the Hahn-Banach theorem, since $\{a_m\}_{m=1}^{\infty}$ it has an extension to a bounded linear functional on $C_c^0(T^*M)$ and, in particular, is given by

$$L(a) = \int a d\mu$$

for some finite radon measure μ.

Now, suppose that $a, b \in C_c^{\infty}(T^*M)$, and $\sup |a - b| < \epsilon$. Then, using (3.2) again,

$$\limsup_{k\to\infty} |\langle \mathrm{Op_h}(a-b)u_{h_{n_k}}, u_{h_{n_k}}\rangle| \leq 2\epsilon.$$

In particular, together with density of $\{a_m\}_{m=1}^{\infty}$ in C_c^{∞}, this implies that μ is the defect measure for $u_{h_{n_k}}$. □

Lemma 4.2. *Suppose that $\{u_{h_n}\}_{n=1}^{\infty}$ satisfies* (4.4) *and has defect measure μ. Then μ is positive.*

Proof. Let $a \in C_c^\infty(T^*M)$ with $a \geq 0$. We claim that

$$\int a d\mu \geq 0,$$

and hence that $\mu \geq 0$. To see this, let $\epsilon > 0$. Let $\delta > 0$. Then, by the easy Gårding inequailty (Lemma 3.5),

$$\lim_{n\to\infty} \langle \mathrm{Op_h}(a) u_{h_n}, u_{h_n}\rangle \geq -\delta \|u_{h_n}\|_{L^2}^2 \geq -C\delta.$$

Since the left hand side is independent of δ, and $\delta > 0$ is arbitrary, the claim follows. □

Lemma 4.3. *Suppose that* $\{u_{h_n}\}_{n=1}^\infty$ *satisfies* (4.4) *and has defect measure* μ*. Then* μ *is finite and*

$$\mu(T^*M) \leq \limsup_{n\to\infty} \|u_{h_n}\|_{L^2}^2.$$

Proof. Now, let $K \subset T^*M$ compact and $a_K \in C_c^\infty(T^*M; [0,1])$ with $a \equiv 1$ on K. Then, since μ is positive, for any $\delta > 0$,

$$\mu(K) \leq \int a d\mu = \lim_{n\to\infty} \langle \mathrm{Op_h}(a_K) u_{h_n}, u_{h_n}\rangle \leq (1+\delta) \sup |a_K| \limsup_{n\to\infty} \|u_{h_n}\|^2 \leq (1+\delta) \limsup_{n\to\infty} \|u_{h_n}\|^2,$$

and, since $\delta > 0$ is arbitrary, we have

$$\mu(K) \leq \limsup_{n\to\infty} \|u_{h_n}\|_{L^2}^2.$$

Letting $K \uparrow T^*M$ completes the proof of the lemma. □

Note that while Theorem 4.1 guarantees the existence of defect measures, it says nothing about uniqueness. Indeed, it is certainly possible to find sequences with subsequences having many different defect measures.

Remark 3. We will often abuse notation and omit the sequence from a family of functions $\{u_h\}_{0<h<h_0}$, saying only that u_h has defect measure μ.

4.2. **Defect measures and partial differential equations.** We now study how the defect measures of solutions to partial differential equations behave.

Lemma 4.4. *Let* $P \in \Psi^m$*. Suppose that* u_h *has defect measure* μ*, and satisfies*

$$Pu = o(1)_{L^2}.$$

Then, $\operatorname{supp} \mu \subset \{\sigma(P) = 0\}$.

Proof. Suppose that $a \in C_c^\infty(T^*M)$, and $\operatorname{supp} a \subset \{\sigma(P) \neq 0\}$. Then, $\operatorname{supp} a \subset \mathrm{ell}(P)$, and hence, by Lemma 3.4, there is $E \in \Psi^{\mathrm{comp}}$ such that

$$\mathrm{Op_h}(a) = EP + O(h^\infty)_{\Psi^{-\infty}}.$$

In particular,

$$\langle \mathrm{Op_h}(a) u_h, u_h\rangle = \langle EPu_h, u_h\rangle + O(h^\infty) = o(1),$$

since $Pu_h = o(1)_{L^2}$, and $E : L^2 \to L^2$ is uniformly bounded. In particular,

$$\int a d\mu = \lim_{h\to 0} \langle \mathrm{Op_h}(a) u_h, u_h\rangle = 0.$$

□

One should think of Lemma 4.4 as saying that a solution to $Pu = 0$ lives microlocally on the set where $\sigma(P) = 0$. This should be familiar from the case of Fourier multipliers, where one can literally say that if $m(D)u = 0$, then $\operatorname{supp}\hat{u} \subset \{m(\xi) = 0\}$.

Our next lemma shows that solutions to partial differential equations are invariant under the flow associated to the partial differential operator.

Lemma 4.5. *Let $P \in \Psi^m$ self adjoint with symbol $p = \sigma(P)$. Suppose that u_h has defect measure μ, and satisfies*

$$Pu = o(h)_{L^2}.$$

Then, $H_p^*\mu = 0$, where H_p is the Hamiltonian flow for p.

Proof. Observe that

$$\begin{aligned}\int H_p a d\mu &= \lim_{h\to 0} \frac{i}{h}\langle [P, \mathrm{Op_h}(a)]u, u\rangle \\ &= \lim_{h\to 0} \frac{i}{h}[\langle (P\,\mathrm{Op_h}(a) - \mathrm{Op_h}(a)P)u, u\rangle \\ &= \lim_{h\to 0} \frac{i}{h}[\langle \mathrm{Op_h}(a)u, Pu\rangle - \langle \mathrm{Op_h}(a)Pu, u\rangle = 0.\end{aligned}$$

□

If we return to our motivating example of propagation (Section 3.1.2), we can now explain why the solution to the Schrödinger equation splits into two packets. We know that defect measure for u are invariant under $-\partial_t - 2\xi\partial_x$. Therefore, since the initial data consists of two packets localized at $(0, -1)$ and $(0, 2)$, we obtain propagation tot he right at speed 2 and propagation to the left at speed 4.

4.3. **Observation for the Helmholtz equation.** We will actually derive estimates on the solution to (4.1) from estimates on Helmholtz equation:

$$(-h^2\Delta_g - 1)u = 0.$$

We introduce the following geometric control condition.

Definition 4.1. Let $U \subset S^*M$. Then U satisfies the geometric control condition if for all $(x, \xi) \in S^*M$, the geodesic through (x, ξ) enters U in finite time.

