Microlocal Analysis for Hyperbolic Boundary Value Problems

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CHAPTER 1

Introduction

These notes are being prepared for a course in Fall 2017 in the Stanford Department of Mathematics. The ultimate goal of the course is to study the propagation of singularities on manifolds with boundary and parametrices for boundary value problems including the Melrose–Taylor parametrix. Throughout we will focus on the Dirichlet type problems, occasionally digressing to discuss the Dirichlet to Neuamnn map which can be applied to a wide variety of boundary conditions.

The material considered here comes largely from the work of Melrose [], Melrose–Sjöstrand [], Taylor [], Zworski [] and Farris []

The basic example to which these notes apply is that of the wave operator $P = \partial_t^2 - \Delta_g$ posed on $\mathbb{R} \times M$ where M is a compact manifold with boundary ∂M.

1. Notation for manifolds with boundary

2. Supported and Extendible distributions

CHAPTER 2

Microlocal Preliminaries

In this chapter, we briefly review the standard calculus of pseudodifferential operators. **[TODO]references**

1. The Kohn–Nirenberg calculus on $\mathbb{R}^d$

1.1. Symbols. We introduce two symbol classes. We say that a family of smooth functions with parameter $h \in [0, 2)$, $a(x, \xi; h) \in C^\infty(\mathbb{R}^{2d})$ lies in $S^m(\mathbb{R}^d \times \mathbb{R}^d)$ if

$$|\partial_x^\alpha \partial_\xi^\beta a(x, \xi; h)| \le C_{\alpha\beta} \langle \xi \rangle^{m-|\beta|}, \qquad \langle \cdot \rangle := (1 + |\cdot|^2)^{\frac{1}{2}}.$$

We say that $a \in S^m$ is *positively homogeneous of order* m and write $a \in S^m_{\mathrm{hom}}$ if there exists $F : \mathbb{R}^d \to (0, \infty)$ so that for $s \ge 1$ and $|\xi| \ge F(x)$,

$$a(x, s\xi) = s^m a(x, \xi).$$

We say that $a \in S^m$ lies in S^m_{phg} if there exist $a_j(x, \xi) \in S^j_{\mathrm{hom}}$, $j = m, m-1, \dots$ and $C_{\alpha\beta N} > 0$ so that

$$\Big|\partial_x^\alpha \partial_\xi^\beta \Big(a - \sum_{j=0}^{N-1} h^j a_{m-j} \Big)\Big| \le C_{\alpha\beta N} h^N \langle \xi \rangle^{m-N}. \tag{1}$$

For $a \in S^m$, we then define the operator

$$\mathrm{Op_h}(a)u := \frac{1}{(2\pi h)^d} \iint e^{i\langle x-y, \xi\rangle/h} a(x, \xi; h) u(y) dy d\xi$$

at first for $u \in \mathcal{S}$ and then by transposition for $u \in \mathcal{S}'$.

We define the set of *pseudodifferential operators of order* m, $\Psi^m_h(\mathbb{R}^d)$ and the *polyhomogeneous pseudodifferential operators* $\Psi^m_{h,\mathrm{phg}}(\mathbb{R}^d)$ by

$$\Psi^m_h(\mathbb{R}^d) := \{A \in \mathcal{L}(\mathcal{S}'; \mathcal{S}') \mid A = \mathrm{Op_h}(a), \text{ for some } a \in S^m(\mathbb{R}^d \times \mathbb{R}^d)\}$$
$$\Psi^m_{h,\mathrm{phg}}(\mathbb{R}^d) := \{A \in \mathcal{L}(\mathcal{S}'; \mathcal{S}') \mid A = \mathrm{Op_h}(a), \text{ for some } a \in S^m_{\mathrm{phg}}(\mathbb{R}^d \times \mathbb{R}^d)\}.$$

Here $\mathcal{L}$ denotes the set of continuous linear operators.

Remark 1. We note that there are many other ways of quantizing symbols in S^m. However for most purposes, they are the same and in particular, they yield the same classes $\Psi^m_h(\mathbb{R}^d)$. $\Psi^m_{h,\mathrm{phg}}(\mathbb{R}^d)$.

For $A \in \Psi^m_h(\mathbb{R}^d)$, we define the principal symbol map $\sigma_m : \Psi^m_h \to S^m/hS^{m-1}$ by

$$\sigma_m(\mathrm{Op_h}(a)) = a + hS^{m-1}.$$

We sometimes write $a(x, hD)$ for the operator $\mathrm{Op_h}(a)$.

We recall a few facts about pseudodifferential operators. Define the norm

$$\|u\|_{H_h^s} := \|\langle hD\rangle^s u\|_{L^2}$$

for all $u \in H^s(\mathbb{R}^d)$.

Lemma 1.1. *Suppose that $A \in \Psi_h^m(\mathbb{R}^d)$. Then for $s \in \mathbb{R}$, $A : H_h^s(\mathbb{R}^d) \to H_h^{s-m}(\mathbb{R}^d)$ and, moreover, there exists $C > 0$ so that*

$$\|Au\|_{H_h^{s-m}} \leq \sup |\langle\xi\rangle^{-m}\sigma_m(A)|\|u\|_{H_h^s} + Ch\|u\|_{H_h^{s-1}}.$$

Lemma 1.2. *Suppose that $a, b \in S^m_{\mathrm{phg}}(\mathbb{R}^d)$ and $\operatorname{supp} a \cap \operatorname{supp} b = \emptyset$. Then for all $N > 0$,*

$$\mathrm{Op_h}(a)\,\mathrm{Op_h}(b) = O_{H_h^{-N}(\mathbb{R}^d)\to H_h^N(\mathbb{R}^d)}(h^N).$$

Lemma 1.3. *Suppose $0 \leq a \in S^m(\mathbb{R}^d)$. Then there exists $c > 0$ so that for all $u \in C_c^\infty(\mathbb{R}^d)$,*

$$\langle \mathrm{Op_h}(a)u, u\rangle_{L^2} \geq -ch\|u\|_{H_h^{\frac{m-1}{2}}}.$$

Next, we observe that for $a \in S^{m_1}$, $b \in S^{m_2}$,

$$\begin{aligned} \mathrm{Op_h}(a)\,\mathrm{Op_h}(b) &= \mathrm{Op_h}(ab) + hR & R &\in \Psi_h^{m_1+m_2-1} \\ h^{-1}[\mathrm{Op_h}(a), \mathrm{Op_h}(b)] &= \mathrm{Op_h}\big(-i\{a,b\}\big) + hR & R &\in \Psi_h^{m_1+m_2-2} \\ \mathrm{Op_h}(a)^* &= \mathrm{Op_h}(\bar{a}) + hR & R &\in \Psi_h^{m_1-1} \end{aligned} \tag{2}$$

where

$$\{a, b\} := \sum_i \partial_{\xi_i} a \partial_{x_i} b - \partial_{x_i} a \partial_{\xi_i} b.$$

Note that the map $\sigma_m : \Psi^m_{phg} \to S^m$ is well defined if we take $\sigma_m(\mathrm{Op_h}(a)) = a_m$ where a_m is from the expansion (1).

1.2. Pseudodifferential operators on compact manifolds without boundary. Let M be a compact manfiold without boundary. Let

$$\Psi^{-\infty}(M) := \mathcal{L}(\mathcal{D}'(M), C^\infty(M))$$

and

$$h^\infty\Psi^{-\infty} := \{A \in \Psi^{-\infty} \mid A = O_{H_h^{-N}\to H_h^N}(h^N),\ \text{for all } N > 0\}.$$

We say that $A : \mathcal{D}'(M) \to \mathcal{D}'(M)$ lies in $\Psi_h^m(M)$ (respectively $\Psi^m_{h,\mathrm{phg}}(M)$) if

(1) For all $\varphi, \psi \in C_c^\infty$ with $\operatorname{supp}\varphi \cap \operatorname{supp}\psi = \emptyset$, $\varphi A \psi \in h^\infty\Psi^{-\infty}$.
(2) If (U, κ) is a coordinate chart with $\kappa : U \to V \subset \mathbb{R}^d$ a diffeomorphsim and $\varphi, \psi \in C_c^\infty(U)$, $\chi \in C_c^\infty(V)$. Then

$$\chi(\kappa^{-1})^*\varphi A\psi\kappa^*\chi \in \Psi_h^m(\mathbb{R}^d),\ (\text{respectively } \Psi^m_{h,\mathrm{phg}}(\mathbb{R}^d)).$$

We say that $a \in S^m(T^*M)$ if in any coordinates (x,ξ) $a(x,\xi) \in S^m(\mathbb{R}^d \times \mathbb{R}^d)$. Similarly, we say $a \in S^m_{\mathrm{phg}}(T^*M)$ if $a(x,\xi) \in S^m_{\mathrm{phg}}(\mathbb{R}^d \times \mathbb{R}^d)$. Then there is a a symbol map $\sigma_m : \Psi^m_h(M) \to S^m(T^*M)/hS^{m-1}(T^*M)$ given by the following procedure. Let $(U_\alpha, \kappa_\alpha)$ be an atlas on M. Then for $\varphi_\alpha^2 \in C_c^\infty(U_\alpha)$ a partition of unity on M,

$$\sigma(A) = \sum_\alpha \tilde{\kappa}_\alpha^* \sigma((\kappa_\alpha^{-1})^* \varphi_\alpha A \varphi_\alpha \kappa_\alpha^*)$$

where $\tilde{\kappa}_\alpha : T^*U \to T^*\mathbb{R}^d$ is the lift of κ_α as a symplectomorphism. Note that σ_m has the following properties

(1) $0 \to h\Psi_h^{m-1} \to \Psi_h^m \xrightarrow{\sigma_m} S^m/hS^{m-1} \to 0$ is exact.
(2) $\sigma(AB) = \sigma(A)\sigma(B)$.
(3) $\sigma([A,B]) = -i\{\sigma(A), \sigma(B)\}$ where $\{\cdot,\cdot\}$ is the poisson bracket.

Finally, there is a non-canonical quantization map $\mathrm{Op_h} : S^m(T^*M) \to \Psi^m_h(T^*M)$ with $\mathrm{Op_h} : S^m_{\mathrm{phg}}(T^*M) \to \Psi^m_{h,\mathrm{phg}}(T^*M)$ so that

(1) For $a \in S^m(T^*M)$, $\sigma_m(\mathrm{Op_h}(A)) = a + hS^{m-1}$.
(2) For all $A \in \Psi^m_h(T^*M)$ (respectively $\Psi^m_{h,\mathrm{phg}}(M)$), there exists $a \in S^m(T^*M)$ (respectively $S^m_{h,\mathrm{phg}}(M)$) so that
$$A - \mathrm{Op_h}(a) \in h^\infty \Psi^{-\infty}.$$
(3) If $a, b \in S^\infty_{\mathrm{phg}}$ and $\operatorname{supp} a \cap \operatorname{supp} b = \emptyset$.
$$\mathrm{Op_h}(a)\,\mathrm{Op_h}(b) \in h^\infty \Psi^{-\infty}(M).$$

We will also write $S^{\mathrm{comp}}(T^*M)$ for symbols compactly supported in T^*M and Ψ_h^{comp} for their quantizaitons.

We will sometimes have reason to use the standard calculus (i.e. $h = 1$). In this case, we write Op for the quantization and Ψ^m for the operators resulting from S^m.

CHAPTER 3

Basic estimates for hyperbolic equations on manifolds with boundary

1. Energy Estimates and Well Posedness

1.1. Estimates without a boundary. We will work in the case of second order operators, but the methods developed here apply equally well to higher order equations.

1.1.1. *First order operators.* We consider the problem

$$P_t := (D_t - \mathrm{Op}(a_t))u = f, \quad 0 < t < T, \qquad u|_{t=0} = u_0 \tag{3}$$

where

(i) $a_t(x,\xi) = a(t,x,\xi)$ belongs to a bounded set in $S^1(\mathbb{R}^d \times \mathbb{R}^d)$ for $0 \le t \le T$
(ii) $t \mapsto a_t$ is continuous with values in $C^\infty(\mathbb{R}^d \times \mathbb{R}^d)$
(iii) $\mathrm{Im}\, a(t,x,\xi) \ge -M$, $0 \le t \le T$.

We start with an energy estimate

LEMMA 1.1. *Let $s \in \mathbb{R}$. Then for $\lambda \in \mathbb{R}$ large enough and all $u \in C^1([0,T]; H^s(\mathbb{R}^d)) \cap C^0([0,T]; H^{s+1}(\mathbb{R}^d))$ and $p \in [1,\infty]$*

$$\Big(\frac{1}{2}\int_0^T \|e^{-\lambda t}u(t,\cdot)\|_{H^s}^p \lambda dt\Big)^{\frac{1}{p}} \le \|u(0,\cdot)\|_{H^s} + 2\int_0^T e^{-\lambda t}\|P_t u\|_{H^s} dt. \tag{4}$$

PROOF. Then consider $E(t) = e^{-2\lambda t}\|u(t)\|_{L^2}^2$.

$$\begin{aligned}
\partial_t E(t) &= 2\,\mathrm{Re}\langle \partial_t[e^{-\lambda t}u], e^{-\lambda t}u\rangle \\
&= 2\,\mathrm{Re}\, e^{-2\lambda t}\langle \partial_t u, u\rangle - 2\lambda E(t) \\
&= -2e^{-2\lambda t}\,\mathrm{Im}\langle D_t u, u\rangle - 2\lambda E(t) \\
&= -2e^{-2\lambda t}\,\mathrm{Im}\langle P_t u, u\rangle - 2e^{-2\lambda t}\,\mathrm{Im}\langle \mathrm{Op}(a_t)u, u\rangle - 2\lambda E(t) \\
&\le 2\|e^{-\lambda t}P_t u\|E^{1/2}(t) + 2(C-\lambda)E(t)
\end{aligned}$$

where in the last line we apply the sharp Gårding inequality (1.3) together with (i), (iii), and the fact that $u(t) \in H^1$. Choosing $\lambda \ge C$, then

$$\partial_t E(t) \le 2\|e^{-\lambda t}P_t u\|E^{1/2}(t).$$

Integrating in time gives

$$\sup_{0\le\tau\le t} E(\tau) \le E(0) + 2\sup_{0\le\tau\le t} E^{1/2}(t)\int_0^t \|e^{-\lambda s}P_s u\|ds.$$

So,

$$\Big(\sup_{0\le\tau\le t} E^{1/2}(\tau) - \int_0^t \|e^{-\lambda s}P_s u\| ds\Big)^2 \le E(0) + \Big(\int_0^t \|e^{-\lambda s}P_s u\| ds\Big)^2$$

and in particular,

$$e^{-Ct}\|u(t)\| \le \|u(0)\| + 2\int_0^t \|e^{-Cs}P_s u\| ds.$$

Then, taking $\lambda > 2C$ large enough and using that $\|e^{-\lambda t/2}\lambda\|_{L^p(\mathbb{R}_t)} \le 2$ for $\lambda > 0$. So,

$$e^{-\lambda t}\|u(t)\| \le e^{(C-\lambda)t}\|u(0)\| + 2\int_0^t e^{-\lambda s}\|P_s u\| e^{(C-\lambda)(t-s)} ds.$$

and taking $\lambda > 2C$, and integrating both sides in t,

$$\left(\int_0^T \Big(e^{-\lambda t}\|u(t)\|\Big)^p dt\right)^{1/p} \le \|e^{-\lambda t/2}\|_{L^p}\|u(0)\| + 2\left(\int_0^T \Big(\int_0^t e^{-\lambda s}\|P_s u\| e^{(C-\lambda)(t-s)} ds\Big)^p dt\right)^{1/p}.$$

Applying Minkowski's inequality then gives

$$\left(\int_0^T \Big(e^{-\lambda t}\|u(t)\|\Big)^p dt\right)^{1/p} \le \|e^{-\lambda t/2}\|_{L^p}\|u(0)\| + \|e^{-\lambda t/2}\|_{L^p}\int_0^T e^{-\lambda s}\|P_s u\| ds$$

Then, since $\|e^{-\lambda t/2}\|_{L^p} \le (2/\lambda)^{1/p}$, the lemma follows for $s = 0$.