Theorem 4.2. *Suppose U satisfies the geometric control condition. Then for all $a \in S^0(T^*M; [0, 1])$ with $|a| > 0$ on U, there are $h_0 > 0$, $C > 0$ such that for all $0 < h < h_0$, $u \in L^2$ with $-h^2\Delta u \in L^2$,*

$$\|u\|_{L^2} \leq C\|\mathrm{Op_h}(a)u\|_{L^2} + Ch^{-1}\|(-h^2\Delta - 1)u\|_{L^2}. \tag{4.5}$$

Proof. As with many defect measure arguments, we argue by contradiction. Suppose that there is no $C > 0$ such that (4.5) holds. Then, there are $h_n \to 0$ $u_{h_n} \in L^2$, with $\|u_{h_n}\|_{L^2} = 1$ such that

$$\|\mathrm{Op_{h_n}}(a)u_n\|_{L^2} + h_n^{-1}\|(-h_n^2\Delta - 1)u_n\|_{L^2} \leq \frac{1}{n}.$$

By Theorem 4.1, we may assume u_{h_n} has defect measure μ. We now drop the index n, observing simply that

$$\mathrm{Op_h}(a)u = o(1)_{L^2}, \qquad (-h^2\Delta - 1)u = o(h)_{L^2}.$$

We first observe that, since $\operatorname{supp}\mu \subset S^*M$ by Lemma 4.4, for $\chi \in C_c^\infty(T^*M)$ with $\chi \equiv 1$ on S^*M, we have

$$\int |a|^2 d\mu = \int_{S^*M} |a|^2 d\mu = \int_{S^*M} |\chi|^2|a|^2 d\mu = \int |\chi|^2|a|^2 d\mu = \lim_{h\to 0} \| \mathrm{Op_h}(\chi)\,\mathrm{Op_h}(a)u\|_{L^2}^2 = 0. \tag{4.6}$$

Next, by Lemma 4.5, $\mu(H_{|\xi|_g^2} b) = 0$ for all $b \in C_c^\infty(T^*M)$. We claim that this implies $\mu \equiv 0$. To see this, we find $b \in C_c^\infty$ such that $H_{|\xi|_g^2} b + |a|^2 b > 0$ on S^*M.

Let $\rho_0 \in S^*M$. Then, by the geometric control assumption $\exp(T_{\rho_0} H_{|\xi|_g^2})(\rho_0) \in U$ for some $T_{\rho_0} < \infty$. Since U is open, this implies that there is a neighborhood V_{ρ_0} of ρ_0 such that $\exp(T_{\rho_0} H_{|\xi|_g^2})(V_{\rho_0}) \subset U$. Now,

$$S^*M \subset \bigcup_{\rho \in S^*M} V_\rho.$$

Therefore, since S^*M is compact, there are V_{ρ_i}, $i = 1, \dots N$ such that

$$S^*M \subset \bigcup_{i=1}^N V_{\rho_i}.$$

In particular, there is $T > 0$ such that for all $\rho \in S^*M$, there is $0 \le t \le T$ such that $\exp(tH_{|\xi|_g^2})(\rho) \in U$.

Now, put

$$c(\rho) = \frac{1}{T}\int_0^T (T-t)|a|^2(\exp(tH_{|\xi|_g^2})(\rho) dt.$$

Then,

$$\begin{aligned} H_{|\xi|_g^2} c(\rho) &= \frac{1}{T}\int_0^T (T-t)\partial_t[|a|^2(\exp(tH_{|\xi|_g^2})(\rho)] dt \\ &= \frac{1}{T}\int_0^T |a|^2(\exp(tH_{|\xi|_g^2})(\rho)) dt - a(\rho) =: \langle |a|^2\rangle_T(\rho) - |a|^2(\rho) \end{aligned}$$

Now, let $\chi \in C_c^\infty(\mathbb{R})$ with $\chi \equiv 1$ near 0 and put

$$b = e^c \chi(|\xi|_g^2 - 1).$$

Then,

$$H_{|\xi|_g^2} b = e^c\chi(|\xi|_g^2 - 1)(\langle |a|^2\rangle_T - |a|^2),$$

and in particular,

$$H_{|\xi|_g^2} b + |a|^2 b = e^c\chi(|\xi|_g^2 - 1)\langle |a|^2\rangle_T > 0 \qquad \text{on } S^*M,$$

where the fact that $\langle |a|^2\rangle_T > 0$ on S^*M follows from our choice of T and that $|a| > 0$ on U.

Now, observe that, since S^*M is compact, using Lemma 4.5, that $\operatorname{supp}\mu \subset S^*M$, and (4.6), we have

$$0 = \int H_{|\xi|_g^2} b d\mu = \int_{S^*M} H_{|\xi|_g^2} b d\mu = \int_{S^*M} (H_{|\xi|_g^2} b + |a|^2 b) d\mu \ge c\mu(S^*M).$$

In particular, $\mu \equiv 0$.

Finally, we need to show that the L^2 normalization of u_n implies $\mu(T^*M) = 1$, a contradiction. For this, we let $\chi \in C_c^\infty(T^*M)$ with $\chi \equiv 1$ on S^*M. Then, by Lemma 3.4, there is $E \in \Psi^{-2}$ such that

$$\mathrm{Op_h}(1-\chi) = EP + O(h^\infty)_{\Psi^{-\infty}}.$$

In particular,

$$1 = \|u\|^2 = \lim_{h\to 0}\langle \mathrm{Op_h}(\chi)u, u\rangle + \lim_{h\to 0}\langle EPu, u\rangle = \int \chi d\mu = \mu(S^*M).$$

□

4.4. **The resolvent for the damped wave equation and exponential decay of energy.** We now return to the damped wave equation (4.1). Recall that the energy, $E(t)$ defined in (4.2) is non-increasing. We will use an approach based on spectral theory to prove exponential decay for the damped wave equation.

Our first step is to observe that, while we cannot effectively use the Fourier transform in all variables, we can take the adjoint Fourier transform in time to obtain the operator

$$P(\tau) := -\Delta_g - i\tau a(x) - \tau^2 \tag{4.7}$$

Note that, the solution to (4.1) may not be tempered backward in time. Therefore, we will not be able to take the Fourier transform over all time for a solution to (4.1). Instead, we take the Fourier transform only of $1_{[0,\infty)}(t)u(t)$. To follow the conventions of scattering theory, we actually take the adjoint Fourier transform and denote, $v(\tau) := \mathcal{F}^*_{t\to\tau}(1_{[0,\infty)}(t)u(t))$.

In particular,

$$P(\tau)v(\tau) = -iu_0\tau + u_1 + au_0.$$

If we knew that $P(\tau)$ was invertible, we could then write

$$v(\tau) = P(\tau)^{-1}[-iu_0\tau + u_1 + au_0.],$$

and for $t \geq 0$,

$$u(t) = \frac{1}{2\pi}\int e^{-it\tau}P(\tau)^{-1}[-iu_0\tau + u_1 + au_0]d\tau,$$

Finally, if $P(\tau)^{-1}$ has an analytic continuation to $\operatorname{Im}\tau > -\beta - \epsilon$ with reasonable estimates, then,

$$u(t) = \frac{1}{2\pi}\int e^{-it(\tau - i\beta)}P(\tau - i\beta)^{-1}[iu_0(\tau - i\beta) + u_1 + au_0]d\tau = O(e^{-\beta t}).$$

We now make this argument rigorous.

We start by showing that the $P(\tau)^{-1}$ is meromorphic in τ.

Lemma 4.6. *Let $a \in C^\infty(M)$ with $a \geq 0$, and a not identically 0. The operator*

$$P(\tau)^{-1} : L^2(M) \to H^2(M)$$

is a meromorphic family of operators with finite rank poles. It has no poles for $\tau \in \mathbb{R}\setminus\{0\}$, a simple pole at $\tau = 0$, and is holomorphic for $\operatorname{Im}\tau > 0$. Moreover, there is $c \in \mathbb{C}$ and an family of *operators $B(\tau): L^2 \to H^2$ analytic in a neighborhood of* 0 *such that*

$$P(\tau)^{-1}u = \frac{\langle u, c\rangle}{\tau} + B(\lambda).$$

Proof. We start by showing that $P(\tau)^{-1}$ is meromorphic.

Observe that, since $\sigma(-h^2\Delta_g + 1) = |\xi|_g^2 + 1 \geq c\langle\xi\rangle^2$, $(-h^2\Delta_g+1)^{-1} : L^2(M) \to H^2(M)$ exists for $0 < h < h_0$. In particular, for $\tau_0 = is$, and $|s|$ large enough,

$$(-\Delta_g - \tau_0^2)^{-1} : L^2(M) \to H^2(M).$$

We now use $(-\Delta_g - \tau_0^2)^{-1}$ to approximate the inverse of $P(\tau)$. In particular,

$$P(\tau) = (-\Delta_g - \tau_0^2)(I + (-\Delta_g - \tau_0^2)^{-1}(\tau_0^2 - \tau^2 - i\tau a(x))),$$

and therefore, $P(\tau)$ is invertible if and only if

$$I + K(\tau) : L^2 \to L^2$$

is invertible with

$$K(\tau) := (-\Delta_g - \tau_0^2)^{-1}(\tau_0^2 - \tau^2 - i\tau a(x)).$$

Now, $K(\tau): L^2 \to H^2$ and therefore, $K(\tau)$ is compact and therefore, $I + K(\tau)$ is a holomorphic family of Fredholm operators. Finally, since

$$\|(-\Delta_g - \tau_0^2)^{-1}\|_{L^2\to H^2} \leq C|\tau_0|^{-2},$$

and

$$K(\tau_0) = -i(-\Delta_g - \tau_0^2)^{-1}\tau_0 a(x),$$

we have

$$\|K(\tau_0)\|_{L^2\to H^2} \leq C|\tau_0|^{-1} < 1,$$

provided $|\tau_0| \gg 1$. In particular, $I + K(\tau_0)$ is invertible, and hence, by the analytic Fredholm theorem, $\tau \mapsto (I + K(\tau))^{-1}$ is meromorphic. This implies the meromorphy of $P(\tau)^{-1}$.

Next, we study the location of the poles of $P(\tau)^{-1}$. We start by showing that there are no poles in $\operatorname{Im}\tau > 0$. Indeed, since $a \geq 0$,

(4.8)
$$\begin{aligned}
\|P(\tau)u\|_{L^2}\|u\|_{L^2} &\geq \max(|\operatorname{Im}\langle P(\tau)u,u\rangle|, |\operatorname{Re}\langle P(\tau)u,u\rangle|)\\
&\geq \max(|\langle(-\operatorname{Re}\tau a - 2\operatorname{Im}\tau\operatorname{Re}\tau)u,u\rangle|, ||\|\nabla_g u\|^2 + [(\operatorname{Im}\tau)^2 - (\operatorname{Re}\tau)^2]\|u\|^2|)\\
&\geq \max(\langle 2\operatorname{Im}\tau|\operatorname{Re}\tau|\|u\|^2, [(\operatorname{Im}\tau)^2 - (\operatorname{Re}\tau)^2]\|u\|^2)\\
&\geq \frac{1}{2}(\operatorname{Im}\tau)^2\|u\|_{L^2}^2.
\end{aligned}$$

In particular, $P(\tau)$ is injective and hence invertible for $\operatorname{Im}\tau > 0$.

Now, for $\tau \in \mathbb{R}\setminus\{0\}$, suppose that $P(\tau)u = 0$. Then we have

$$0 = \operatorname{Im}\langle P(\tau)u,u\rangle = -i\tau\langle au,u\rangle_{L^2}.$$

In particular, $u \equiv 0$ on $\operatorname{supp} a$ and hence $(-\Delta - \tau^2)u = 0$. But, this is a contradiction by unique continuation for the Laplacian. Thus, there are no poles in $\mathbb{R}\setminus\{0\}$.

Finally, we study the pole at 0. By (4.8), $P(\tau)^{-1}$ has a pole of at most order two at 0. Therefore, there are A_{-1}, A_{-2} operators of finite rank such that

$$P(\tau)^{-1} = \frac{A_{-2}}{\tau^2} + \frac{A_{-1}}{\tau} + B(\tau),$$

where $B(\tau)$ is analytic near $\tau = 0$. Therefore, for $\tau \neq 0$,

$$\tau^2 = P(\tau)A_{-2} + \tau P(\tau)A_{-1} + \tau^2 P(\tau)B(\tau).$$

But, since $B(\tau$ and A_{-1} are bounded, this implies $P(0)A_{-2} = 0$. In particular, $-\Delta_g A_{-2} = 0$, so the range of A_{-2} is contained n the constants i.e. there is $\psi \in L^2$ such $A_{-2}u = \langle u, \psi\rangle$.

Similarly,

$$\tau^2 = A_{-2}P(\tau) + \tau A_{-1}P(\tau) + \tau^2 B(\tau)P(\tau),$$

so $A_{-2}P(0) = 0$, and hence $\langle -\Delta u, \psi\rangle = 0$ for all $u \in H^2$. In particular, $-\Delta_g \psi = 0$, and hence $\psi \equiv c_{-2}$ for some $c_{-2} \in \mathbb{C}$.

With this in had, we can check that

$$0 = P'(0)A_{-2} + P(0)A_{-1}, \qquad 0 = A_{-2}P'(0) + A_{-1}P(0).$$

Using the first equality, we have that

$$-\Delta_g A_{-1}u = -ia(x)\langle u, c_{-2}\rangle.$$

But, since $a \geq 0$, and a is not identically 0, this implies $c_{-2} = 0$ and hence $A_{-2} = 0$. Now that we have $A_{-2} = 0$, we can argue as above to see that $-\Delta_g A_{-1} \equiv 0$, $-A_{-1}\Delta_g \equiv 0$, and hence that there is $c_{-1} \in \mathbb{C}$ such that $A_{-1}u = \langle u, c_{-1}\rangle$. This completes the proof. □

We now introduce the assumption on a necessary to guarantee exponential decay of energy. We say that $a \in C^\infty(M;\mathbb{R})$ satisfies the geometric control condition if for all $(x,\xi) \in S^*M$, there is $T > 0$ such that

$$a(\pi_M(\exp(TH_{|\xi|_g^2})(x,\xi))) > 0.$$

In particular, $T^*_{(\operatorname{supp} a)^o}M$ satisfies the geometric control condition.

Next, we use the observation estimate (4.5) to give estimates on $P(\tau)^{-1}$ in a strip near the real axis.