To finish the proof, apply the $s = 0$ case to $\tilde{A}_t = \mathrm{Op}(\langle\xi\rangle^s) A_t \,\mathrm{Op}(\langle\xi\rangle^{-s})$ with $u = \mathrm{Op}(\langle\xi\rangle^s)u$. ☐

THEOREM 1.1 (Well-posedness for first order equations). *Let (i)-(iii) hold and $s \in \mathbb{R}$. Then for all $f \in L^1((0,T); H^s(\mathbb{R}^d))$ and $\phi \in H^s(\mathbb{R}^d)$, there is a unique solutions $u \in C([0,T]; H^s(\mathbb{R}^d))$ of* (3) *and* (4) *holds.*

PROOF. We start with uniqueness. Suppose (3) holds with $\phi = 0$ and $f = 0$. Then, since $u \in C([0,1]; H^s(\mathbb{R}^d))$, $\mathrm{Op}(a_t)u \in C([0,T]; H^{s-1}(\mathbb{R}^d))$ and hence $\partial_t u \in C([0,T]; H^{s-1}(\mathbb{R}^d))$. That is, $u \in C^1([0,T]; H^{s-1})$ and in particular, (4) implies that $u = 0$.

To show existence, we apply the energy estimate to the adjoint problem. Suppose $v \in C_c^\infty((-\infty,T)\times\mathbb{R}^d)$. Then, observe that by (4) applied with $t \mapsto T-t$ gives for any $r \in \mathbb{R}$,

$$\sup_{t\in[0,T]} \|v(t)\|_{H^{-r}(\mathbb{R}^d)} \le C\int_0^T \|(D_t - \mathrm{Op}(a_t)^*)v\|_{H^{-r}(\mathbb{R}^d)} dt.$$

Thus, for $f \in L^1([0,T]; H^r(\mathbb{R}^d))$, $\phi \in H^r(\mathbb{R}^d)$,

$$\begin{aligned}\Big|\int_0^T \langle f(t), v(t)\rangle_{\mathbb{R}^d} dt - i\langle\phi, v(0)\rangle\Big| &\le (\|f(t)\|_{L^1([0,T];H^r(\mathbb{R}^d))} + \|\phi\|_{H^r}) \sup_{[0,T]} \|v(t)\|_{H^{-r}(\mathbb{R}^d)} \\ &\le C(\|f(t)\|_{L^1([0,T];H^s(\mathbb{R}^d))} + \|\phi\|_{H^r}) \\ &\qquad \int_0^T \|(D_t - \mathrm{Op}(a_t)^*)v\|_{H^{-r}(\mathbb{R}^d)} dt.\end{aligned} \tag{5}$$

By the Hahn-Banach theorem there exists $u \in L^\infty([0,T];H^r(\mathbb{R}^d))$ so that

$$\int_0^T \langle u, (D_t - \mathrm{Op}(a_t)^*)v\rangle dt = \int_0^T \langle f(t), v(t)\rangle dt - i\langle \phi, v(0)\rangle$$

for all $v \in C_c^\infty((-\infty,T)\times\mathbb{R}^d)$ in particular, u solves (3) as a distribution, so all we need to do is show that u has the desired regularity.

Let $f_\epsilon \in \mathcal{S}$ and $\phi_\epsilon \in \mathcal{S}$ have $f_\epsilon \to f_0$ in $L^1([0,T];H^s)$ and $\phi_\epsilon \to \phi_0$ in H^s. Then, applying the above arguments, we obtain $u_\epsilon \in L^\infty([0,T];H^s(\mathbb{R}^d))$ with u_ϵ solving (3) for f_ϵ and ϕ_ϵ. Then, $u_\epsilon \in L^\infty([0,T];H^{s+2})$ which implies $\mathrm{Op}(a_t)u_\epsilon \in L^\infty([0,T];H^{s+1})$. The equation then implies $D_t u_\epsilon \in L^\infty([0,T];H^{s+1})$ and in particular, $u_\epsilon \in C^0([0,T];H^s)$. Using the equation again then implies $\partial_t u_\epsilon \in C^0([0,T];H^s)$. In particular, (4) applies to u_ϵ. Therefore,

$$\sup_{[0,T]} \|u_\epsilon(t,\cdot) - u_{\epsilon'}(t,\cdot)\|_{H^s} \le C\Big(\|\phi_\epsilon - \phi_{\epsilon'}\|_{H^s} + \int_0^T \|f_\epsilon - f_{\epsilon'}\|_{H^s} dt\Big).$$

Hence, since ϕ_ϵ and f_ϵ are Cauchy, u_ϵ is Cauchy in $C^0([0,T];H^s(\mathbb{R}^d))$ and in particular converges to $u_0 \in C^0([0,T];H^s(\mathbb{R}^d))$ satisfying (4) with $\phi = \phi_0$ and $f = f_0$. By similar arguments after using (3), u_ϵ is also Cauchy in $C^1([0,T];H^{s-1}(\mathbb{R}^d))$ and hence $u_0 \in C^1([0,T];H^{s-1}(\mathbb{R}^d))$. Therefore, u_0 is the desired solution. □

1.2. Second Order Operators. We now consider second order operators. The same techniques apply to m^{th} order operators of the same type, but for simplicity we work only with second order. We want to study,

$$P = \sum_{j=0}^2 P_j(t,x,D_x)D_t^j$$

where

(i) $P_m = 1$ and $P_j \in C^\infty(\mathbb{R};\Psi^{2-j}_{\text{phg}}(\mathbb{R}^d))$ with $\sigma_{2-j}(P_j) = p_j(t,x,\xi)$, $\sigma_2(P) = p$
(ii) The zeros of the map

$$f : \tau \mapsto \langle\xi\rangle^{-2} p(t,x,\langle\xi\rangle\tau,\xi)$$

are uniformly simple on $\mathbb{R}\times\partial\overline{T^*\mathbb{R}^d}$. That is,

$$|\partial_\tau \langle\xi\rangle^{-2}p(t,x,\langle\xi\rangle\tau,\xi)|^2 + |p(t,x,\langle\xi\rangle\tau,\xi)|^2 > 0, \quad \text{for } \tau\in\mathbb{R},\ (t,x,\xi)\in\mathbb{R}^{d+1}\times\partial\overline{T^*\mathbb{R}^d}.$$

Let $\tilde\lambda_1, \tilde\lambda_2 \in C^\infty(\mathbb{R}\times\partial\overline{T^*\mathbb{R}^d})$ be the zeros of f on $\partial\overline{T^*\mathbb{R}^d}$ and $\lambda_i \in S^1_{\text{phg}}(\mathbb{R}^{d+1}\times\mathbb{R}^d)$ with $\langle\xi\rangle^{-1}\lambda_i|_{\partial\overline{T^*\mathbb{R}^d}} = \tilde\lambda_i$ and $\langle\xi\rangle a^{-1}|\lambda_1 - \lambda_2| > 0$. Then define $\Lambda_i = \lambda_i(t,x,D)$.

The goal is then to factor P in terms of $D_t - \Lambda_i$ with an error involving no derivatives in time. For this, write

$$P = \sum_{j=0}^2 P_j D_t^j = \sum_{j=0}^2 [(D_t - \Lambda_i)P_j + [P_j, D_t] + \Lambda_i P_j]D_t^{j-1}$$

where for $k<0$, $D_t^k = 0$. Then, $[P_j, D_t] \in C^\infty(\mathbb{R};\Psi^{2-j})$, so this is a sum of the form

$$P = (D_t - \Lambda_i)\sum_{j=0}^2 P_j D_t^{j-1} + \sum_{j=0}^1 \tilde P_j D_t^j$$

with $\tilde{P}_j \in S^{2-j}$. Iterating this procedure gives

$$P = (D_t - \Lambda_i)E_i + R$$

where $E_i = \sum_{k=0}^{1} E_{ik} D_t^k$ with $E_{ik} \in C^\infty(\mathbb{R}; \Psi^{1-k})$ and $R \in C^\infty(\mathbb{R}; \Psi^2)$. Taking the symbol of both sides we see that

$$p(t, x, \tau, \xi) = (\tau - \lambda_i)\sigma_1(E_i)(t, x, \tau, \xi) + \sigma_2(R)(t, x, \xi)$$

In particular, setting $\tau = \lambda_i$, we see that $\sigma_2(R)(t, x, \xi) = 0$ and hence $R \in C^\infty(\mathbb{R}; \Psi^1)$. Moreover, $\sigma(E_i) = \tau - \lambda_j$ $j \neq i$.

We summarize this in the next lemma

LEMMA 1.2. *Suppose that P satisfies (i) and (ii) and let λ_i $i = 1, 2$ as above . Then there exist $\Lambda_i(x, D')$, $\tilde{\Lambda}_i(x, D')$ with $\sigma(\Lambda_i) = \lambda_i = \sigma(\tilde{\Lambda}_i)$ so that*

$$P = (D_t - \Lambda_1)(D_t - \Lambda_2) + R(x, D') = (D_t - \tilde{\Lambda}_1)(D_t - \tilde{\Lambda}_2) + \tilde{R}(x, D')$$

with $R, \tilde{R} \in S^1$.

With this factorization in place, we can prove the energy estimate.

LEMMA 1.3. *Let $s \in \mathbb{R}$, $T > 0$, and $u \in C^1([0, T]; H^s) \cap C^0([0, T]; H^{s+1})$ with $Pu \in L^1((0, T); H^s)$. Then*

$$\sup_{0 \le t \le T} \sum_{j<2} \|D_t^j u(t, \cdot)\|_{H^{s+1-j}} \le C_{s,T}\Big(\sum_{j<2} \|D_t^j u(0, \cdot)\|_{s+1-j} + \int_0^T \|Pu(t, \cdot)\|_s dt\Big). \tag{6}$$

PROOF. By the estimate (4), there exists $C > 0$ uniform in $p \in [1, \infty]$, $\lambda > 0$ large so that

$$\left(\int_0^T \lambda\big[e^{-\lambda t}\|E_i u(t, \cdot)\|_{H^s}\big]^p dt\right)^{\frac{1}{p}} \le C\Big(\|E_i u(0, \cdot)\|_{H^s} + \int_0^T e^{-\lambda t}(\|Pu\|_{H^s} + C\|u\|_{H^{s+1}})dt\Big). \tag{7}$$

We next estimate u and $D_t u$ in terms of $E_1 u$ and $E_2 u$. For this, observe that

$$\tau^k = \frac{\lambda_2^k(\tau - \lambda_1) - \lambda_1^k(\tau - \lambda_2)}{\lambda_2 - \lambda_1}$$

and hence with

$$Q_1^k = (\lambda_1 - \lambda_2)^{-1}(x, D)\Lambda_2^k \in C^\infty(\mathbb{R}; \Psi^0)$$
$$Q_2^k = (\lambda_2 - \lambda_1)^{-1}(x, D)\Lambda_1^k \in C^\infty(\mathbb{R}; \Psi^0)$$
$$\sigma_k(Q_1^k E_1 + Q_2^k E_2) = \tau^k$$

and we have

$$D_t^k = Q_1^k E_1 + Q_2^k E_2 - \sum_{j=0}^{1} R_j D_t^j, \qquad R_j \in C^\infty(\mathbb{R}; \Psi^{k-1-j}).$$

So, for each t,

$$\sum_{k=0}^{1} \|D_t^k u\|_{H^{s+1-k}} \le C \sum_i \|E_i u\|_{H^s} + C \sum_{k=0}^{1} \|D_t^k u\|_{H^{s-k}}.$$

Next, observe that

$$\|D_t u\|_{H^{s-1}} \le \|E_1 u\|_{H^{s-1}} + \|\Lambda_2 u\|_{H^{s-1}} + C\|u\|_{H^{s-1}}$$

and hence

$$\sum_{k=0}^{1} \|D_t^k u\|_{H^{s+1-k}} \le C \sum_i \|E_i u\|_{H^s} + C\|u\|_{H^s}. \tag{8}$$

Using (7), together with (4) applied to u with $a_t = 0$,

$$\Big(\int_0^T \lambda[e^{-\lambda t} \sum_{k=0}^{1} \|D_t^k u\|_{H^{s+1-k}}]^p dt\Big)^{\frac{1}{p}} \le \tag{9}$$

$$C_{s,T}\Big(\sum_i \|E_i u(0)\|_{H^s} + \|u(0)\|_{H^s}\Big) + C\int_0^T e^{-\lambda t}(\|Pu\|_{H^s} + \sum_{k=0}^{1} \|D_t^k u\|_{H^{s+1-k}})dt.$$

Letting $p = 1$ and $\lambda \gg 1$, in (9), the last term can be absorbed in the left hand side and we have

$$\int_0^T \lambda[e^{-\lambda t} \sum_{k=0}^{1} \|D_t^k u\|_{H^{s+1-k}}]dt \le C_{s,T}\Big(\sum_{k=0}^{1} \|D_t^k u(0)\|_{H^{s+1-k}} + \int_0^T \|Pu\|_{H^s} dt\Big).$$

Inserting this into the right hand side of (9) gives

$$\Big(\int_0^T \lambda[e^{-\lambda t} \sum_{k=0}^{1} \|D_t^k u\|_{H^{s+1-k}}]^p dt\Big)^{\frac{1}{p}} \le C_{s,T}\Big(\sum_{k=0}^{1} \|D_t^k u(0)\|_{H^{s+1-k}} + \int_0^T \|Pu\|_{H^s} dt\Big).$$

□

We now turn to the well posedness for 2nd order equations.

Theorem 1.2. *Assume that (i) and (ii) hold. Then for $f \in L^1((0,T);H^s)$, $u_j \in H^{s+1-j}$ there exists a unique solution $u \in C^1([0,T];H^s) \cap C^0([0,T];H^{s+1})$ to*

$$Pu = f \ \text{in}\ 0 < t < T, \qquad D_t^j u|_{t=0} = u_j, \ \text{for}\ j < 2.$$

Proof. As before, the uniqueness follows from the energy estimate. Therefore, we need only study existence. Suppose $v \in C_c^\infty((-\infty,T)\times\mathbb{R}^d)$ and $u \in C^\infty(\mathbb{R}^{d+1})$. Then

$$\int_0^T \langle u, P^* v\rangle dt = \int_0^T (Pu, v)dt - i \sum_{j+k<2} \langle D_t^j u(0), E_t^k P_{j+k+1}^* v(0)\rangle \tag{10}$$

Since, P^* satisfies (i) and (ii), integrating backwards in time and applying (7),

$$\sum_{k<2} \|D_t^k v(t)\|_{H^{-s-k}} \le C\int_0^T \|P^* v\|_{H^{-1-s}}.$$

Thus,

$$\Big|\int_0^T (f,v)dt - i\sum_{j+k<2} \langle u_j, E_t^k P_{j+k+1}^* v(0)\rangle\Big| \le C\Big(\int_0^T \|f\|_{H^s} dt + \sum_j \|u_j\|_{H^{s+1-j}}\Big)\int_0^T \|P^* v\|_{H^{-1-s}}.$$

The Hahn Banach theorem implies that there exists $u \in L^\infty((0,T); H_{s+1})$ such that (10) holds with $Pu = f$, $D_t^j u(0) = u_j$.