Lemma 4.7. *Suppose that a satisfies the geometric control condition. Then there are $C > 0$ and $\beta > 0$ such that for*

$$|\operatorname{Re}\tau| \geq 1, \qquad |\operatorname{Im}\tau| \geq -\beta,$$

we have

$$\|P(\tau)^{-1}\|_{L^2 \to H^j} \leq C|\tau|^{j-1}, \qquad j = 0, 1, 2 \tag{4.9}$$

Proof. Since $P(\tau)^{-1}$ has no poles in

$$\{\tau \,:\, \operatorname{Im}\tau \geq 0, \tau \neq 0\},$$

may assume that $|\operatorname{Re}\tau| > \tau_0$ and $|\operatorname{Im}\tau| \leq \beta$. We will consider $\operatorname{Re}\tau > \tau_0$, the other case being similar. Let $h = \frac{1}{\operatorname{Re}\tau}$, then, by Theorem 4.2, there is $C > 0$ such that for τ_0 large enough,

$$\begin{aligned}\|u\|_{L^2} &\leq C\|au\|_{L^2} + h^{-1}\|(-h^2\Delta_g - 1)u\|_{L^2} \\ &\leq C\|au\| + Ch^{-1}\|h^2(-\Delta_g - ia\tau - \tau^2)u\|_{L^2} + Ch\|a\tau u\|_{L^2} + Ch\|[(\operatorname{Re}\tau)^2 - \tau^2]u\|_{L^2} \\ &\leq C\|au\| + Ch\|P(\tau)u\|_{L^2} + C\|au\|_{L^2} + C(\beta + \beta^2)\|u\|_{L^2}.\end{aligned}$$

Choosing $\beta > 0$ small enough, we have

$$\|u\|_{L^2} \leq C\|au\|_{L^2} + Ch\|P(\tau)u\|_{L^2}. \tag{4.10}$$

Now, observe that, since $a \geq 0$,

$$\|au\|_{L^2}^2 \leq \sup|a|\langle au, u\rangle_{L^2},$$

and therefore, Now,

$$\|P(\tau)u\|_{L^2}\|u\|_{L^2} \geq |\operatorname{Im}\langle P(\tau)u, u\rangle\| = \operatorname{Re}\tau\langle au, u\rangle - C\operatorname{Re}\tau\beta\|u\|_{L^2}^2 \geq \operatorname{Re}\tau(c\|au\|^2 - C\beta\|u\|_{L^2}^2).$$

In particular, using this in (4.10), and shrinking β if necessary, we have

$$\|u\|_{L^2} \leq Ch\|P(\tau)u\|_{L^2},$$

which implies the required L^2 estimate by the Fredholm alternative.

To obtain the estimate for $j = 2$, we write

$$\|-\Delta_g u\|_{L^2} \leq \|i\tau au\|_{L^2} + \|\tau^2 u\|_{L^2} \leq C|\tau|^{2-j}\|P(\tau)u\|_{L^2}.$$

The estimate for $j = 1$ follows by interpolation. □

We now study the long time behavior of solutions to the damped wave equation.

Lemma 4.8. *Suppose that a satisfies the geometric control condition. Then there are $C > 0$ and $\beta > 0$ such that for all u solving* (4.1)*,*

$$E(T) \leq Ce^{-\beta T}, \qquad T \geq 0.$$

Proof. Let u solve (4.1), $\chi \in C^\infty(\mathbb{R}; [0, 1])$ with $\chi \equiv 1$ on $[1, \infty)$ and $\operatorname{supp}\chi \subset (0, \infty)$. Rather than taking the sharp cutoff $1_{[0,\infty)}(t)$, we put $v(t) = \chi(t)u(t)$, and

$$\check{v}(\tau) := \int e^{it\tau}v(t)dt,$$

where the integral is understood as the Fourier transform acting on $\mathcal{S}'$. Then,

$$P(\tau)\check{v}(\tau) = \check{f}(\tau),$$

where

$$f(t) = \chi''(t)u(t) + 2\chi'(t)\partial_t u(t) - a(t)\chi'(t)u(t).$$

Note that by the energy estimate, Lemma 4.1,

$$\|f(t)\|_{L^2(\mathbb{R}_t;L^2)} \leq C(\|u_0\|_{H^1} + \|u_1\|_{L^2}).$$

It is easy to see that $\check{v}(\tau)$ is analytic in $\operatorname{Im}\tau > 0$. This has two consequences. First,

$$v(t) = \frac{1}{2\pi}\int_{\operatorname{Im}\tau=1} e^{-it\tau}\check{v}(\tau)d\tau.$$

Second, since $\tau \mapsto P(\tau)$ is analytic, and f is compactly supported in t, $\check{f}(\tau)$ is analytic in τ,

$$P(\tau)\check{v}(\tau) = \check{f}(\tau), \qquad \operatorname{Im}\tau \geq 0,$$

and

$$\|\check{f}(\tau + i\beta)\|_{L^2(\mathbb{R}_\tau;L^2(M))} \leq C_\beta\|f\|_{L^2(\mathbb{R}_t;L^2(M))} \leq C(\|u_0\|_{H^1} + \|u_1\|_{L^2}).)$$

Now, since $P(\tau)^{-1}$ is meromorphic and has no poles in $\operatorname{Im}\tau > 0$, we have

$$\check{v}(\tau) = P(\tau)^{-1}\check{f}(\tau), \qquad \operatorname{Im}\tau > 0,$$

and

$$v(t) = \frac{1}{2\pi}\int_{\operatorname{Im}\tau=1} e^{-it\tau}P(\tau)^{-1}\check{f}(\tau)d\tau.$$

Our goal is to deform the contour from $\operatorname{Im}\tau = 1$ to $\operatorname{Im}\tau = -\beta$ for some $\beta > 0$. Note that, since $P(\tau)^{-1}$ has no poles on in $\operatorname{Im}\tau \geq 0 \setminus \{0\}$, and the estimate (4.9) holds. When can choose $\beta > 0$ such that the only pole in $\operatorname{Im}\tau \geq \beta$ is at 0.

We will justify deforming the contour below, but, once we can, we will obtain

(4.11)
$$\begin{aligned} v(t) &= \frac{1}{2\pi}\int e^{-it\tau-\beta t}P(\tau - i\beta)^{-1}\check{f}(\tau - i\beta)d\tau + i\operatorname{Res}_{\tau=0} e^{-it\tau}P(\tau)^{-1}\check{f}(\tau),\\ -D_t v(t) &= \frac{1}{2\pi}\int e^{-it\tau-\beta t}(\tau - i\beta)P(\tau - ic)^{-1}\check{f}(\tau - i\beta)d\tau + iD_t\operatorname{Res}_{\tau=0}\tau e^{-it\tau}P(\tau)^{-1}\check{f}(\tau), \end{aligned}$$

Using the first equality, we have by the Plancherel formula that

$$\begin{aligned} &\|e^{\beta t}(v(t) - i\operatorname{Res}_{\tau=0} e^{-it\tau}P(\tau)^{-1}\check{f}(\tau))\|_{L^2(\mathbb{R}_t;H^1)}\\ &= C\|P(\tau - i\beta)^{-1}\check{f}(\tau - i\beta)\|_{L^2(\mathbb{R}_\tau;H^1(M))} \qquad (4.12)\\ &\leq C\|\check{f}(\tau - i\beta)\|_{L^2(\mathbb{R}} \leq C\|f\|_{L^2(\mathbb{R}_t;L^2(M)} \leq C(\|u_0\|_{H^1} + \|u_1\|_{L^2}) \end{aligned}$$

Next, recall that $P(\tau)^{-1}$ has a simple pole at 0, and the residue at zero is given in Lemma 4.6. Therefore,

$$\operatorname{Res}_{\tau=0} e^{-it\tau}P(\tau)^{-1}\check{f}(\tau) = \langle\check{f}(0), c\rangle_{L^2(M)}$$

for some $c \in \mathbb{C}$. In particular, using (4.12), we have

$$\|e^{\beta t}\nabla_g v(0)\|_{L^2(\mathbb{R}_t;L^2(M))} \leq C(\|u_0\|_{H^1} + \|u_1\|_{L^2}).$$

Using again that the residue at 0 is simple,

$$D_t\operatorname{Res}_{\tau=0} e^{-it\tau}P(\tau)^{-1}\check{f}(\tau) = 0, \qquad (4.13)$$

Therefore, combining (4.13), (4.11), and Plancherel's formula again, we have

$$\begin{aligned} &\|e^{\beta t}D_t((v(t)))\|_{L^2(\mathbb{R}_t;L^2)}\\ &= C\|(\tau - i\beta)P(\tau - i\beta)^{-1}\check{f}(\tau - i\beta)\|_{L^2(\mathbb{R}_\tau;L^2(M))}\\ &\leq C\|\check{f}(\tau - i\beta)\|_{L^2(\mathbb{R}_\tau;L^2(M))} \leq C\|f\|_{L^2(\mathbb{R}_t;L^2(M)} \leq C(\|u_0\|_{H^1} + \|u_1\|_{L^2}). \end{aligned}$$

Now, since $E(t)$ is decreasing, we have for $T\geq 2$,

$$\begin{aligned}E(T)&\leq \frac{1}{2}\int_{T-1}^{T}E(t)dt\\&=\frac{1}{2}(\|\nabla_g v\|^2_{L^2([T-1,T];L^2(M))}+\|\partial_t v\|^2_{L^2([T-1,T];L^2(M))})\\&\leq Ce^{-\beta(T-1)}(\|e^{\beta t}\nabla_g v\|_{L^2(\mathbb{R};L^2(M))}+\|e^{\beta t}\partial_t v\|_{L^2(\mathbb{R};L^2(M))}\quad \leq Ce^{-\beta(T-1)}(\|u_0\|_{H^1}+\|u_1\|_{L^2})\\&\leq Ce^{-\beta T}E(0).\end{aligned}$$

Increasing C if necessary, and using that $E(t)$ is decreasing, we can assume that

$$E(T)\leq Ce^{-\beta T}E(0),\qquad 0\leq T<\infty.$$

It remains only to justify the contour deformation. For this, recall that on $\operatorname{Im}\tau=1$,

$$\|P(\tau)^{-1}\check{f}(\tau)\|_{L^2(M)}\leq C|\tau|^{-1}\|\check{f}(\tau)\|_{L^2(M)}.$$

Therefore, by Cauchy-Schwarz

$$\begin{aligned}&\lim_{R\to\infty}\Big\|\int_{\operatorname{Im}\tau=1,|\operatorname{Re}\tau|\geq R}e^{-it\tau}P(\tau)^{-1}\check{f}(\tau)d\tau\Big\|_{L^2(M)}\\&\leq C\lim_{R\to\infty}\int_{|s|\geq R}\|P(\tau+i)^{-1}\check{f}(\tau+i)\|_{L^2(M)}d\tau\\&\leq C\lim_{R\to\infty}\Big(\int_{|s|\geq R}C(1+|s|)^{-2}ds\int_{|s|\geq R}\|\check{f}(s+i)\|^2_{L^2(M)}ds\Big)^{\frac{1}{2}}=0.\end{aligned}$$

Next, let $\gamma_{\pm,R}:=\{\pm R+is\,:\,-\beta\leq s\leq 1\}$. Then,

$$\limsup_{R\to\infty}\Big\|\int_{\gamma_{\pm,R}}e^{-it\tau}P(\tau)^{-1}\check{f}(\tau)d\tau\Big\|_{L^2(M)}\leq\limsup_{R\to\infty}\frac{C}{R}\int_{-\beta}^{1}\|\check{f}(\pm R+is)\|_{L^2(M)}ds=0,$$

since the analyticity of f implies $f\in L^\infty_{\mathrm{loc}}(\tau;L^2(M))$.

This completes the proof since now the formulas (4.11) are justified. ☐

4.5. **Exercises.**

Exercise 4.1. Compute the defect measures for the following families of functions:

1. $u_h=(2\pi h)^{-\frac{d}{4}}e^{-|x-x_0|^2/(2h)}e^{i\langle x-x_0,\xi_0\rangle/h}$, $x_0,\xi_0\in\mathbb{R}^d$.
2. $u_h=\chi(x)e^{i\varphi(x)/h}$, $\chi\in C_c^\infty$, $\varphi\in C^\infty(\mathbb{R}^d;\mathbb{R})$.
3. $u_h=\chi(x)e^{i\varphi(x)/h^\alpha}$, $\chi\in C_c^\infty$, $\varphi\in C^\infty(\mathbb{R}^d;\mathbb{R})$, $\alpha\in\mathbb{R}\setminus\{1\}$.

Exercise 4.2. Show that the assumption on U in theorem 4.2 cannot be removed. That is, if there is $(x,\xi)\in S^*M$ such that the geodesic through (x,ξ) does not enter U, then the estimate (4.5) is false.

Exercise 4.3. Show that if a does not satisfy the geometric control condition, then the conclusions of Lemma 4.8 are false.

5. Weyl law

Our next application of the theory of pseudodifferential operators is to prove what is known as the Weyl law. Let (M,g) be a compact, Riemannian manifold, with Laplacian $-\Delta_g$, and $V\in C^\infty(M;\mathbb{R})$. Standard estimates (one version of which we will review below) show that

$$P:=-\Delta_g+V:L^2(M)\to L^2(M)$$

is self-adjoint and has spectrum consisting of only eigenvalues $\{\mu_j\}_{j=1}^\infty\subset\mathbb{R}$ with $\mu_j\to\infty$. We will be interested in the following question: Can we estimate

$$N_P(\mu):=\#\{j\,:\,\mu_j\leq\mu\}.$$

There are many approaches to this question including Dirichlet–Neumann bracketing, the wave method, the heat method, and the method of complex powers. The wave method generally produces the sharpest results, but requires many more tools than we currently have. We will not pursue any of the above methods, instead using a functional calculus approach.