As before the remaining task is to improve the regularity of u and in fact to show that there is a solution with the desired regularity. We proceed as in the proof of Theorem 1.1. That is, it is enough to show that if $f \in \mathcal{S}$ and $u_j \in \mathcal{S}$, then the resulting solution has the desired regularity. So, suppose that u is the function given above solving

$$Pu = f \in \mathcal{S}, \qquad D_t^j u|_{t=0} = u_j \in \mathcal{S}.$$

Then, for any s, $u \in L^\infty([0,T]; H^{s+1})$ and using the factorization of P

$$(D_t - \Lambda_i)E_i u = f + R_j u$$

So, $E_i u \in L^\infty([0,T]; H^s)$ and hence $D_t E_i u \in L^\infty([0,T]; H^{s-1})$, so $E_i u \in C^0([0,T]; H^{s-1})$. Now, we have by (8) that $D_t u \in L^\infty([0,T]; H^{s-1})$ and hence using the equation $D_t^2 u \in L^\infty([0,T]; H^{s-2})$. Therefore, $D_t u \in C^0([0,T]; H^{s-2})$ and $u \in C^0([0,T]; H^{s-1})$. In particular,

$$u \in \bigcap_s \bigcap_{j=0}^{1} C^j([0,T]; H^{s-j}).$$

Applying the equation proves that in fact u is smooth. □

2. Strict Hyperbolicity

Now, we consider more general second order operators on a manifold X without boundary.

Definition 2.1. Let $P \in \mathrm{Diff}^m(X)$ with principal symbol p is *strictly hyperbolic with respect to* $\phi \in C^\infty(X; \mathbb{R})$ if $p(x, d\phi(x)) \neq 0$ and the map

$$\mathbb{R} \ni \tau \mapsto p(x, \xi + \tau d\phi(x))$$

has m distinct real roots for all $x \in X$ and $\xi \in (T_x^*X \setminus 0) \setminus \mathbb{R}d\phi(x)$.

Example 2.1. *Consider* $X = \mathbb{R}_t \times M_x$ *and* $P = -\partial_t^2 + \Delta_g$. *Then,* $p(t,x,\tau,\xi) = \tau^2 - |\xi|_g^2$. *Consider* $\phi(t,x) = t$. *Then,* $d\phi = dt$ *and so*

$$p(t,x,d\phi,0) = 1 \neq 0, \qquad p(t,x,\tau d\phi,\xi) = \tau^2 - |\xi|_g^2$$

So, P *is strictly hyperbolic with respect to* $\phi = t$.

Our main aim for the rest of this section is to study well posedeness of the Cauchy for P and in particular,

Theorem 2.1. *Let* P *be a differential operator of order* 2 *with* C^∞ *coefficients in* X *and let* $Y \subset X$ *be an open, precomapct subset. Assume that* P *is strictly hyperbolic with respect to* $\phi \in C^\infty(X; \mathbb{R})$ *and define*

$$X_+ = \{x \in X \mid \phi(x) > 0\}, \qquad X_0 := \{x \in X \mid \phi(x) = 0\}.$$

(i) *If* $f \in H^s_{\mathrm{loc}}(X)$ *has support in the closure of* X_+ *then there exists* $u \in H^{s+1}_{\mathrm{loc}}(X)$ *with support in the closure of* X_+ *such that* $Pu = f$ *in* Y.

(ii) If $s \geq 0$, v is a vector field with $v\phi = 1$ near X_0, $f \in \overline{H}^s_{\text{loc}}(X_+)$ and $u_j \in H^{s+1-j}_{\text{loc}}(X_0)$, $j < 2$, then there exists $u \in \overline{H}^{s+1}_{\text{loc}}(X_+)$ such that $Pu = f$ in $X_+ \cap Y$ and $v^j u = u_j$in $X_0 \cap Y$. Moreover u satisfies for every $K \subset X$ compact with K, **[TODO]estimates**

2.0.1. *Anisotropic Sobolev spaces.* We will work locally assuming that $X = \mathbb{R}^d$ and $\phi(x) = x_1$, but before we do so, it will be useful to have certain anisotropic sobolev spaces. For a distribution $u \in \mathcal{S}(\mathbb{R}^d)$ and $m, s \in \mathbb{R}$, we define the norm

$$\|u\|^2_{m,s} := (2\pi)^{-n} \int |\hat{u}(\xi)|^2 (1 + |\xi|^2)^m (1 + |\xi'|^2)^s d\xi$$

We then define $H^{m,s}(\mathbb{R}^d)$ as the closure of $\mathcal{S}$ with respect to the $\|\cdot\|_{m,s}$ norm.

We will actually want to work in the half space $\mathbb{R}^d_+ := \{(x_1, x') \in \mathbb{R}^d \mid x_1 > 0\}$. For this, we use the following notation. For a space of distriutions $F(\mathbb{R}^d)$, we define

$$\overline{F}(\mathbb{R}^d_+) := \{U|_{\mathbb{R}^d_+} \mid U \in F(\mathbb{R}^d)\}.$$

We also define

$$\dot{F}(\overline{\mathbb{R}^d_+}) := \{U \in F(\mathbb{R}^d) \mid \operatorname{supp} U \subset \overline{\mathbb{R}^d_+}\}.$$

We will mostly be concerned with $\overline{H}^{m,s}(\mathbb{R}^d_+)$ and $\dot{H}^{m,s}(\overline{\mathbb{R}^d_+})$. For $u \in \overline{H}^{m,s}(\mathbb{R}^d_+)$, we define the norm

$$\|u\|_{m,s} := \inf\{\|U\|_{m,s} \mid U \in H^{m,s}(\mathbb{R}^d),\, U|_{\mathbb{R}^d_+} = u\}.$$

We define spaces analogously on a manifold X with boundary.

We will need the following lemma

LEMMA 2.1. *Let $u \in \overline{S}'(\mathbb{R}^d_+)$. Then $u \in \overline{H}^{m,s}(\mathbb{R}^d_+)$ if and only if $u \in \overline{H}^{m-1,s+1}(\mathbb{R}^d_+)$ and $D_1 u \in \overline{H}^{m-1,s}(\mathbb{R}^d_+)$;*

$$\frac{1}{2}\|u\|^2_{m,s} \leq \|D_1 u\|^2_{m-1,s} + \|u\|^2_{m-1,s+1} \leq \|u\|^2_{m,s}.$$

Furthermore, $u \in \overline{H}^{m,s}(\mathbb{R}^d_+)$ if and only if $u \in \overline{H}^{m,s-1}$ and $D_j u \in \overline{H}^{m,s-1}(\mathbb{R}^d_+)$ for $1 < j \leq d$ and

$$\|u\|^2_{m,s} = \|u\|^2_{m,s-1} + \sum_{j=2}^d \|D_j u\|^2_{m,s-1}.$$

PROOF. Let $u \in \overline{H}^{m,s}(\mathbb{R}^d_+)$ and $U \in H^{m,s}(\mathbb{R}^d)$ with $U|_{\mathbb{R}^d_+} = u$. Then,

$$\|U\|^2_{m,s} = \|D_1 U\|^2_{m-1,s} + \|U\|^2_{m-1,s+1} = \|U\|^2_{m,s-1} + \sum_{j=2}^d \|D_j U\|^2_{m,s-1}$$

is checked easily from the definition. Therefore, $u \in \overline{H}^{m-1,s+1}$, $D_1 u \in \overline{H}^{m-1,s}$, $u \in \overline{H}^{m,s-1}$, and $D_j u \in \overline{H}^{m,s-1}$ and

$$\|D_1 u\|^2_{m-1,s} + \|u\|^2_{m-1,s+1} \leq \|u\|^2_{m,s}$$

$$\|u\|^2_{m,s-1} + \sum_{j=2}^d \|D_j u\|^2_{m,s-1} \leq \|u\|^2_{m,s}$$

We now need to show the other inclusions. Together with the estimates, they are obtained from the following lemma.

LEMMA 2.2. *Let* $\Lambda_{m,s}(\xi) = (\langle\xi'\rangle + i\xi_1)^m\langle\xi'\rangle^s$. *Then* $\Lambda_{m,s}(D)$ *is an isomorphism of* $\dot{\mathcal{S}}(\overline{\mathbb{R}^d_+})$ *extending by continuity to* $L^2(\mathbb{R}^d_+) \to \dot{H}^{-m,-s}(\overline{\mathbb{R}^d_+})$. *Moreover,* $\bar{\Lambda}_{m,s}(D)$ *extends by continuity from* $\overline{C^\infty_c}(\mathbb{R}^d_+)$ *to an isomorphism from* $\overline{H}^{m,s}(\mathbb{R}^d_+)$ *to* $L^2(\mathbb{R}^d_+)$. *In particular,*

$$\|u\|_{m,s} = \|\bar{\Lambda}_{m,s}(D)u\|_{L^2(\mathbb{R}^d_+)}$$

and $\overline{H}^{m,s}(\mathbb{R}^d_+)$ *consists of those* $u \in \overline{S}'(\mathbb{R}^d_+)$ *with* $\bar{\Lambda}_{m,s}u \in L^2$.

PROOF. One can easily check that $\operatorname{supp}\mathcal{F}^{-1}(\Lambda_{m,s}) \subset \overline{\mathbb{R}^d_+}$ and $\operatorname{supp}\mathcal{F}^{-1}(\bar{\Lambda}_{m,s}) \subset \overline{\mathbb{R}^d_-}$. (For example by the Paley-Weiner theorem.) Therefore, $\Lambda_{m,s}(D)$ maps $\dot{\mathcal{S}}(\overline{\mathbb{R}^d}_+)$ continuously to itself and has inverse $\Lambda_{-m,-s}(D)$. The extension to L^2 then follows since $|\Lambda_{m,s}(\xi)|^2 = \langle\xi\rangle^{2m}\langle\xi'\rangle^{2s}$. The second statement follows by duality. □

□

We record the following consequence of Lemma 2.2 for later use.

COROLLARY 2.1. *Suppose* $u \in \dot{H}^{-m,-s}(\overline{\mathbb{R}^d_+})$. *Then there exist* $u_0 \in \dot{H}^{1-m,-s-1}(\overline{\mathbb{R}^d_+})$ *and* $u_1 \in \dot{H}^{1-m,-s}$ *so that*

$$u = u_0 + D_1 u_1.$$

PROOF. Observe that $\Lambda_{m,s}(D) = (\langle D'\rangle + iD_1)\Lambda_{m-1,s}$. So, since $\Lambda_{m,s} : L^2 \to \dot{H}^{-m,-s}$ is an isomorphism, there exists $v \in L^2$ such that

$$u = \Lambda_{m,s}v = (\langle D'\rangle + iD_1)\Lambda_{m-1,s}v$$

and observing that

$$\langle D'\rangle\Lambda_{m-1,s}v \in \dot{H}^{1-m,-s-1}, \qquad i\Lambda_{m-1,s}v \in \dot{H}^{1-m,-s}.$$

□

2.0.2. *The local problem.* We start with the case $X = \mathbb{R}^d$ and $\phi(x) = x_1$. Then,

$$P = \sum_{j=0}^{2} P_j(x_1, x', D_{x'})D^j_{x_1}, \qquad P_j \in \mathrm{Diff}^{2-j}$$

and by the strict hyperbolicity assumption $0 < |P_2|$. Hence, dividing by a nonzero smooth function we may assume that $P_2 = 1$. Moreover, since P is a homogeneous polynomial of degree 2 in (ξ_1, ξ') for which that map $\xi_1 \mapsto p(x, \xi_1, \xi')$ has two distinct real zeros, we may assume P satisfies the assumptions of Theorem 1.2 for $x \in Y$.

Let $\lambda_1(x, \xi'), \lambda_2(x, \xi')$ be the roots of $p(x, \cdot, \xi')$. Fix $\chi \in C^\infty_c(\mathbb{R}^d)$ with $\chi \equiv 1$ in Y and define

$$\tilde{p} = \prod_{j=1}^{2}(\xi_1 - \tilde{\lambda}_j), \qquad \tilde{\lambda}_i = \chi\lambda_j + (1-\chi)j|\xi'|.$$

Now, let $\tilde{P} \in \mathrm{Diff}^2(\mathbb{R}^d)$ with principal symbol $\tilde{p}$. Then $\tilde{P}$ satisfies the hypotheses of Theorem 1.2 for $x \in \mathbb{R}^d$ and has constant coefficients outside a compact set.

LEMMA 2.3. *Suppose that P is strictly hyperbolic with respect to x_1 in $X \subset \mathbb{R}^d$ and let $Y \subset X$ be open and precompact. If $f \in \overline{H}^{s,t}(\mathbb{R}^d_+)$ and $s \geq 0$, $u_j \in H^{s+t+1-j}(\mathbb{R}^{d-1})$, $j < 2$, then there exists $u \in \overline{H}^{s+1,t}(\mathbb{R}^d_+)$ such that $D_1^j u \in C^0(\mathbb{R}; H^{s+t+1-j}(\mathbb{R}^{d-1}))$ when $x_1 \geq 0$ and*

$$Pu = f \text{ in } Y \cap \mathbb{R}^d_+, \quad D_1^j u = u_j \text{ in } Y \cap (\mathbb{R}^{d-1} \times \{0\}),\ j < 2.$$

Moreover, for $\chi \in C_c^\infty(X)$ with $\chi \equiv 1$ on Y, $C_\chi > 0$ so that we can find such a u with

$$\|u\|_{H^{s+1,t}(\mathbb{R}^d_+)} \leq C_\chi(\|\chi f\|_{s,t} + \sum_j \|\chi u_j\|_{H^{s+t+1-j}})$$

PROOF. Fix $T > \sup\{x_1 \mid x \in \operatorname{supp}\chi\}$. Then, since $s \geq 0$, $f \in L^1((0,T); H^{s+t+1-j})$. Hence, by theorem 1.2 there exists $v \in C^0([0,T]; H^{s+t+1}) \cap C^1([0,T]; H^{s+t})$ with $\tilde{P}v = \chi f$ on $0 < x_1 < T$ and $D_1^j v|_{x_1=0} = \chi u_j$. Note that, letting $X_T := (0,T) \times \mathbb{R}^{d-1}$, by Lemma 2.1 $v \in \overline{H}^{1,s+t}(X_T)$. Moreover, $\tilde{P}v \in \overline{H}^{s,t}(X_T)$.

Now, observe that by Lemma 2.1

$$D_1^j v \in \overline{H}^{k+1-j,s+t-k}(X_T), \qquad k = 0, j < 2.$$

Suppose this holds for $k < s$. Then,

$$D_1^2 v = \tilde{P}v - \sum_{j<2} \tilde{P}_j D_1^j v \in \overline{H}^{k,s+t-k-1}(X_T).$$

In particular, $D_1 v \in \overline{H}^{k+1,s+t-k-1}$ and $v \in \overline{H}^{k+2,s+t-k-1}$ by Lemma 2.1. So, by induction $v \in \overline{H}^{s+1,t}(X_T)$ as desired. Letting $u = \chi v$ gives the desired result. □

Next we give a local version of the first part of Theorem 2.1.

LEMMA 2.4. *Assume P is strictly hyperbolic with respect to x_1 in $X \subset \mathbb{R}^d$ and let $Y \subset X$ be open and precompact. If $f \in \dot{H}^{s,t}(\overline{\mathbb{R}^d_+})$, then there exists $u \in \dot{H}^{s+1,t}(\overline{\mathbb{R}^d_+})$ such that $Pu = f$ in Y.*

PROOF. First, suppose $s \geq 0$. Then, by Lemma 2.3, there exists $v \in \overline{H}^{s+1,t}(\{x_1 > -1\})$ with

$$\tilde{P}v = f,\ -1 < x_1 < T, \qquad D_t^j v|_{x_1=-1} = 0.$$

Observe then that since $\operatorname{supp} f \subset \overline{\mathbb{R}^d_+}$, the energy estimate (6) implies $\operatorname{supp} v \subset \overline{\mathbb{R}^d_+}$ and hence, extending v to $x_1 < -1$ by 0, $v \in \dot{H}^{s+1,t}(\overline{\mathbb{R}^d_+})$.

Thus, we have verified the theory for $s \geq 0$. To prove it for $s < 0$, we proceed by induction. Assume the theorem holds when s is replace by $s+1$. Then by Corollary 2.1,

$$f = f_0 + D_1 f_1$$

where $f_0 \in \dot{H}^{s+1,t-1}$ and $f_1 \in \dot{H}^{s+1,t}$. Now, there exist $u_0 \in \dot{H}^{s+2,t-1}$ and $u_1 \in \dot{H}^{s+2,t}$ such that

$$\tilde{P}u_0 = f_0, \qquad \tilde{P}u_1 = f_1.$$

Now, $U = u_0 + D_1 u_1 \in \dot{H}^{s+1,t}$ and

$$\tilde{P}U - f = [\tilde{P}, D_1]u_1 \in \dot{H}^{s+1,t-1}$$

since $[\tilde{P}, D_1]$ of order at most 1 in the D_1 derivatives and 2 in all derivatives. Then, we have $v \in \dot{H}^{s+2,t-1}$ with $\tilde{P}v = [\tilde{P}, D_1]u_1$ and $u = U - v$ is the desired solution. □

In order to finish the proof of Theorem 2.1, we need some uniqueness for solutions

LEMMA 2.5. *Suppose P is strictly hyperbolic in X with respect to ϕ. Then every $x_0 \in X$ has a fundamental system of neighborhoods V such that $u \in \mathcal{D}'(V)$, $\phi \geq \phi(x_0)$ in $\operatorname{supp} u$, and $Pu = 0$ in V implies $u = 0$ in V.*

PROOF. Take coordinates so that $x_0 = 0$ and $\phi(x) = \phi(0) + x_1 - |x'|^2$. This is possible by the Morse lemma. Then let

$$V_\epsilon = \{x \mid |x_1| < \epsilon^2, |x'| < \epsilon\}.$$

Now, $\xi_1 \mapsto \langle \xi' \rangle^{-2} p(0, \langle \xi' \rangle \xi_1, \xi')$ has distinct real zeros on $\partial \overline{T_0^* \mathbb{R}^{d-1}}$ so the same is true for $x \in V_\epsilon$ provided $0 < \epsilon < \epsilon_0$. Moreover, this ϵ_0 can be chosen uniformly on compact subset of X.

Now, by Lemma 2.4 for $g \in C_c^\infty(V_\epsilon)$, we can find $v \in C^\infty(V_\epsilon)$ with $P^* v = g$ in V_ϵ and $v = 0$ for $x_1 > \epsilon^2 - \delta$ where $\delta > 0$ is chosen so that $g = 0$ on $x_1 > \epsilon^2 - \delta$. Then,

$$0 = (Pu, v) = (u, P^* v) = (u, g)$$

and in particular u is 0 in V_ϵ. □

We finally prove the Theorem 2.1

PROOF. Let $X_\nu \subset X$ be coordinate patches and $Y_\nu \subset X_\nu$ be precompact so that $\bar{Y} \subset \cup_\nu Y_\nu$. For each ν, by Lemma 2.4 we can find $u_\nu \in H^{s+1}_{\mathrm{loc}}(X_\nu)$ vanishing in $\phi < 0$ so that $Pu_\nu = f$ in Y_ν. We choose a covering of $X_0 \cap \bar{Y}$ by open sets V_μ with the properties claimed in Lemma 2.5 and choose V_μ small enough that that if $V_\mu \cap V_{\mu'} \neq \emptyset$, $V_\mu \cup V_{\mu'} \subset Y_\nu$ for some ν. We the define $u = u_\nu$ in V_μ whenever $V_\mu \subset Y_\nu$. Then u is well defined in $V = \cup_\mu V_\mu$ and $Pu = f$ there.

Now, let $\chi \in C_c^\infty(X)$ with $\chi \equiv 1$ in a neighborhood, W of $X_0 \cap \bar{Y}$ so that $\bar{W} \subset V$. Then $P(\chi u) = f$ in W. In particular, there exists $g \in H^s_{\mathrm{loc}}(X)$ and $\epsilon > 0$ so that $g = 0$ when $\phi < \epsilon$ and $P(\chi u) = f - g$ in Y. Therefore, we need to prove the statement with f replaced by g and ϕ replaced by $\phi - \epsilon$.

Since $\bar{Y}$ is compact, and the ϵ_0 in the proof of Lemma 2.5 can be chosen uniformly on compact sets, the ϵ above can be chosen uniformly with ϕ replaced by $\phi - t$ for $t \in [0, \sup_Y \phi(x)]$. Then, since the theorem is trivial if $Y \subset \{\phi < 0\}$, the proof is complete after a finite number of iterations of the argument above.

For the second part of the theorem we appeal to Lemma 2.3 instead of Lemma 2.4

□

COROLLARY 2.2. *Suppose that for $a < b$, $X_{ab} := \{a < \phi(x) < b\}$ is precompact. Then the solution given by Theorem 2.1 is unique when $Y = X_{ab}$.*

PROOF. Suppose that $Pu = 0$ in X_{0b}, $\operatorname{supp} u \subset \{\phi(x) \geq 0\}$. Then, by construction $\operatorname{supp} u \subset \{\phi > \epsilon\}$ where ϵ is uniform on X_{ab}. In particular, iterating finitely many times, $u = 0$.

For uniqueness with $s \geq 0$ an initial conditions imposed, suppose that $Pu = 0$ in X_{0b}, $u \in \overline{H}^{1,t}(X_{0b})$ with $D^j u|_{t=0} = 0$ for $j < 2$. Then, extending u by 0 into $\phi < 0$, we have $u \in H^{1,t}(X)$ we have $\operatorname{supp} u \subset \{\phi(x) \geq 0\}$ and hence as before $u = 0$. □

3. Boundaries

We will focus on the case $m = 2$ throughout these notes. In this case, it is possible to associate a Lorentzian metric to p. Multiplying P by $p(x, d\phi(x))^{-1}$, we assume that $p(x, d\phi) = 1$. Since $m = 2$, in coordinates (x, ξ) on T^*M, we can write

$$p(x, \xi) = \langle G^{-1}(x)\xi, \xi\rangle$$

where $G^{-1}(x)$ is a symmetric invertible matrix. In fact, The invertibility of G follows from the fact that p is strictly hyperbolic with respect to ϕ. In fact, if $G^{-1}(x)\xi = 0$, then

$$p(x, \xi + \tau d\phi) = \langle G^{-1}(x)(\xi + \tau d\phi), (\xi + \tau d\phi)\rangle = \tau^2\langle G^{-1}(x)d\phi, d\phi\rangle = \tau^2 p(x, d\phi) = \tau^2$$

which has a double root at 0.

We then define a symmetric bilinear form on T_x^*M by

$$\langle \xi, \eta\rangle_g =: \langle G^{-1}(x)\xi, \eta\rangle$$

which we identify in the usual way with a metric on T_xX i.e.

$$\langle V, W\rangle_g =: \langle G(x)V, W\rangle.$$

We now classify tangent vectors into three types.

DEFINITION 3.1. We say that $V \in T_xX$

(1) is *time-like* if $\langle V, V\rangle_g > 0$

(2) is *space-like* if $\langle V, V\rangle_g < 0$

(3) is *null* if $\langle V, V\rangle_g = 0$.

We then say that a hypersurface H is spacelike, timelike or null if $p(\nu) < 0$, $p(\nu) > 0$, or $p(\nu) = 0$ respectively where ν a conormal to H.

Throughout this section, we will also make the assumptions that

$$X \ni x \mapsto \phi(x) \text{ is proper}, \qquad \partial X \text{ is time-like.} \tag{11}$$

The proof of Theorems 4.1 and **??** will follow as usual from certain a-priori estimates on the solution.

3.1. Estimates with a boundary.

References: Hormande:III 24.1

4. Existence and Uniqueness

References: Hormande:III 24.1

THEOREM 4.1. *Let* $f \in \bar{H}^s_{\mathrm{loc}}(X^o)$*,* $u_0 \in H^{s+1}(\partial X)$ *where* $s \geq 0$ *and assume* u_0*,* f *vanish on* $\{\phi < a\}$*, that* p *is strictly hyperbolic with respect to* ϕ *and* (11) *holds. Then there is a unique* $u \in \bar{H}^{s+1}_{\mathrm{loc}}(X^o)$ *such that*

$$\begin{cases} Pu = f & \textit{in } X^o \\ u = u_0 & \textit{on } \partial X \\ u = 0 & \textit{on } \phi < a. \end{cases}$$

Moreover u *satisfies for every* $a' < a < b < b'$*,* **[TODO]estimates**

$$\|u\|_{\bar{H}^{s+1}(a<\phi<b)} \leq C(\|f\|_{\bar{H}^s(a'<\phi<b')} + \|u_0\|_{H^{s+1}(\partial X)}).$$

CHAPTER 4

Propagation of Singularities

In this chapter we study an operator $P \in \mathrm{Diff}^2(X)$ for X a manifold with boundary ∂X and interior X^o. We assume throughout that ∂X is non-characteristic for P. That is, p does not vanish on $N^*\partial X$. For propagation in X^o, we will allow P to have an imaginary principal symbol. However, when it comes time to study the problem near the boundary, we will insist that the symbol be real valued.

1. Propagation in the bulk

2. Propagation of singularities for strictly hyperbolic problems

We now prove the propagation of singularities result for pseudodifferential operators.

Theorem 2.1. *Let X be a compact manifold and $P \in \Psi^m_{\mathrm{phg}}(X)$ with $\sigma(P) = p - iq$ with p, q real valued. Suppose that $A, B, B_1 \in \Psi^0_{\mathrm{phg}}(M)$ such that*

(1) for all $(x_0, \xi_0) \in \overline{\mathrm{WF}}(A)$, there exist $T > 0$ so that $\exp(-T\langle\xi\rangle^{-m+1}H_p)(x_0,\xi_0) \in \overline{ell}(B)$,

$$\exp(-t\langle\xi\rangle^{-m+1}H_p)(x_0,\xi_0) \in ell(B_1),\ 0 \le t \le T.$$

(2) $q \ge 0$ on $\mathrm{WF}(B_1)$. Then, for all $u \in \mathcal{D}'(M)$ if $B_1Pu \in H^{s-m-1}(X)$ and $Bu \in H^s$ then $Au \in H^s$ and for all $N > 0$, there exists $C_N > 0$ such that

$$\|Au\|_{H^s} \le C\|B_1Pu\|_{H^{s-m-1}} + C\|Bu\|_{H^s} + C_N\|u\|_{H^{-N}}.$$

2.1. Construction of the escape function. The idea will be to use positivity of the commutator $[P, A]$ to obtain estimates on u in terms of P. In order to do this, we will produce a so-called escape function which is increasing along the flow.

Lemma 2.1. *Let X be a compact manifold without boundary and A, B, B_1 be as in Theorem 2.1. Then there exists $0 \le g \in C^\infty(\partial\overline{T^*X})$ such that there exists $\beta \ge 0$ with*

$$g > 0 \text{ on } \bar{\mathrm{WF}}(A), \qquad \langle\xi\rangle^{-m+1}H_p g \le -\beta g \text{ in a neighborhood of } \partial\overline{T^*X} \setminus ell(B).$$

Proof. Let $\varphi_t(x_0,\xi_0) := \exp(t\langle\xi\rangle^{-m+1}H_p)(x_0,\xi_0)$. We start with the case $\bar{\mathrm{WF}}(A) = \{(x_0,\xi_0)\}$. We may assume that $\langle\xi\rangle^{-m+1}H_p(x_0,\xi_0) \neq 0$ since otherwise any $g \ge 0$ will due. Now, let $T > 0$ so that $\varphi_{-T}(x_0,\xi_0) \in \bar{ell}(B)$. Tke $\Sigma \subset \partial\overline{T^*X}$ a hypersurface through (x_0,ξ_0) transverse to H_p. Then, there exists a neighborhood V of (x_0,ξ_0) in Σ and $\delta > 0$ so that

$$\Phi(t,q) : (-T-\delta,\delta) \times V \ni (t,q) \mapsto \varphi_t(q) \in \overline{T^*X}.$$

is a diffeomorphism onto its image and

$$\Phi(-T-\delta,-T+\delta,V)\subset e\bar{l}l(B),\qquad \Phi(-T-\delta,\delta,V)\subset\bar{\ell}(B_1)$$

Let $0\le\psi(t)\in C_c^\infty(-T-\delta,\delta)$ such that $\psi'(t)\le-\beta\psi$ off of $(-T-\delta/2,T+\delta/2)$ and $\chi(0)>0$. Then, fix $0\le\chi\in C_c^\infty(V)$ and define

$$g(x,\xi)=(\psi\otimes\chi)(\Phi^{-1}(x,\xi)).$$

and extend g by 0 outside of the image of Φ.

Notice that $g>0$ in a neighborhood of (x_0,ξ_0), so by compactness of $\bar{\mathrm{WF}}(A)$, there exist $(x_n,\xi_n)\in\overline{T^*X}$ such that

$$g=\sum_n g_{x_n,\xi_n}>0\qquad\text{on }\bar{\mathrm{WF}}(A),$$

Moreover,

$$H_pg=\sum_n H_pg_{x_n,\xi_n}\le-\beta\sum_n g_{x_n,\xi_n}=-\beta g\text{ in a neighborhood of }\overline{T^*X}\setminus\bar{\ell}(B).$$

. □

We now prove the propagation of singularities estimate

Proof. Fix a volume form on M and write $\operatorname{Im}A=\frac{A-A^*}{2i}$, $\operatorname{Re}A=\frac{A+A^*}{2}$. Let $G=\operatorname{Op}(\langle\xi\rangle^{s+\frac{1-m}{2}}g)$ and $E=\operatorname{Op}(\langle\xi\rangle^{s+\frac{1-m}{2}})$ for some metric on X. For $u\in C^\infty(M)$ consider

$$\operatorname{Im}\langle Pu,G^*Gu\rangle=\frac{\operatorname{Im}\langle\operatorname{Re}Pu,G^*Gu\rangle+\operatorname{Re}\langle\operatorname{Im}Pu,G^*Gu\rangle}{2}$$

[TODO]finish see Dyatlov–Zworski

□

3. The b-wavefront set

[TODO]

4. Propagation near the boundary

With Theorem 2.1 in place, we have a complete understanding of how singularities propagate in X^o (at least provided that H_p is not radial!). Therefore, it remains to understand the behavior near the boundary of X. Throughout this section, we will use **[TODO]** to change coordinates so that

$$p=\xi_1^2-r(x,\xi')$$

where $p(x,\xi)$ is the symbol of p.

We denote by $\pi:T^*_{\partial X}X\to T^*\partial X$ the projection through $N^*\partial X$ and write $\Sigma:=\{\rho\in T^*X\mid p(\rho)=0\}$. We then divide $T^*\partial X$ into three regions, the *elliptic*, *hyperbolic*, and *glancing* regions

$$\begin{aligned}&\mathcal{E}:=\{q\in T^*\partial X\mid\pi^{-1}(q)\cap\Sigma=\emptyset\},\\&\mathcal{H}:=\{q\in T^*\partial X\mid\#(\pi^{-1}(a)\cap\Sigma)=2\},\\&\mathcal{G}:=T^*\partial X\setminus(\mathcal{E}\cup\mathcal{H}).\end{aligned}\tag{12}$$

We will study each of these regions separately. There is no propagation in $\mathcal{E}$ and the propagation in $\mathcal{H}$ results in broken bicharaceteristics. The propagation through $\mathcal{G}$ is subtle and will require a great deal of analysis.