5.1. Basic properties of P.

Recall that in local coordinates,

$$\Delta_g:=\frac{1}{\sqrt{\bar g}}\sum_{i,j=1}^d\partial_{x^i}(g^{ij}\sqrt{\bar g}\partial_{x_j}),$$

where g^{ij} is the inverse of the metric, and $\bar g=|\det g_{ij}|$. Recall also that

$$\sigma(P)=|\xi|_g^2+V(x).$$

Lemma 5.1 (Elliptic Regularity)**.** *Let $K\Subset\mathbb{C}$. Then, for $k\in\mathbb{R}$, there is $C>0$ such that for all $N\in\mathbb{R}$, there is $C_N>0$ such that for all $z\in K$, and $u\in L^2(M)$ satisfying*

$$(P-z)u=f$$

in the sense of distributions,

$$\|u\|_{H_h^{k+2}}\leq C\|f\|_{H_h^k}+C_N\|u\|_{H_h^{-N}}.$$

Proof. Let $\chi\in C_c^\infty(\mathbb{R})$ with $\chi\equiv1$ for $|t|\leq T$. Then, there is $T(K)$ such that for $z\in K$,

$$||\xi|_g^2+V(x)-z|\geq c|\xi|_g^2,\qquad |\xi|_g\geq T(K).$$

In particular, by Lemma 3.4, there is $Q(z)\in\Psi^{-2}$, uniformly bounded for $z\in K$, such that

$$Q(z)(P-z)=\mathrm{Op_h}(1-\chi(|\xi|_g))+O(h^\infty)_{\Psi^{-\infty}}.$$

In particular, using Corollary 3.1,

$$\begin{aligned}\|u\|_{H_h^{k+2}}&\leq\|\mathrm{Op_h}(1-\chi(|\xi|_g))u\|_{H_h^{k+2}}+\|\mathrm{Op_h}(\chi(|\xi|_g))u\|_{H_h^{k+2}}\\&\leq\|Q(z)(P-z)u\|_{H_h^{k+2}}+C_N\|u\|_{H_h^{-N}}+O(h^\infty)\|u\|_{H_h^{-N}}\\&\leq C\|(P-z)u\|_{H_h^k}+C_N\|u\|_{H_h^{-N}},\end{aligned}$$

and the lemma is proved. □

Lemma 5.2. *The operator P is symmetric and for each $z \in \mathbb{C} \setminus \mathbb{R}$,*

$$P - z : H_h^2(M) \to L^2(M)$$

is invertible and for $k \geq 0$,

$$\|(P-z)^{-1}\|_{H_h^k \to H_h^{k+2}} = O(1 + |\operatorname{Im} z|^{-1}).$$

Proof. We check symmetry in local coordinates. Indeed, let U be a local coordinate patch and $u, v \in C_c^\infty(U)$ with coordinates y. Then,

$$\begin{aligned}
\int_M \Delta_g u \bar{v} dx &= \int \bar{g}^{-1/2} \sum_{i,j=1}^d \partial_{y^i}(g^{ij}\bar{g}^{1/2}\partial_{y^j} u(y))\bar{v}(y)\bar{g}^{1/2} dy \\
&= -\int \sum_{ij=1}^d g^{ij}\bar{g}^{1/2}\partial_{y^j} u(y)\overline{\partial_{y^i} v} dy \\
&= \int u(y)\overline{\bar{g}^{-1/2}\sum_{ij=1}^d \partial_{y^j}(g^{ij}\bar{g}^{1/2}\partial_{y^i} v)\bar{g}^{1/2} dy} \\
&= \int_M u\overline{\Delta_g v} dx
\end{aligned}$$

Next, we show that for $z \in \mathbb{C} \setminus \mathbb{R}$, $P - z$ is injective. Indeed, suppose that $u \in H_h^2(M)$, and $(P - z)u = 0$. Then, by Lemma 5.1, $u \in C^\infty(M)$, and hence,

$$0 = \operatorname{Im}\langle (P-z)u, u\rangle = -\operatorname{Im} z \|u\|_{L^2}^2. \tag{5.1}$$

In particular, since $z \in \mathbb{C} \setminus \mathbb{R}$, $\|u\|_{L^2} = 0$, and hence $u = 0$.

To prove surjectivity, suppose $v \in L^2(M)$ is orthogonal to $(P - z)C^\infty(M)$. Then, for all $u \in C^\infty$,

$$\langle (P-z)u, v\rangle_{L^2} = 0.$$

In particular, $(P - \bar{z})v = 0$ in the sense of distributions. But then, by Lemma 5.1, $v \in C^\infty$, and hence, since $(P - \bar{z})$ is injective, $v = 0$.

We have now shown that $P - z : H_h^2(M) \to L^2(M)$ is invertible, and, by the integration by parts used in (5.1),

$$\|u\|_{L^2(M)} \leq |\operatorname{Im} z| \|(P-z)u\|_{L^2(M)}.$$

Therefore, by Lemma 5.1,

$$\|u\|_{H_h^{k+2}(M)} \leq C\|(P-z)u\|_{H_h^k(M)} + \|u\|_{L^2(M)} \leq C(1 + |\operatorname{Im} z|^{-1})\|(P-z)u\|_{H_h^k(M)},$$

and the proof is complete. □

We can now describe the spectrum of P.

Lemma 5.3. *The operator P with domain $C^\infty(M)$ is essentially self-adjoint and the domain of the closure is $H^2(M)$. Furthermore, there is an orthonormal basis $\{u_j(h)\}_{j=1}^\infty \subset C^\infty(M)$ of $L^2(M)$ consisting of eigenfunctions of P, i.e. satisfying*

$$P(h)u_j = E_j(h)u_j(h),$$

where $E_j \to \infty$ *as* $j \to \infty$*.*

Proof. **Essential self-adjiontness**

Let $\bar{P}$ be the closure of P with domain $C^\infty(M)$. (Existence of this closure follows from the fact that P is densely defined and symmetric.) Now, suppose that $u \in \mathcal{D}(\bar{P})$, i.e. $\bar{P}u \in L^2(M)$. Let $u_n \in C^\infty(M)$ with $u_n \to u \in L^2(M)$ and $Pu_n \to \bar{P}u$. Then,

$$f + iu = (\bar{P} + i)u = \lim_{n\to\infty}(P + i)u_n.$$

In particular,

$$\|u_n\|_{H^2(M)} \leq C\|(P + i)u_n\|_{L^2(M)} \to \|f + iu\|_{L^2(M)} < \infty.$$

In particular, u_n is bounded in $H^2(M)$ and hence, there is a subsequence and $v \in H^2(M)$ such that $u_n \underset{H^2}{\rightharpoonup} v$. On the other hand, $u_n \to u$ in L^2, and hence $u = v \in H^2(M)$. Thus, we have shown that $\mathcal{D}(\bar{P}) \subset H^2(M)$.

On the other hand, if $u \in H^2(M)$, then there is $u_n \in C^\infty$ with $u_n \to u$ in H^2 then it is easy to check that Pu_n converges in L^2, and hence that $\bar{P}u \in L^2$. so that $u \in \mathcal{D}(\bar{P})$. In particular, $\mathcal{D}(\bar{P}) = H^2_h$.

Now, to determine the adjoint of $\bar{P}$, recall that $v \in \mathcal{D}(\bar{P}^*)$ if and only if for all $u \in H^2_h(M)$,

$$|\langle \bar{P}u, v\rangle_{L^2}| \leq C_v\|u\|_{L^2}.$$

Let $v_n \in C^\infty$ with $v_n \to v$. Then, fix $u \in C^\infty$, Then,

$$\langle u, Pv_n\rangle = \langle Pu, v_n\rangle \to \langle Pu, v_n\rangle.$$

In particular, for all $u \in C^\infty(M)$,

$$\lim_{n\to\infty} |\langle u, Pv_n\rangle_{L^2}| = |\langle \bar{P}u, v\rangle_{L^2}| \leq C_v\|u\|_{L^2}.$$

Now, as a distribution, $Pv_n \to Pv \in H^{-2}$, and hence,

$$|\langle u, Pv\rangle_{L^2} \leq C_v\|u\|_{L^2}.$$

This implies that $Pv \in L^2$ and hence $v \in H^2_h(M)$. The fact that $\mathcal{D}(\bar{P}^*) = H^2_h(M)$ now follows easily, and symmetry implies that $\bar{P}^* = \bar{P}$.