4.1. The elliptic region. Our first task will be to prove the analog of the fact that if X is a compact manifold without boundary, then $\mathrm{WF}(u) \subset \mathrm{WF}(Pu) \cup \{p = 0\}$. Note that the same proof shows that this continues to hold in the interior of a manifold with boundary and in particular, $\mathrm{WF_b}(u)|_{X^o} \subset \mathrm{WF_b}(Pu)|_{X^o}$. Now, with $\Sigma := \{p = 0\} \subset T^*X$, define $\tilde{\Sigma} = \iota(\Sigma) \subset \tilde{T}^*X$ where $\iota : T^*X^o \to \tilde{T}^*X$ is the natural inclusion extended to T^*X. One might hope that analog of the elliptic regularity statement would be

$$\mathrm{WF_b}(u) \subset \tilde{\Sigma} \cup \mathrm{WF_b}(Pu).$$

However, it is easy to see that such a statement cannot hold without imposing some boundary conditions and so, the correct statement should be

$$\mathrm{WF_b}(u) \subset \tilde{\Sigma} \cup \mathrm{WF_b}(Pu) \cup \mathrm{WF}(u|_{\partial X}). \tag{13}$$

Now, notice that in coordinates where $\partial X = \{x_1 = 0\}$

$$\tilde{\Sigma}|_{\partial X} = \{(x', \xi') \in T^*\partial X \mid (0, x', \xi_1, \xi') \in \Sigma\} = \mathcal{H} \cup \mathcal{G}$$

and so to obtain (13) it is enough to show

$$\mathrm{WF_b}(u) \cap \mathcal{E} \subset \big(\mathrm{WF_b}(Pu) \cup \mathrm{WF}(u|_{\partial X})\big) \cap \mathcal{E}.$$

In order to study the elliptic region, we start with a local problem. In particular, we assume that $X \subset \overline{\mathbb{R}^d}_+$ with $\partial X \subset \{x_1 = 0\}$. We then consider

$$P = D_{x_1}^2 + b(x, D')D_{x_1} + c(x, D').$$

We will start by proving elliptic regularity for such an operator.

Lemma 4.1. *Suppose that $u \in \overline{H}^1_{\mathrm{comp}}$, $Pu \in \overline{H}^{-1}$, and $u|_{x_1=0} = 0$ and $T^*\{x_1 = 0\} \subset \mathcal{E}$. Then,*

$$\|u\|_{\overline{H}^1} \le C(\|Pu\|_{\overline{H}^{-1}} + \|u\|_{L^2}).$$

We first need,

Lemma 4.2. *For $u \in \dot{H}^1_{\mathrm{comp}}$,*

$$(D_{x_1}^2 u, u)_{H^{-1}, H^1} = \|D_{x_1} u\|^2.$$

Proof. Since $u \in \dot{H}^1_{\mathrm{comp}}$, we may extend u by 0 to $U \in H^1(\mathbb{R}^d)$. Then, let $u_\epsilon \to U$ in H^1 with $u_\epsilon \in C_c^\infty$ and $\operatorname{supp} u_\epsilon \subset \{x_1 \ge 0\}$. For example, take $\psi \in C_c^\infty(\mathbb{R}^d)$ with $\operatorname{supp} \psi \subset \{x_1 > 0\}$, $\int \psi = 1$ and let $u_\epsilon = \epsilon^{-d}\psi_\epsilon * u$.

$$(D_{x_1}^2 u_\epsilon, u_\epsilon)_{H^{-1}, H^1} = \|D_{x_1} u_\epsilon\|^2.$$

Now,

$$\begin{aligned} |(D_{x_1}^2 u, u) - (D_{x_1}^2 u_\epsilon, u_\epsilon)| &= |(D_{x_1}^2 (u - u_\epsilon), u) + (D_{x_1}^2 u_\epsilon, u - u_\epsilon)| \\ &\le \|u - u_\epsilon\|_{H^1}(\|u\|_{H^1} + \|u_\epsilon\|_{H^1}) \to 0. \end{aligned}$$

Also,

$$\|D_{x_1}u_\epsilon\|_{L^2}^2 \to \|D_{x_1}u\|_{L^2}^2,$$

completing the proof. □

We now prove Lemma 4.2

PROOF. We proceed using standard energy estimates.

$$(Pu,u)_{\overline{H}^{-1},\dot{H}^1} = (D_{x_1}^2 u,u) + (b(x,D')D_{x_1}u,u) + (c(x,D')u,u)$$
$$\|D_{x_1}u\|_{L^2}^2 + (D_{x_1}u, b^*(x,D')u) + (c(x,D')u,u)$$

Now, since $T^*\{x_1=0\} \subset \mathcal{E}$, there exists $\delta > 0$ so that

$$c(x,\xi') \geq \frac{b^2(x,\xi')}{4} + \delta|\xi'|^2.$$

Therefore, there exists $\delta_0 > 0$ so that

$$(c(x,D')u,u) - \frac{(1+\delta_0)}{4}\|b(x,D')u\|_{L^2}^2 \geq \frac{\delta}{2}\|\nabla_{x'}u\|_{L^2}^2 - C\|u\|_{L^2}^2.$$

On the other hand

$$|(D_{x_1}u, b^*(x,D')u)| \leq \|D_{x_1}u\|_{L^2}\|b^*(x,D')u\|_{L^2} \leq \frac{1+\delta_0}{4}\|b(x,D'u\|_{L^2}^2 + \frac{1}{1-\delta_0}\|D_{x_1}u\|_{L^2}^2.$$

So,

$$|(Pu,u)| \geq c(\|D_{x_1}u\|_{L^2}^2 + \|\nabla_{x'}u\|_{L^2}^2) - C\|u\|_{L^2}^2.$$

In particular,

$$c\|u\|_{H^1}^2 \leq \|Pu\|_{H^{-1}}\|u\|_{H^1} + (C+c)\|u\|_{L^2}^2 \leq \epsilon^{-1}\|Pu\|_{H^{-1}}^2 + \epsilon\|u\|_{H^1}^2 + C\|u\|_{L^2}^2$$

So, choosing $0 < \epsilon < c/2$,

$$\|u\|_{H^1}^2 \leq C(\|Pu\|_{H^{-1}}^2 + \|u\|_{L^2}^2).$$

□

Next, let $E : H^s(\{x_1=0\}) \to \overline{H}^{s+\frac{1}{2}}$ be the extension operator. Let $u \in \overline{H}^1$ with $u|_{\{x_1=0\}} = u_0 \in H^{\frac{1}{2}}$. Then, $u - Eu_0 \in \overline{H}^1$ with $u|_{x_1=0} = 0$ Moreover,

$$\|Eu_0\|_{\overline{H}^1} \leq \|u_0\|_{H^{1/2}}.$$

So,

$$\|P(u-Eu_0)\|_{\overline{H}^{-1}} \leq \|Pu\|_{\overline{H}^{-1}} + C\|u_0\|_{H^{1/2}}.$$

In particular,

LEMMA 4.3. *For $u \in \overline{H}^1$, with $u|_{x_1=0} = u_0$,*

$$\|u\|_{\overline{H}^1} \leq C(\|Pu\|_{\overline{H}^{-1}} + \|u_0\|_{H^{1/2}} + \|u\|_{L^2}).$$

Next, we improve the regularity of u.

LEMMA 4.4. *Suppose* $u \in L^2(\mathbb{R}^d_+)$, $u|_{x_1=0} = u_0 \in H^{1/2}$ *and* $Pu \in H^{-1}$. *Then,*

$$\|u\|_{H^1} \leq C(\|Pu\|_{H^{-1}} + \|u_0\|_{H^{1/2}}).$$

PROOF. Let $\chi \in C_c^\infty(\mathbb{R}^{n-1})$ with $\chi \equiv 1$ near 0. Then put $u_\epsilon = \chi(\epsilon D')u$. Then $u_\epsilon \to u \in L^2$, $Pu_\epsilon \to Pu \in H^{-1}$, $u_\epsilon \in H^{0,t}$ for all t. Now,

$$Pu_\epsilon = \chi(\epsilon D')Pu + [\chi(\epsilon D'), b(x, D')]D_{x_1}u + [\chi(\epsilon D'), c(x, D')]u \in H^{-1,t}$$

for any t. In particular,

$$D^2_{x_1}u_\epsilon \in H^{-1,t-1}, \qquad D_{x_1}u_\epsilon \in H^{-1,t}.$$

Therefore, $D_{x_1}u_\epsilon \in H^{0,t-1}$ and then, since $u_\epsilon \in H^{0,t-2}$, $u \in H^{1,t-2}$. In particular, choosing $t \geq 2$, $u_\epsilon \in H^1$. Hence,

$$\|u_\epsilon\|_{H^1} \leq C(\|Pu_\epsilon\|_{H^{-1}} + \|u_\epsilon\|_{L^2} + \|u_\epsilon|_{x_1=0}\|_{H^{1/2}}) \leq 2C(\|Pu\|_{H^{-1}} + \|u\|_{L^2} + \|u_0\|_{H^{1/2}}).$$

In particular, there exists a subsequence so that $u_\epsilon \rightharpoonup \tilde{u}$ in $\overline{H}^1$. But $u_\epsilon \to u \in L^2$, so $u \in \overline{H}^1$ and the estimate continues to hold. □

LEMMA 4.5. *Suppose* $s \geq 0$, $u \in \overline{H}^{s,t}(\mathbb{R}^d_+)$, $u|_{x_1=0} = u_0 \in H^{s+t+\frac{1}{2}}$. *Then*

$$\|u\|_{\overline{H}^{s+1,t}} \leq C(\|Pu\|_{H^{s-1,t}} + \|u\|_{H^{s,t}} + \|u|_{x_1=0}\|_{H^{s+t+\frac{1}{2}}}).$$

PROOF. We start with $t = 0$, Observe that

$$P(\langle D'\rangle^s)u = \langle D'\rangle^s Pu + [b(x, D'), \langle D'\rangle^s]D_{x_1}u + [c(x, D'), \langle D'\rangle^s]u$$

. So,

$$\|\langle D'\rangle^s u\|_{H^1} \leq C(\|\langle D'\rangle^s Pu\|_{H^{-1}} + \|u\|_{H^{0,s}} + \|\langle D'\rangle^s u_0\|_{H^{1/2}}.$$

In particular,

$$\|u\|_{H^{1,s}} \leq C\|Pu\|_{H^{-1,s}} + \|u\|_{H^{s,0}} + \|u_0\|_{H^{s+\frac{1}{2}}}.$$

Iterating as before, we then obtain Therefore,

$$\|u\|_{H^{s+1,0}} \leq C\|Pu\|_{s-1,0} + \|u_0\|_{H^{\frac{1}{2}+s}}.$$

Now, for $t \neq 0$,

$$\|\langle D'\rangle^t u\|_{H^{s+1,0}} \leq C(\|\langle D'\rangle^t Pu\|_{H^{s-1,0}} + \|u\|_{H^{s,t}} + \|\langle D'\rangle^t u_0\|_{H^{s+\frac{1}{2}}}$$

which concludes the proof. □

Finally,

LEMMA 4.6. *Suppose* $T^*\{x_1 = 0\} \subset \mathcal{E}$, $u \in \mathcal{N}(\overline{\mathbb{R}^d}_+)$,

$$Pu = f \in \mathcal{N}(\overline{\mathbb{R}^d}_+), \qquad u|_{x_1=0} = u_0.$$

Then,

$$\mathrm{WF_b}(u)|_{T^*\{x_1=0\}} = \mathrm{WF_b}(f)|_{\{x_1=0\}} \cup \mathrm{WF_b}(u_0).$$

PROOF. Here, the inclusion of the right hand side in the left is automatic from the definition of $\mathrm{WF_b}$. Therefore, we need only show that the right hand side is contained in the left. Suppose that $(0, \xi_0') \notin \mathrm{WF_b}(f)|_{\{x_1=0\}} \cup \mathrm{WF_b}(u_0)$. Fix $\delta > 0$ so that

$$\{0 < x_1 \le \delta\} \cap \{x \mid (x, \xi_1, 0) \in \mathrm{WF}(u) \cup \mathrm{WF}(f)\} = \emptyset.$$

In particular, this is possible since $\mathrm{WF_b}(u)|_{\{x_1=0\}} \subset T^*\{x_1 = 0\}$ and $\mathrm{WF_b}(u)$ is closed. Therefore, for $x_1 > 0$ small and $(x, \xi) \in \mathrm{WF}(u)$, $x_1|\xi_1| < |\xi'|$.

Now, let

$$W' := \mathrm{WF_b}(f)|_{\{x_1=0\}} \cup \mathrm{WF_b}(u_0) \cup \{(x, \xi') \mid 0 < x_1 \le \delta,\ (x, \xi) \in \mathrm{WF}(f) \text{ for some } \xi_1\}.$$

This is closed since $\mathrm{WF}_b(f)$ is. Let $\chi \in S^\infty(\mathbb{R} \times T^*\mathbb{R}^{d-1})$ with $\chi(x, \xi') = 0$ for $x_1 \ge \delta$ and order $-\infty$ in a conic neighborhood of W'. We call such a χ a goodc cutoff. Then, since $\mathrm{WF}(f)$ avoids the $N^*\{x_1 = c\}$ for $c \le \delta$, $\chi(x, D')f \in C^\infty$ **[TODO]references**.

Now, choose s, t such that for $\chi \in S^j$ good,

$$\chi(x, D')u \in \overline{H}^{s,t-j}_{\mathrm{loc}}.$$

Now, for $\chi \in S^0$, good

$$P\chi(x, D')u = \chi(x, D')f + [P, \chi(x, D')]u \in \overline{H}^{s-1,t}.$$

Indeed,

$$[P, \chi(x, D')]u = P_0(x, D')\chi_0(x, D') + D_{x_1}P_1(x, D')\chi_1(x, D')$$

where $P_i\chi_i$ is a good cutoff in S^{1-i}.

By iteration we obtain then that

$$\chi(x, D')u \in \overline{H}^{s+1,t-1}$$

and in particular,

$$\chi(x, D')u \in \overline{H}^{s_0,t_0}$$

with $s_0 \ge 0$.

Now, observe that

$$\chi(x, D')u|_{x_1=0} = \chi(x, D')u_0 \in C^\infty,$$

So

$$P\chi(x, D')u \in \overline{H}^{s_0-1,t_0}, \qquad \chi(x, D')u|_{x_1=0} \in C^\infty.$$

Therefore, by Lemma 4.5, $\chi(x, D')u \in \overline{H}^{s_0+1,t_0}$ and in particular, $\chi(x, D')u \in C^\infty$. Now, since $(0, \xi_0') \notin W'$, we may take $\chi(x, \xi')$ good with $\chi(0, \xi_0') = 1$ and hence by **[TODO]reference**, $(0, \xi_0') \notin \mathrm{WF_b}(u)$. □

Finally, we return to the general situation and prove our main theorem for singularities there.

THEOREM 4.1. *Suppose that $P \in \mathrm{Diff}^2(X)$ with ∂X non-characteristic for P. Suppose $f \in \mathcal{N}(X)$ and*

$$Pu = f \in X^o, \qquad u|_{\partial X} = u_0$$

Then,

$$\mathrm{WF_b}(u)|_{\partial X} \cap \mathcal{E} = (\mathrm{WF}_b(f)|_{\partial X} \cup \mathrm{WF}(u_0)) \cap \mathcal{E}.$$

[TODO]for this theorem to make sense we need to prove earlier that if ∂X is non-char, $Pu = f \in \mathcal{N}(X)$ then $u \in \mathcal{N}(X)$. We did this in class

PROOF. Again, the inclusion of the left-hand side in the right is trivial. So, suppose we work in coordinates where $(0, \xi_0') \in \mathcal{E}$, $(0, \xi_0') \notin \mathrm{WF}_b(f)|_{\partial X} \cup \mathrm{WF}(u_0)$. Then, observe that in these coordinates

$$P = P_2(x) D_{x_1}^2 + P_1(x, D') D_{x_1} + P_0(x, D')$$

with $P_i \in S^{2-i}$ and $P_2 \neq 0$ near 0. Therefore, let $\chi \in C_c^\infty(X)$ with $\chi \equiv 1$ near 0 and let $\psi \in C_c^\infty(\mathbb{R}^{d-1})$ with $\psi \equiv 1$ in a conic neighborhood of $\{\xi' \mid \xi'/|\xi'| = \xi_0'/|\xi_0'|, |\xi| \geq 1\}$. Let

$$\tilde{P} = D_{x_1}^2 + \sum_{j=0}^{1} ([P_0(x)^{-1} \chi(x) \psi(D') P_j(x, D') + P_0(0)^{-1}(1 - \chi(x)\psi(D')) P_j(0, \xi_0'/|\xi_0'||D'|)] D_{x_1}^j \tag{14}$$

In particular, for $g \in \mathcal{N}(X)$,

$$(0, \xi_0') \notin \mathrm{WF}_{\mathrm{b}}([\tilde{P} - P]g).$$

Therefore,

$$(0, \xi_0') \notin \mathrm{WF}_{\mathrm{b}}(\tilde{P}u)$$

Moreover, $T^*\{x_1 = 0\} \subset \mathcal{E}_{\tilde{P}}$ for χ supported in a small enough neighborhood of 0 and ψ in a small enough neigbhorood of ξ_0'. Therefore, Lemma 4.6 applies and $(0, \xi_0') \notin \mathrm{WF}_{\mathrm{b}}(u)$ as desired. □

[TODO]prove the next thing We next state a quantitative analog of Theorem 4.1.

THEOREM 4.2. *Suppose that $P \in \mathrm{Diff}^2(X)$ with ∂X non-characteristic for P. Suppose $f \in \mathcal{N}(X)$ and*

$$Pu = f \in X^o, \qquad u|_{\partial X} = u_0$$

Then, for $A \in \Psi_b^0(X)$, $B \in \Psi_b^0(X)$, with $\mathrm{WF}_{\mathrm{b}}(A) \subset \mathrm{ell}(B)$ and $\mathrm{WF}_{\mathrm{b}}(B) \subset \mathcal{E} \cap \tilde{\Sigma} = \emptyset$, we have

$$\|Au\|_{\overline{H}^s} \leq C(\|BPu\|_{\overline{H}^{s-2}} + \|Bu|_{\partial X}\|_{H^{s-\frac{1}{2}}}).$$

4.2. The Hyperbolic region. Now that we have microlocal elliptic regularity in place, we want to study the analog of Theorem 2.1. In particular, we want to understand how singularities may propagate inside $\tilde{\Sigma}$. This will happen in two steps. First, we study $\mathcal{H} \subset T^*\partial X$ and only then $\mathcal{G}$. In $\mathcal{H}$, many methods are available to prove propagation of singularities. We will use a factorization method similar to what was used to prove well posedness for hyperbolic equations. For other approaches see **[TODO]references**.

Recall that we may choose coordinates so that $\partial X = \{x_1 = 0\}$ and

$$p(x, \xi) = \xi_1^2 - r(x, \xi').$$

If $(x_0', \xi_0') \in \mathcal{H}$, then the roots of $\xi_1 \mapsto p(0, x_0', \xi_1, \xi_0')$ are $\pm\sqrt{r_0(x_0', \xi_0')}$ where $r_0(x', \xi') := r(0, x', \xi')$. Note then that the $\pm$ root is a covector pointing in/out of X. Moreover, since

$$H_p = 2\xi_1 \partial_{x_1} - H_r,$$

$\gamma_\pm(t) : \exp(\pm t H_p)(0, x', \pm\sqrt{r_0(x', \xi')}, \xi')$ is a bicharacteristic for p in X^o for t in $(0, \epsilon)$.

Definition 4.1. We define a *compressed broken bicharacteristic for p* as a continuous map

$$\gamma(t) : I \to \tilde{\Sigma} \setminus \mathcal{G}$$

where $I \subset \mathbb{R}$ is an interval and

- When $\gamma(t) \in \tilde{\Sigma} \cap T^*X^o$, $\gamma(t) \in C^1$ and $\gamma'(t) = H_p(\gamma(t))$.
- $\{t \in I \mid \gamma(t) \in \mathcal{H}\}$ is a discrete subset of I.

Locally near a point $(x', \xi') \in \mathcal{H}$, such a broken bicharacteristic is given by

$$\gamma(t) = \begin{cases} \iota(\exp(tH_p)(0, x', \sqrt{r_0(x', \xi')}, \xi')) & 0 \le t \le \epsilon \\ \iota(\exp(tH_p)(0, x', -\sqrt{r_0(x', \xi')}, \xi')) & -\epsilon \le t \le 0. \end{cases}$$

Our next theorem says that singularities of solutions to $Pu = 0$ with $u|_{\partial X} = 0$ are invariant along such broken bicharacteristics.

Theorem 4.3. *Suppose that* $P \in \mathrm{Diff}^2(X)$ *with* ∂X *non-characteristic for* P. *Suppose* $f \in \mathcal{N}(X)$ *and*

$$Pu = f \in X^o, \qquad u|_{\partial X} = u_0$$

Suppose that $\gamma : I \to \tilde{\Sigma}$ *is a broken bicharacteristic such that* $\gamma(I) \cap (\mathrm{WF}_b(f) \cup \mathrm{WF}(u_0)) = \emptyset$, *then for any* $t_0 \in I$ *if* $\gamma(t_0) \notin \mathrm{WF}_{\mathrm{b}}(u)$ *then*

$$\gamma(I) \notin \mathrm{WF}_{\mathrm{b}}(u).$$

Notice that for $\gamma(I) \cap T^*\partial X = \emptyset$, Theorem 4.3 is an easy consequence of Theorem 2.1. Therefore, in the proof we may work locally near a point $t \in I$ such that $\gamma(t) \in \mathcal{H}$. Without loss of generality, we therefore assume $\rho_0 = \gamma(0) \in \mathcal{H}$. Then, since $\rho \in \mathcal{H}$, freezing coefficients as in (14), we may replace P by $\tilde{P}$ such that $T^*\partial X \subset \mathcal{H}$ so that $\tilde{P}u$ has the same wavefront set properties as Pu near ρ_0.

We continue to call the operator P for brevity and observe that

$$p(x, \xi) = \xi_1^2 - r(x, \xi')$$

with $r(x, \xi') < c|\xi'|^2$. In particular, P is strictly hyperbolic with respect to x_1 and we may apply Lemma 1.2. However, it will be necessary to upgrade this factorization.

Lemma 4.7. *There exist* $\Lambda_\pm(x, D')$, $\tilde{\Lambda}_\pm(x, D') \in S^1$ *with symbols* $\sigma(\Lambda_\pm) = \sigma(\tilde{\Lambda}_\pm) = \pm\sqrt{r}$ *so that*

$$P = (D_{x_1} - \Lambda_+)(D_{x_1} - \Lambda_-) + E(x, D') = (D_{x_1} - \tilde{\Lambda}_-)(D_{x_1} - \tilde{\Lambda}_+) + \tilde{E}(x, D')$$

with $E, \tilde{E} \in S^{-\infty}$.

Proof. We have from Lemma 1.2 that there exist $\Lambda_{0,-}(x, D') \in S^1$, $\Lambda_{0,+}(x, D') \in S^1$ and $E_0(x, D') \in S^1$ such that

$$P = (D_{x_1} - \Lambda_{0,+})(D_{x_1} - \Lambda_{0,-}) + E_0(x, D')$$

with $\sigma(\Lambda_{0,\pm}) = \pm\sqrt{r(x, \xi')}$. Suppose that there exist $\Lambda_{j,-}(x, D') \in S^1$, $\Lambda_{j,+}(x, D') \in S^1$ and $E_j(x, D') \in S^{1-j}$ such that

$$P = (D_{x_1} - \Lambda_{j,+})(D_{x_1} - \Lambda_{j,-}) + E_j(x, D')$$

with $\sigma(\Lambda_{j,\pm}) = \pm\sqrt{r(x,\xi')}$. We will show that the error can be improved to $E_{j+1} \in S^{-j}$. For this, let

$$\lambda_{j+1}(x,\xi') = -\frac{\sigma(E_j)(x,\xi')}{2\sqrt{r(x,\xi')}} \in S^{-j}.$$

Then

$$\begin{aligned}
\lambda_{j+1}(x,D')(D_{x_1}-\Lambda_{j,-}) + E_j &= \lambda_{j+1}(x,D')(\Lambda_{j,+}-\Lambda_{j,-}) + E_j + \lambda_{j+1}(x,D')(D_{x_1}-\Lambda_{j,+})\\
&= \lambda_{j+1}(x,D')(\Lambda_{j,+}-\Lambda_{j,-}) + E_j\\
&\quad\quad + (D_{x_1}-\Lambda_{j,+})\lambda_{j+1}(x,D') + [\lambda_{j+1}(x,D'), D_{x_1}-\Lambda_{j,+}]\\
&= E_{j+1,1}(x,D')\\
&\quad\quad + (D_{x_1}-\Lambda_{j,+}-\lambda_{j+1}(x,D'))\lambda_{j+1}(x,D') + E_{j+1,2}(x,D')
\end{aligned}$$

where $E_{j+1,1}, E_{j+1,2} \in S^{-j}$.

In particular,

$$\begin{aligned}
P &= (D_{x_1}-\Lambda_{j,+})(D_{x_1}-\Lambda_{j,-}) + E_j(x,D')\\
&= (D_{x_1}-\Lambda_{j,+})(D_{x_1}-\Lambda_{j,-}) - \lambda_{j+1}(x,D')(D_{x_1}-\Lambda_{j,-})\\
&\quad\quad + (D_{x_1}-\Lambda_{j,+}-\lambda_{j+1}(x,D'))\lambda_{j+1}(x,D') + E_{j+1,2}(x,D') - E_{j+1,1}(x,D')\\
&= (D_{x_1}-\Lambda_{j,+}-\lambda_{j+1}(x,D'))(D_{x_1}-\Lambda_{j,-}+\lambda_{j+1}(x,D')) + E_j(x,D')
\end{aligned}$$

with $E_j \in S^{-j}$. Let $\Lambda_{j+1,\pm} = \Lambda_{j,\pm} \pm \lambda_{j+1}(x,D')$. Then defining $\Lambda_\pm \sim \Lambda_{0,\pm} \pm \sum_{j\geq 1}\lambda_j$ gives the desired factorization.

Repeating the arguments starting with the $\tilde{\Lambda}$ factorization completes the proof □

Next, we construct an operator $Q(x,D')$ with desirable microlocalization properties such that $[D_{x_1}-\Lambda_+, Q] \in S^{-\infty}$. This will allow us to complete the proof of Theorem 4.3 by effectively microlocalizating $Pu = f$ to a broken bicharacteristic.

LEMMA 4.8. *Fix $\rho_0 \in \mathcal{H}$, $\gamma : I \to \tilde{\Sigma}$ a broken bicharacteristic with $\gamma(0) = \rho_0$ and let U be a conic neighborhood of $\gamma([0,\infty)\cap I)$. Then there exist $\epsilon > 0$ and a conic neighborhood $V \subset T^*\partial X$ so that for any $\tilde{q}_+ \in S^0(T^*\partial X)$ supported in V, there exists $Q_+(x,D') \in S^0$ with symbol q_+ such that $q_+(0,\rho) = \tilde{q}_+$,*

$$\mathrm{WF}(Q_+) \cap \{x_1 \leq \epsilon\} \subset \{(x,\xi') \mid (x, \sqrt{r(x,\xi')}, \xi') \in U\} \tag{15}$$

and

$$[D_{x_1}-\Lambda_+, Q_+] \in S^{-\infty}. \tag{16}$$

Similarly, let U be a conic neighborhood of $\gamma((-\infty,0]\cap I)$. Then there exist $\epsilon > 0$, $V \subset T^*\partial X$ a conic neighborhood of ρ_0 so that for $\tilde{q}_- \in S^0(T^*\partial X)$ supported in V there exists Q_- with $\sigma(Q_-)(0,\rho) = \tilde{q}_-$,

$$\mathrm{WF}(Q_-) \cap \{x_1 \leq \epsilon\} \subset \{(x,\xi') \mid (x, -\sqrt{r(x,\xi')}, \xi') \in U\} \tag{17}$$

and

$$[D_{x_1}-\tilde{\Lambda}_-, Q_-] \in S^{-\infty}. \tag{18}$$

Proof. Fix $\tilde{q}_0 \in S^0(T^*\partial X)$ with $\operatorname{supp} q_0 \subset U \cap T^*\partial X$ and $q_0(\rho_0) = 1$. Then, since ∂_{x_1} is transverse to $\{x_1 = 0\}$, there exists $q_0 \in C^\infty([0,\epsilon) \times T^*\partial X)$ solving

$$(\partial_{x_1} - H_{\sqrt{r}})q_0 = 0, \qquad q_0|_{x_1=0} = \tilde{q}_0.$$

Moreover, by homogeneity, $q_0 \in C^\infty([0,\epsilon); S^0(T^*\partial X))$. Let $Q_0 = q_0(x, D')$. Then,

$$[D_{x_1} - \Lambda_+, Q_0] = E_0(x, D') \in S^{-1}.$$

Now, since on $\gamma((0,\infty) \cap I)$, $H_p = g(\partial_{x_1} - H_{\sqrt{r}})$, choosing $\operatorname{supp} \tilde{q}_0$ small enough, we may assume $\operatorname{supp} q_0 \cap \{x_1 \le \epsilon\}$ is contained in the RHS of (15) and hence $\mathrm{WF}(E_0) \cap \{x_1 \le \epsilon\}$ is contained in the RHS of (15) Assume now that we have Q_j with $\sigma(q_j)(\rho_0) = 1$, $\mathrm{WF}(Q_j) \subset U$ and

$$[D_{x_1} - \Lambda_+, Q_j] = E_j(x, D') \in S^{-1-j}. \tag{19}$$

Let $q_j \in S^{-j}$ solve

$$(\partial_{x_1} - H_{\sqrt{r}})q_j = -\sigma(E_j), \qquad q_j|_{x_1=0} = 0.$$

Then, $\operatorname{supp} q_j \cap \{x_1 \le \epsilon\}$ is contained in the RHS of (15). Putting $Q_{j+1} = Q_j + q_j(x, D')$. Then we have (19) with j replaced by $j+1$. Letting $Q_+ \sim \sum_j q_j$ completes the proof of (16).