Eigenfunctions

To understand the spectrum of P, consider the operator $(\bar{P} + M) : L^2 \to L^2$, then for $u \in C^\infty(M)$,

$$\langle(\bar{P} + M)u, u\rangle = \|\nabla_g u\|^2_{L^2} + M\|u\|_{L^2} + \langle Vu, u\rangle.$$

Therefore, for $M \geq 2\sup|V|$,

$$\|u\|_{L^2} \leq C\|(\bar{P} + M)u\|_{L^2}.$$

Since $(\bar{P} + M)$ is self-adjoint, the same is true for P^*.

Now, Lemma 5.1 implies that

$$\|u\|_{H^2_h} \leq C\|(\bar{P} + M)u\|_{L^2}, \qquad u \in C^\infty(M)$$

and hence, together with self-adjointness, this shows that $\bar{P}+M : H_h^2 \to L^2$, invertible. Thus, $(P+M)^{-1} : L^2 \to L^2$ is compact and the spectral theorem for compact, self-adjoint operators completes the proof.

□

5.2. **The functional calculus.** We now recall that, for a self adjoint operator, P, we can define $f(P)$ for $f : \mathbb{R} \to \mathbb{C}$ using the spectral theorem. In particular, for $P = -h^2\Delta_g + V$, we have

$$f(P)v = \sum_{j=1}^{\infty} f(E_j(h))\langle v, u_j\rangle_{L^2(M)} u_j, \tag{5.2}$$

where $\{u_j\}_{j=1}^{\infty}$ are the eigenfunctions of P. In this section, we will shows that if f is sufficiently nice, then $f(P)$ is a pseudodifferential operator.

Before we proceed, we introduce the notion of an almost analytic extension of a function $f \in \mathcal{S}(\mathbb{R})$. We say that $\tilde{f} \in C^\infty(\mathbb{C})$ is an almost analytic extension of f if

$$\tilde{f}|_{\mathbb{R}} = f, \qquad \bar{\partial}_z \tilde{f}(z) = O(|\operatorname{Im} z|^\infty), \operatorname{supp} \tilde{f} \subset \{|\operatorname{Im} z| \le 1\} \qquad \bar{\partial}_z := \frac{1}{2}(\partial_x + i\partial_y).$$

You will show in exercise 5.1, that every Shwartz function has an almost analytic extension.

The first crucial tool is the Helffer-Sjöstrand formula

Lemma 5.4. *Let $f \in \mathcal{S}$. Then for any almost analytic extension of f, $\tilde{f}$,*

$$f(P) = \frac{1}{\pi i}\int_{\mathbb{C}} \bar{\partial}_z \tilde{f}(z)(P-z)^{-1} dm_{\mathbb{C}},$$

where $m_{\mathbb{C}}$ denotes the Lebesgue measure on $\mathbb{C}$.

Proof. Let $B(t,\epsilon)$ be the disk in $\mathbb{C}$ of radius ϵ around t. Then,

$$\begin{aligned}
\frac{1}{\pi}\int_{\mathbb{C}} \bar{\partial}_z \tilde{f}(z)(t-z)^{-1} dm_{\mathbb{C}} &= \frac{1}{\pi}\lim_{\epsilon\to 0^+}\int_{\mathbb{C}\setminus B(t,\epsilon)} \bar{\partial}_z \tilde{f}(z)(t-z)^{-1} dm_{\mathbb{C}} \\
&= \frac{1}{\pi}\lim_{\epsilon\to 0^+}\int_{\mathbb{C}\setminus B(t,\epsilon)} \bar{\partial}_z (\tilde{f}(z)(t-z)^{-1}) dm_{\mathbb{C}} \\
&= \frac{1}{2\pi i}\lim_{\epsilon\to 0}\oint_{\partial B(t,\epsilon)} \tilde{f}(z)(t-z)^{-1} dz \\
&= \frac{1}{2\pi i}\lim_{\epsilon\to 0}\oint_{\partial B(t,\epsilon)} (f(z)+O(\epsilon))(t-z)^{-1} dz = f(t).
\end{aligned}$$

In the third line we have used Green's formula:

$$-\frac{i}{2}\oint_{\partial B(t,\epsilon)} \tilde{f}(t-x-iy)^{-1}(dx+idy) = \int_{\mathbb{C}\setminus B(t,\epsilon)} \frac{1}{2}(\partial_x + i\partial_y)\tilde{f}(t-x-iy)^{-1} dxdy.$$

The lemma now follows from putting $t = E_j$ and using (5.2). □

We now show that for $f \in \mathcal{S}$, $f(P) \in \Psi^{-\infty}$.

Lemma 5.5. *Suppose that $f \in \mathcal{S}(\mathbb{R})$. Then, $f(P) \in \Psi^{-\infty}$ and $\sigma(f(P)) = f(\sigma(P))$.*

Proof. First, let

$$q_0(z) = (|\xi|_g^2 + V(x) - z)^{-1}.$$

Then, by Lemma Theorem 3.2

$$(P - z)\operatorname{Op_h}(q_0(z)) = I + hOp(r_1(z)),$$

where $r_1 \in S^{-1}$ satisfying

$$|\partial_x^\alpha \partial_\xi^\beta r_1(z)| \le C_{\alpha\beta}|\operatorname{Im} z|^{-K_{1,\alpha\beta}}\langle\xi\rangle^{-1-|\beta|}.$$

Suppose we have $Q_N(z) = \sum_{j=0}^N q_j(z)$ with

$$|\partial_x^\alpha \partial_\xi^\beta q_j(z)| \le C_{\alpha\beta}|\operatorname{Im} z|^{-K_{j,\alpha\beta}}\langle\xi\rangle^{-1-|\beta|},$$

and

$$(P - z)\operatorname{Op_h}(Q_N(z)) = I + h^N \operatorname{Op_h}(r_N(z)),$$

where $r_N \in S^{-N}$ satisfies

$$|\partial_x^\alpha \partial_\xi^\beta r_j(z)| \le C_{\alpha\beta}|\operatorname{Im} z|^{-K_{j,\alpha\beta}}\langle\xi\rangle^{-j-|\beta|}.$$

Then, putting $q_{N+1}(z) = (|\xi|_g^2 + V(x) - z)^{-1} r_N(z)$, we can increase N by 1.