Using $\partial_{x_1} + H_{\sqrt{r}}$ rather than $\partial_{x_1} - H_{\sqrt{r}}$ completes the proof of (18). □

Proof of Theorem 4.3. Let γ be a broken bicharacteristic with $\rho_0 = \gamma(0) \in \mathcal{H}$. Since $f \in \mathcal{N}(X)$, shrinking $\epsilon > 0$ if necessary, we may assume that for $0 < x_1 \le \epsilon$, $(x,\xi) \in \mathrm{WF}(f) \cup \mathrm{WF}(u)$ implies $\xi' \neq 0$. Shrinking $\epsilon > 0$ if necessary, let $Q_+ \in C^\infty([0,\epsilon); \Psi^0(\partial X)$ be as in Lemma 4.8. Then,

$$(D_{x_1} - \Lambda_+)Q_+(D_{x_1} - \Lambda_-) = Q_+P + R(x, D') \tag{20}$$

where $R \in C^\infty([0,\epsilon]; \Psi^{-\infty})$. In particular, since for $0 < x_1 \le \epsilon$ $(x,\xi) \in \mathrm{WF}(f) \cup \mathrm{WF}(u)$ implies $\xi' \neq 0$, we have in particular,

$$(D_{x_1} - \Lambda_+)Q_+(D_{x_1} - \Lambda_-)u \in C^\infty(x_1 \le \epsilon). \tag{21}$$

In particular, for $x_1 > 0$, $\mathrm{WF}(Q_+(D_{x_1} - \Lambda_-)u)$ is arbitrarily close to $\xi_1 = r(x,\xi')^{1/2}$ and contained in U. Therefore, for B_+ so that $\gamma([0,\epsilon) \cap \{\frac{\epsilon}{2} \le x_1 \le \epsilon\} \subset \ell(B_+)$,

$$\|Q_+(D_{x_1} - \Lambda_-)u\|_{H^{0,s}(\epsilon/2 \le x_1 \le \epsilon)} \le C\|B_+u\|_{H^{s+1}}.$$

Now, by Lemma 1.1, **[TODO]be more precise here**

$$\begin{aligned}
&\|Q_+(D_{x_1} - \Lambda_-)u(x_1,\cdot)\|_{H^s} \\
&\qquad \le C(\|Q_+(0,x',D')(D_{x_1} - \Lambda_-)u\|_{H^{0,s}(\frac{\epsilon}{2} \le x_1 \le \epsilon)} + \|Q_+Pu\|_{H^{0,s}} + \|R(x,D')u\|_{H^{0,s}} \\
&\qquad \le C(\|B_+u\|_{H^{s+1}} + \|Q_+Pu\|_{H^s}).
\end{aligned}$$

Also, using (21) repeatedly,

$$\|Q_+(D_{x_1} - \Lambda_-)u\|_{H^{s,0}} \le C(\|B_+u\|_{H^{s+1}} + \|Q_+Pu\|_{H^s}).$$

Now, let Q_- as in (17) with $Q_-(0,x',D') = Q_+(0,x',D')$. Then,

$$[Q_-(D_{x_1} - \tilde{\Lambda_+}) - Q_+(D_{x_1} - \Lambda_-)u]|_{x=0} = Q_-(0,x',D')(\Lambda_- - \tilde{\Lambda}_+)u_0.$$

Observe that

$$(D_{x_1}-\tilde{\Lambda}_-)Q_-(D_{x_1}-\tilde{\Lambda}_+)=Q_-P+\tilde{R} \tag{22}$$

with $\tilde{R}\in C^\infty([0,\epsilon];\Psi^{-\infty})$. Therefore, letting $A\in\Psi^0(\partial X)$ with $\mathrm{WF}(Q_-(0,x',D'))\subset\mathrm{ell}(A)$,

$$\begin{aligned}\|Q_-(D_{x_1}-\tilde{\Lambda}_+)u(x_1,\dot{})\|_{H^s}\\ &\le C(\|Q_-(0,x',D')(D_{x_1}-\tilde{\Lambda}_+)u|_{x_1=0}\|_{H^s}+\|Q_-Pu\|_{H^{0,s}}+\|\tilde{R}(x,D')u\|_{H^{0,s}}\\ &\le C(\|Au_0\|_{H^{s+1}}+\|B_+u\|_{H^{s+1}}+\|Q_+Pu\|_{H^s}+\|u\|_{H^{-N}}).\end{aligned}$$

Using (22) repeatedly then gives

$$\|Q_-(D_{x_1}-\tilde{\Lambda}_+)u\|_{H^{s,0}}\le C(\|Au_0\|_{H^{s+1}}+\|B_+u\|_{H^{s+1}}+\|Q_+Pu\|_{H^s}+\|u\|_{H^{-N}}).$$

Now, let $\tilde{Q}_0(x,D')\in C^\infty([0,\epsilon];\Psi^0)$ have $\mathrm{WF}(Q_0)\subset\mathrm{ell}(Q_-)\cap\mathrm{ell}(Q_+)$. Then, by the elliptic parametrix construction

$$\tilde{Q}_0(x,D')(D_{x_1}-\tilde{\Lambda}_+)=E_-Q_-(D_{x_1}-\tilde{\Lambda}_+)+R_1$$
$$\tilde{Q}_0(x,D')(D_{x_1}-\Lambda_-)=E_+Q_+(D_{x_1}-\Lambda_-)+R_2$$

with $E_\pm\in C^\infty([0,\epsilon];\Psi^0)$ and $R_i\in C^\infty([0,\epsilon];\Psi^{-\infty})$. Hence,

$$\tilde{Q}_0(x,D')(\Lambda_--\tilde{\Lambda}_+)=E_-Q_-(D_{x_1}-\tilde{\Lambda}_+)-E_+Q_+(D_{x_1}-\Lambda_-)+R_3$$

Finally, since $\tilde{\Lambda}_+-\Lambda_-\in C^\infty([0,\epsilon];\Psi^1)$ is elliptic,

$$\|\tilde{Q}_0(x,D')u\|_{H^{s,1}}\le C(\|Au_0\|_{H^{s+1}}+\|B_+u\|_{H^{s+1}}+\|Q_+Pu\|_{H^s}+\|u\|_{H^{-N}})$$

Now, for Q_0 with $\mathrm{WF}(Q_0)\subset\mathrm{ell}(\tilde{Q}_0)$,

$$PQ_0(x,D')u=Q_0Pu+[Q_0,P]u$$

So,

$$\|PQ_0u\|_{H^{s-1,0}}\le C\|Q_+Pu\|_{H^s}+\|\tilde{Q}_0u\|_{H^{s,0}}$$

Therefore,

$$\|D_{x_1}^2Q_0u\|_{H^{s-1,0}}\le C\|Q_+Pu\|_{H^s}+\|\tilde{Q}_0u\|_{H^{s,1}}$$

and hence

$$\|D_{x_1}Q_0u\|_{H^{s,0}}\le C\|Q_+Pu\|_{H^s}+\|\tilde{Q}_0u\|_{H^{s,1}}.$$

Finally, this implies

$$\|Q_0u\|_{H^{s+1,0}}\le C\|Q_+Pu\|_{H^s}+\|\tilde{Q}_0u\|_{H^{s,1}}.$$

Therefore,

$$\|Q_0u\|_{H^{s+1}}\le C(\|Au_0\|_{H^{s+1}}+\|B_+u\|_{H^{s+1}}+\|Q_+Pu\|_{H^s}+\|u\|_{H^{-N}}).$$

In particular, since Q_+ and Q_- are elliptic at ρ_0, for any $C(x,D')$ with $\mathrm{WF}(C)$ sufficiently close to ρ_0,

$$\|Cu\|_{H^{s+1}}\le C(\|Au_0\|_{H^{s+1}}+\|B_+u\|_{H^{s+1}}+\|Q_+Pu\|_{H^s}+\|u\|_{H^{-N}}). \tag{23}$$

Now, suppose that $\gamma(I)\cap(\mathrm{WF_b}(f)\cup\mathrm{WF}(u_0))=\emptyset$ and $\gamma(\epsilon)\notin\mathrm{WF_b}(u)$. Let U a conic neighborhood of $\gamma([0,\epsilon])$ so that $U\cap(\mathrm{WF_b}(f)\cup\mathrm{WF}(u_0))=\emptyset$. Choose $\epsilon>0$ small enough so that $\gamma((0,\epsilon])\cap\mathrm{WF}(f)=\emptyset$. Then, there exists A elliptic at ρ_0 so that $Au_0\in C^\infty$ and $B_+(x,D')$ elliptic on $\gamma(\epsilon)$ so that $B_+u\in C^\infty$. Moreover, for Q_+ supported in U, $Q_+Pu\in C^\infty$. In particular, (23) implies $\rho_0 \notin \mathrm{WF}_\mathrm{b}(u)$. Switching the roles of $Q_\pm$, we also obtain that if $\gamma(-\epsilon) \notin \mathrm{WF}_\mathrm{b}(u)$, then $\rho_0 \notin \mathrm{WF}_\mathrm{b}(u)$ which completes the proof.

□

[TODO]state estimates

5. The generalized bicharacteristic flow

6. The Weyl law on a manifold with boundary

7. Microlocal defect measures

CHAPTER 5

Equivalence of Glancing Hypersurfaces

CHAPTER 6

The Melrose–Taylor Parametrix

1. The Friedlander Model and the Ansatz

1.1. The Friedlander Model. As a first, step, we consider the Friedlander model. This toy example guides us when we consider the general case. The Friedlander model is given by

$$P=(hD_{y_1})^2-y_1+hD_{y_2}\qquad \partial X=\{y_1=0\}.$$

Suppose that

$$Pu=0\qquad u|_{\partial\Omega}=f \tag{24}$$

Then, taking the semiclassical Fourier transform in the x' variables gives

$$(-h^2\partial_{y_1}^2-y_1+\eta_2)\mathcal{F}_{h,y'}u(y_1,\eta')=0\qquad \mathcal{F}_{h,y'}u(0,\eta')=\mathcal{F}_h(f)(\eta').$$

The solution to this problem for $\mu=0$ is

$$u=(2\pi h)^{-d+1}\int\frac{A(h^{-2/3}(-y_1+\eta_2))}{A(h^{-2/3}\eta_2)}e^{\frac{i}{h}\langle y',\eta'\rangle}\mathcal{F}_h(f)(\eta')d\eta'$$

where A is a solution to the Airy equation. Let $\zeta_0:=-y_1+\eta_2$ and $\theta_0=\langle y',\eta'\rangle$.

Our goal will be to model the parametrices on the Friedlander model and therefore to seek solutions u to

$$P(x,hD)u=0$$

of the form

$$u=(2\pi h)^{-d+1}\int[g_0A(h^{-2/3}\zeta)+ih^{1/3}g_1A'(h^{-2/3}\zeta)]e^{i\theta/h}\mathcal{F}_h(f)(\eta')d\eta' \tag{25}$$

where f is a function on ∂X. We will then correct the boundary values by applying an FIO in the boundary variables which replaces the Fourier transform in the Friedlander model.

REMARK 2. Note that it is necessary to add the A' term since it is not possible to cancel A' with an amplitude times A. However, since A solves a second order ODE, it is possible to cancel A'' with an amplitude times A.

1.2. Eikonal and Transport Equations. First, we consider a general differential operator

$$P(x,hD)=\sum a_{jk}(x)hD_jhD_k+\sum b_j(x)hD_j+c(x)$$

with $a_{jk}=a_{kj}$ applied to (25).

For A an Airy function, we have, letting f_j denote $\partial_j f$, and $\zeta_h = h^{-2/3}\zeta$

$$hD_j\left(gA(\zeta_h)e^{\frac{i}{h}\theta}\right) = \theta_j gA(\zeta_h)e^{\frac{i}{h}\theta} - ihg_jA(\zeta_h)e^{\frac{i}{h}\theta} - ih^{1/3}\zeta_j gA'(\zeta_h)e^{\frac{i}{h}\theta}$$

$$\begin{aligned} hD_khD_j\left(gA(\zeta_h)e^{\frac{i}{h}\theta}\right) = & \\ & \left[(\theta_k\theta_j - \zeta_j\zeta_k\zeta)g - ih(\theta_kg_j + \theta_jg_k + \theta_{jk}g) - h^2g_{jk}\right]A(\zeta_h)e^{\frac{i}{h}\theta} \\ & - ih^{1/3}\left[(\theta_j\zeta_k + \zeta_j\theta_k)g - ih(g_j\zeta_k + \zeta_jg_k + \zeta_{jk}g)\right]A'(\zeta_h)e^{\frac{i}{h}\theta} \\ hD_j\left(gA'(\zeta_h)e^{\frac{i}{h}\theta}\right) = & \\ & \theta_jgA'(\zeta_h)e^{\frac{i}{h}\theta} - ihg_jA'(\zeta_h)e^{\frac{i}{h}\theta} - ih^{-1/3}\zeta_j\zeta gA(\zeta_h)e^{\frac{i}{h}\theta} \\ hD_khD_j\left(gA'(\zeta_h)e^{\frac{i}{h}\theta}\right) = & -ih^{-1/3}\left[(\theta_j\zeta_k + \theta_k\zeta_j)\zeta g\right. \\ & \left. -ih(g_j\zeta_k\zeta + g_k\zeta_j\zeta + \zeta_{jk}\zeta g + \zeta_j\zeta_kg)\right]A(\zeta_h)e^{\frac{i}{h}\theta} \\ & + \left[(\theta_j\theta_k - \zeta_j\zeta_k\zeta)g - ih(\theta_{kj}g + \theta_jg_k + \theta_kg_j) - h^2g_{jk}\right]A'(\zeta_h)e^{\frac{i}{h}\theta} \end{aligned}$$

So,

$$\begin{aligned} & P(g_0A(\zeta_h)e^{\frac{i}{h}\theta}) \\ & = \begin{bmatrix} (\langle ad\theta, d\theta\rangle - \zeta\langle ad\zeta, d\zeta\rangle + \langle b, d\theta\rangle + c)g_0 \\ -ih(2\langle ad\theta, dg_0\rangle - P_2\theta g_0 + \langle b, dg_0\rangle) + h^2P_2g_0 \end{bmatrix} A(\zeta_h)e^{\frac{i}{h}\theta} \\ & \qquad - ih^{1/3}\begin{bmatrix} (2\langle ad\theta, d\zeta\rangle + \langle b, d\zeta\rangle)g_0 \\ -ih(2\langle ad\zeta, dg_0\rangle - (P_2\zeta)g_0) \end{bmatrix} A'(\zeta_h)e^{\frac{i}{h}\theta} \end{aligned}$$

$$\begin{aligned} & P(ih^{1/3}g_1A'(\zeta_h)e^{\frac{i}{h}\theta}) = \\ & \begin{bmatrix} \zeta(2\langle ad\theta, d\zeta\rangle + \langle b, d\zeta\rangle)g_1 \\ -ih(2\zeta\langle ad\zeta, dg_1\rangle + \langle ad\zeta, d\zeta\rangle g_1 - \zeta(P_2\zeta)g_1) \end{bmatrix} A(\zeta_h)e^{\frac{i}{h}\theta} \\ & + ih^{1/3}\begin{bmatrix} (\langle ad\theta, d\theta\rangle - \zeta\langle ad\zeta, d\zeta\rangle + \langle b, d\theta\rangle + c)g_1 \\ -ih(2\langle ad\theta, dg_1\rangle - (P_2\theta)g_1 + \langle b, dg_1\rangle) + h^2P_2g_1 \end{bmatrix} A'(\zeta_h)e^{\frac{i}{h}\theta} \end{aligned}$$

where $a_{jk} = a_{jk}(x)$, $P_2 = h^{-2}(P - \langle b, hD\rangle - c(x))$ and $\langle\cdot,\cdot\rangle$ denotes the euclidean inner product.