In particular, with $E_M(z) = \operatorname{Op_h}(Q_M(z))$, $R_{M+1}(z) = h^{M+1}\operatorname{Op_h}(r_{M+1}(z))$, we have

$$(P - z)^{-1} = E_M(z) - (P - z)^{-1} R_{M+1}(z),$$

and, one can check that for any s, there are C_s, $K_{M+1,s}$ such that

$$\|R_{M+1}\|_{H_h^s \to H_h^{s+M+1}} \le C_s h^{M+1} |\operatorname{Im} z|^{-K_{M+1,s}}.$$

Therefore, by Lemma 5.4,

$$\begin{aligned}
f(P) &= \frac{1}{\pi}\int_{\mathbb{C}} \bar\partial_z \tilde f(z)(P - z)^{-1} dm_{\mathbb{C}} \\
&= \frac{1}{\pi}\int_{\mathbb{C}} \bar\partial_z \tilde f(z)(E_M(z) - (P - z)^{-1} R_{M+1}(z)) dm_{\mathbb{C}} \\
&= \frac{1}{\pi}\int_{\mathbb{C}} (\operatorname{Op_h}(\bar\partial_z \tilde f(z) Q_M(z)) + \bar\partial_z \tilde f(z)(P - z)^{-1} R_{M+1}(z)) dm_{\mathbb{C}} \\
&= \operatorname{Op_h}(f(|\xi|^2 + V) + \sum_{j=1}^M h^j p_j) + \frac{1}{\pi}\int_{\mathbb{C}} \bar\partial_z \tilde f(z)(P - z)^{-1} R_{M+1}(z)) dm_{\mathbb{C}},
\end{aligned}$$

with $p_j \in S^{-j}$. Now, observe that

$$\begin{aligned}
&\frac{1}{\pi}\int_{\mathbb{C}} |\bar\partial_z \tilde f(z)| \|(P - z)^{-1} R_{M+1}(z))\|_{H_h^s \to H_h^{s+M+3}} dm_{\mathbb{C}} \\
&\le C_N \int_{|\operatorname{Im} z| \le 1} |\operatorname{Im} z|^N \langle \operatorname{Re} z\rangle^{-N} h^{M+1} |\operatorname{Im} z|^{-K_M} dm_{\mathbb{C}} \le C h^{M+1},
\end{aligned}$$

for N chosen large enough. Therefor, putting

$$F \sim \operatorname{Op_h}(f(|\xi|^2 + V) + \sum_{j=1}^\infty h^j p_j),$$

we have

$$f(P) - F = O(h^\infty)_{\Psi^{-\infty}}.$$

So far, we have shown that $f(P) \in \Psi^{-1}$. To see that $f(P) \in \Psi^{-\infty}$, observe that

$$g_k(t) := (t+i)^k f(t) \in \mathcal{S},$$

and therefore, $g_k(P) \in \Psi^{-1}$. Now, since $(P+i)^k \in \Psi^{2k}$ is elliptic, and hence $(P+i)^{-k} \in \Psi^{-2k}$, and

$$f(P) = (P+i)^{-k} g_k(P) \in \Psi^{-2k-1}.$$

Since k is arbitrary, the proof is complete. □

To prove the Weyl law, we will need the trace class properties of pseudodifferential operators found in Exercise 5.2.

Theorem 5.1 (Weyl law)**.** *Let M be smooth, compact Riemannian manifold with dimension d, P as above and $a < b$. Then,*

$$N(a,b,h) := \#\{E(h) \,:\, a \leq E(h) \leq b\} = \frac{1}{(2\pi h)^d} \operatorname{Vol}_{T^*M}\{a \leq |\xi|_g^2 + V(x) \leq b\} + o(h^{-d}).$$

Proof. Fix $\epsilon > 0$ and let $\chi_\pm \in C_c^\infty(\mathbb{R};[0,1])$ with $\chi_+ \equiv 1$ on $[a,b]$ and $\operatorname{supp}\chi_+ \subset (a-\epsilon, b+\epsilon)$, and $\chi_- \equiv 1$ on $[a+\epsilon, b-\epsilon]$, and $\operatorname{supp}\chi_- \subset (a,b)$. Then, observe that

$$\operatorname{tr}\chi_\pm(P) = \sum_j \chi_\pm(E_j).$$

Therefore, since $0 \leq \chi_-(t) \leq 1_{[a,b]}(t) \leq \chi_+(t) \leq 1$, we have

$$\operatorname{tr}\chi_-(P) \leq N(a,b,h) \leq \operatorname{tr}\chi_+(P). \tag{5.3}$$

Now, we estimate $\operatorname{tr}\chi_\pm(P)$ using Lemma 5.5 and Exercise 5.2. Indeed, $\chi_\pm(P) \in \Psi^{-\infty}$ with $\sigma(\chi_\pm(P)) = \chi_\pm(|\xi|_g^2 + V(x))$. Therefore,

$$\operatorname{tr}\chi_\pm = \frac{1}{(2\pi h)^d} \int \chi_\pm(|\xi|_g^2 + V) dx d\xi + O_\epsilon(h^{-d+1}).$$

Using this in (5.2), we have

$$\frac{1}{(2\pi)^d} \int \chi_-(|\xi|_g^2 + V) dx d\xi + O_\epsilon(h) \leq h^d N(a,b,h) \leq \frac{1}{(2\pi)^d} \int \chi_+(|\xi|_g^2 + V) dx d\xi + O_\epsilon(h^1).$$

Thus, using the right hand inequality, we have

$$\limsup_{h\to 0} h^d N(a,b,h) \leq \frac{1}{(2\pi)^d} \int \chi_+(|\xi|_g^2 + V) dx d\xi,$$

and since the left hand side is independent of ϵ, we may send $\epsilon \to 0$ and use the dominated convergence theorem to obtain

$$\limsup_{h\to 0} h^d N(a,b,h) \leq \frac{1}{(2\pi)^d} \frac{1}{(2\pi)^d} \operatorname{Vol}_{T^*M}\{a \leq |\xi|_g^2 + V(x) \leq b\}.$$

Similarly,

$$\frac{1}{(2\pi)^d} \operatorname{Vol}_{T^*M}\{a \leq |\xi|_g^2 + V(x) \leq b\} \leq \liminf_{h\to 0} h^d N(a,b,h),$$

which completes the proof. □

5.3. **Exercises.**

Exercise 5.1. Show that every $f \in \mathcal{S}(\mathbb{R})$ has an almost analytic extension. By considering the function

$$\tilde{f}(x+iy) = \frac{1}{2\pi}\int e^{i(x+iy)\xi}\hat{f}(\xi)\chi(y\xi)\chi(x)d\xi,$$

where $\chi \in C_c^\infty((-1,1))$ with $\chi \equiv 1$ on $[-\frac{1}{2}, \frac{1}{2}]$.

Exercise 5.2. Show that if M is a compact manifold with dimension d, then for any $m > d$ and $P \in \Psi^{-m}(M)$, P is trace class and

$$\mathrm{tr}(P) = \frac{1}{(2\pi h)^{-d}}\int \sigma(P)(x,\xi)dxd\xi + O(h^{d-1}).$$

Appendix A. Elementary operator estimates

Lemma A.1 (Schur test for boundedness)**.** *Suppose that A is an operator wither kernel K, and*

$$\sup_x \int |K(x,y)|dy \le C_1, \qquad \sup_y \int |K(x,y)|dx \le C_2,$$

then

$$\|A\|^2_{L^2\to L^2} \le C_1C_2.$$

Proof. Observe that

$$|Au(x)|^2 \le \Big(\int |K(x,y)u(y)|dy\Big)^2 \le \int |K(x,y)||u(y)|^2dy\int |K(x,y)|dy \le C_1\int |K(x,y)||u(y)|^2dy.$$

Therefore,

$$\|Au\|^2_{L^2} \le C_1\int |K(x,y)||u(y)|^2dydx \le C_1C_2\|u\|^2_{L^2}$$

□

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