Now, applying P under the integral in (25) gives the eikonal equations

$$\begin{cases} \langle ad\theta, d\theta\rangle - \zeta\langle ad\zeta, d\zeta\rangle + \langle b, d\theta\rangle + c = 0 \\ 2\langle ad\theta, d\zeta\rangle + \langle b, d\zeta\rangle = 0 \end{cases} . \tag{26}$$

Writing

$$\phi^{\pm} = \theta \pm \frac{2}{3}(-\zeta)^{3/2}, \tag{27}$$

the eikonal equations are equivalent to the standard equation

$$p(x, d\phi^{\pm}) = 0.$$

Now, suppose that ζ has the form $\sum_{n\geq 0}\zeta_n h^n$ and θ has the form $\sum_{n\geq 0}\theta_n h^n$ and

$$g_i \sim \sum_n g_i^{[n]}(x,\eta')h^n.$$

Then the transport equations have the form

$$(28)\qquad \begin{cases} \begin{array}{l} 2\langle ad\theta_0, dg_0^{[n]}\rangle + 2\zeta_0\langle ad\zeta_0, dg_1^{[n]}\rangle + \langle b, dg_0^{[n]}\rangle \\ +\langle ad\zeta_0, d\zeta_0\rangle g_1^{[n]} - P_2\theta_0 g_0^{[n]} - \zeta_0(P_2\zeta_0)g_1^{[n]} \end{array} = F_1^{[n]}(\theta,\zeta,g_i^{[m]<[n]},\mu) \\ \\ \begin{array}{l} 2\langle ad\zeta_0, dg_0^{[n]}\rangle - 2\langle ad\theta_0, dg_1^{[n]} - \langle b, dg_1^{[n]}\rangle\rangle \\ \quad -(P_2\zeta_0)g_0^{[n]} + (P_2\theta_0)g_1^{[n]} \end{array} = F_2^{k,m}(\theta,\zeta,g_i^{[m]<[n]},\mu). \end{cases}$$

More generally, we consider transport equations of the form

$$(29)\qquad \begin{cases} \begin{array}{l} 2\langle ad\theta_0, dg_0\rangle + 2\zeta_0\langle ad\zeta_0, dg_1\rangle + \langle b, dg_0\rangle \\ \quad +\langle ad\zeta_0, d\zeta_0\rangle g_1 + B_1 g_0 + \zeta_0 B_2 g_1 \end{array} = F_1 \\ \\ 2\langle ad\zeta_0, dg_0\rangle - 2\langle ad\theta_0, dg_1\rangle - \langle b, dg_1\rangle + B_2 g_0 - B_1 g_1 = F_2 \end{cases}$$

Then, these equations are equivalent to

$$(30)\qquad 2\langle ad\phi^{\pm}, d_x g^{\pm}\rangle + \langle b, dg^{\pm}\rangle + G^{\pm}g^{\pm} = F^{\pm}$$

where

$$g^{\pm} = g_0 \pm (-\zeta_0)^{1/2}g_1 \qquad G^{\pm} = B_1 \mp (-\zeta_0)^{1/2}B_2 \qquad F^{\pm} = F_1 \mp (-\zeta_0)^{1/2}F_2.$$

2. Geometric preliminaries

We will start from the equivalence of glancing hypersurfaces **[TODO]prove**. In particular, suppose S is a symplectic manifold of dimension $2d$ with $P = \{p = 0\}, Q = \{q = 0\} \subset S$ hypersurfaces (i.e. embedded submanifolds of codimension 1). Suppose that there is $\rho \in P \cap Q$ so that

$$(31)\qquad \begin{gathered} dp, dq, \text{ are linearly independent at } \rho, \\ \{p,q\}(\rho) = 0, \\ \{p,\{p,q\}\} \neq 0, \qquad \{q,\{q,p\}\}(\rho) \neq 0 \end{gathered}$$

Then there exists a neighborhood U of ρ and a symplectomorphism $\kappa : U \to T^*\mathbb{R}^d$ with

$$\kappa(U \cap P) \subset \{\eta_1^2 - y_1 + \eta_2 = 0\}, \qquad \kappa(U \cap Q) \subset \{y_1 = 0\}, \qquad \kappa(\rho) = (0,0).$$

Now, we can quotient P and Q by their respective Hamiltonian fibrations. That is, we write for $\rho_0, \rho_1 \in Q$,

$$\rho_0 \sim \rho_1 \text{ if there exists } t \in \mathbb{R} \text{ with } \exp(tH_q)(\rho_0) = \rho_1$$

to define the space $Q/\mathbb{R}H_q$. We then define a symplectic form on $Q/\mathbb{R}H_q$ by

$$\sigma_\partial = \pi_*^Q \sigma|_Q$$

where $\pi^Q : Q \to Q/\mathbb{R}H_q$ is the natural projection map. An application of Darboux's theorem shows that σ_∂ is closed and non-degenerate.

EXAMPLE 2.1. *If $S = T^*\mathbb{R}^d$ and $Q = \{x_1 = 0\}$, then $H_q = -\partial_{\xi_1}$ and therefore, $(0, x_0', \xi_0) \sim (0, x_1', \xi_1)$ if $x_0' = x_1'$ and $\xi_0' = \xi_1'$. Hence, in this case $Q/\mathbb{R}H_q$ is canonically isomorphic to $T^*\{x_1 = 0\}$.*

In the relevant case, $S = T^*X$ for some manifold x, and $Q = \{(x,\xi) \mid x \in \partial X\}$. We write $\pi_X \rho = x_0$ when it is necessary to refer to the base variable.

Now, since κ is symplectic and sends Q to $\{x_1 = 0\}$, it sends the flow lines of H_q in Q to those of H_{x_1} in $\{x_1 = 0\}$. In particular, κ induces a map $\kappa_\partial : Q/\mathbb{R}H_q \to T^*\{x_1 = 0\}$. Moreover, κ_∂ is a symplectomorphism.

We will actually assume that

$$H_p(\rho) \text{ is not tangent to } T^*_{x_0}X. \tag{32}$$

This is the case for the wave equation.

We do this so that

$$(\kappa_\partial^{-1})^*(d\eta_i), \qquad i = 2, \dots d \text{ are linearly independent on } T^*_{x_0}\partial X \text{ at } \rho. \tag{33}$$

To see this, observe that by (32) $d\pi^Q H_p$ is not tangent to $T^*_{x_0}\partial X$. Now, let $L := \kappa_\partial(T^*_{x_0}\partial X)$. Then, since

$$\kappa_* H_p(\rho) = g(\rho)(-\partial_{\eta_1} + \partial_{y_2})$$

for some nonvanishing g, $(\kappa_\partial)_* d\pi^Q H_p = g(\rho)\partial_{y_2}$ and we have that ∂_{y_2} is not tangent to L at 0. In particular, since L is Lagrangian (as the image of a Lagrangian under a symplectomorphism), this implies $d\eta_2|_L(0) \neq 0$. Therefore, making a symplectic change of variables on $T^*\{x_1 = 0\}$ fixing (y_2, η_2), it is possible to arrange that (33) holds. We then extend this change of variables (independently of y_1, η_1) to $T^*\mathbb{R}^d$ leaving the normal form completely unchanged .

Now, define the map

$$Y_0 : P \ni \rho \mapsto (\eta_2(\kappa(\rho)), \dots \eta_d(\kappa(\rho))) \in \mathbb{R}^{d-1}$$

and

$$Y : P \ni \rho \mapsto (\pi_X(\rho), Y_0(\rho)) \in X \times \mathbb{R}^{d-1}.$$

2.1. Folds. Let M and N be smooth manifolds. We say that $f : M \to N \in C^\infty$ is a *fold at* m if $\dim \ker df(m) = \dim \operatorname{Coker} df(m) = 1$ and the Hessian of f at m_0 is not equal to 0. The fold set of f is defined as the set $\mathcal{F} := \{m \in M \mid f \text{ is a fold at } m\}$.

As an aside we prove the following

LEMMA 2.1. *Let $f : M \to N$ be a fold at m. Then there exists coordinates t near m on M and s near $f(m)$ on N so that*

$$f(t_1, \dots t_n) = (t_1, \dots t_n^2).$$

PROOF. Choose coordinates y on N so that with $f = (f_1, \dots f_n)$ $df_n(m) = 0$. This is possible since $\dim \operatorname{Coker} df > 0$. Moreover, since $\dim \operatorname{Coker} df = 1$, df_i $i = 1, \dots n-1$ are linearly independent and we may use $x_i = f_i$ $i = 1, \dots n-1$ as coordinates on M so that

$$f(x) = (x_1, \dots x_{n-1}, f_n(x).$$

and $x(m) = 0$. Then, $df_n(0) = 0$ and since the Hessian of f is non-zero, $\partial^2_{x_n} f_n(0) \neq 0$. So, by the implicit function theorem, $\partial_{x_n} f_n(y) = 0$ has a unique solution $x_n = g(x')$ where $x' = (x_1, \dots x_{n-1})$ and $g(0) = 0$. Setting $x_n = x_n - g(x')$, we have $\partial_{x_n} f_n = 0$ when $x_n = 0$ so

$$f_n(x) = f_n(x', 0) + x_n^2 F(x)$$

where $F \in C^\infty$ with $F(0) \neq 0$. Now, replacing y_n by $y_n - f_n(y', 0)$, we have

$$f(x) = (x_1, \dots x_{n-1}, x_n^2 F(x)).$$

Finally, switching the sign of y_n if necessary, we may assume $F(0) > 0$ and replace x_n by $x_n F^{1/2}(x)$ to obtain

$$f(x) = (x_1, \dots x_{n-1}, x_n^2).$$

□

As a consequence, we obtain

LEMMA 2.2. *Suppose $f : M \to N$ has a fold at m. Then there is a neighborhood V of m and a C^∞ map $\iota : V \to V$ so that $\iota^2 = \mathrm{Id}$, $\iota \neq \mathrm{Id}$ and $f \circ \iota = f$. That is, ι is an involution preserving f.*

PROOF. Take $\iota(y_1, \dots, y_n) = (y_1, \dots, -y_n)$ with the coordinates from Lemma 2.1. □

2.2. The structure of Y.

LEMMA 2.3. *The map Y is a fold at ρ with fold set meeting Q transversally at $\eta_2 = 0$.*

PROOF. Consider $\rho \in \{p = 0\} \cap \{H_q p = 0\}$. Now, since $H_q H_q p \neq 0$, $\{H_q p = 0\}$ is a smooth hypersurface transverse to H_q and we may take coordinates (x_1, x') so that $H_q p = x_1$ and $H_q = \partial_{x_1}$ and $0 \mapsto \rho$. Therefore, $p = \frac{1}{2} x_1^2 + g(x')$. Now, $dp(0) \neq 0$. Therefore, $dg(0) \neq 0$ and we may change coordinates so that $x_2 = 2g$ so that

$$p = \frac{1}{2}(x_1^2 + x_2), \qquad H_q = \partial_{x_1}.$$

So, on $P \cap Q$,

$$\ker d\pi^Q(\rho) = \partial_{x_1}, \qquad \operatorname{Coker} d\pi^Q(\rho) = \partial_{x_2}.$$

Moreover, putting $\phi(s) = (s, -s^2, 0)$, $\phi' = H_q \in \ker d\pi^Q$ and

$$\pi_{\operatorname{Coker} d\pi^Q}(\pi^Q \circ \phi(s))'' = -2 \neq 0.$$

Therefore, $\pi^Q|_{P \cap Q}$ has a fold at ρ.

Now, $Y|_{P\cap Q}$ is π^Q followed by replacement of the fiber variables by ξ_i. This is well defined since κ sends the flow lines of H_q in Q to those of H_{x_1} in $\{x_1 = 0\}$. That is, $(\xi_2, \dots \xi_d)$ depend only on $\pi^Q(\rho)$. By the transversality (33), this replacement is a diffeomorphism and hence preserves the fold. Now, to see that Y itself has a fold, observe that dq and dp are independent at ρ and hence $dq|_P(\rho) \neq 0$. Therefore, $q = x_1$ can be used as a coordinate on P. Hence, since $dY\partial_{x_1}$ is independent of $dY|_{P\cap Q}$, Y has a fold at ρ. This also implies that the fold set intersects Q transversally. To see that this happens at $\eta_2 = 0$, simply observe that $Q \cap P \cap \{H_q p = 0\} \subset Y^{-1}(\{\eta_2 = 0\})$. □

3. Solution of Eikonal Equations

3.1. The transverse case. We first suppose that H_p is transverse to $Q = \{x_1 = 0\}$ at $\rho_0 \in Q \cap P$ and solve the standard eikonal equation

$$p(x, d\phi(x,\eta)) = 0, \qquad \phi(0, x', \eta) = \langle x', \eta\rangle. \tag{34}$$

We choose choose coordinates so that $\rho_0 = (0, 0)$.

In this case, (since $H_q p(\rho_0) \neq 0$) the leaves $\Lambda_{\eta_0'} = \{\eta' = \eta_0'\}$ are transverse to P in $P \cap Q$ and hence there is a local diffeomorphism $T^*\{x_1 = 0\} \to P \cap Q$ near ρ_0. In particular, $\pi^Q|_{P\cap Q}$ is a diffeomorphism near ρ_0. Moreover, since $H_p q(\rho_0) \neq 0$, the H_p flow-out of $\Lambda_{\eta_0'} \cap P$, denoted $\Lambda^p_{\eta_0}$ gives a foliation of P by Lagrangian leaves.

To see that these leaves are indeed Lagrangian, observe that

$$\sigma(H_p, \cdot) = dp(\cdot).$$

So, if V is tangent to $P \cap Q$, it is in particular tangent to $\{p = 0\}$ and hence

$$\sigma(H_p, V) = 0$$

which together with $\sigma|_{\Lambda_{\eta_0'}} = 0$ implies $\Lambda^p_{\eta_0}$ is Lagrangian.

Next, note that $\Lambda^p_{\eta_0}$ project diffeomoprhically to X and hence we may use the base as coordinates on $\Lambda^p_{\eta_0}$. Now, let $\alpha = \xi dx$ denote the canonical one form (so that $d\alpha = \sigma$). Then, $\alpha|_{\Lambda^p_{\eta_0}}$ is closed and hence there exists $\phi(x, \eta) \in C^\infty$ fixed by

$$\begin{cases} d\phi(\cdot, \eta) = \alpha & \text{on } \Lambda^p_\eta \\ \phi(0, \eta) = 0 & \text{on } T^*_{\pi(\rho_0)}\{x_1 = 0\}. \end{cases}$$

The second condition fixes a normalization for ϕ on each leaf $\Lambda^p_{\eta_0}$. Now, ϕ is smooth and by construction

$$\{\Lambda^p_\eta = (x, d_x\phi)\}.$$

Therefore,

$$p(x, d_x\phi) = 0, \qquad d_x\phi(0, x', \eta) - \eta = 0$$

and in particular, since $\phi(0, 0, \eta) = 0$, $\phi(0, x', \eta) = \langle x', \eta\rangle$.

Another way to see this is by using Darboux's theorem to find a symplectomorphism so that

$$\kappa(U \cap P) \subset \{\eta_1 = 0\}, \qquad \kappa(U \cap Q) \subset \{y_1 = 0\}.$$

We then define Y as above and set

$$L^p_{\eta_0} := \{\rho \in P \mid Y(\rho) = (\cdot, \eta_0)\}.$$

Then, $L^p_{\eta_0}$ is Lagrangian since

$$\kappa(U \cap L^p_{\eta_0}) = V \cap \{\eta_1 = 0\} \cap \{(\xi_2, \dots, \xi_d) = \eta_0\}$$

with V an open neighborhood of $(0,0)$. Note also that $L^p_{\eta_0}$ foliate P since $\{\eta' = \eta_0\}$ foliate $\{\eta_1 = 0\}$.

Now, $\alpha|_{L^p_{\eta_0}}$ is closed and in particular, since Q is transverse to $L^p_{\eta_0}$, there exists $\Phi_{\eta_0} \in C^\infty(L^p_{\eta_0})$ with

$$d\Phi_{\eta_0} = \alpha|_{L^p_{\eta_0}}, \qquad \Phi_{\eta_0}|_{Q \cap L^p_{\eta_0}} = f(\eta_0).$$

Since $L^p_{\eta_0}$ foliate P, we therefore have for any smooth f, a smooth function $\Phi \in C^\infty(P)$ with

$$d(\Phi|_{L^p_{\eta_0}}) = \alpha|_{L^p_{\eta_0}}, \qquad \Phi|_{A \cap L^p_{\eta_0}} = f(\eta_0).$$

Now, since $H_p q \neq 0$, Y is a diffeomorphism and in particular,

$$\Phi = Y^* \theta, \qquad \theta : Y(P) \to \mathbb{R} \in C^\infty.$$

That is, since $Y(P) \subset \{\eta_1 = 0\}$, $\theta = \theta(x, \xi')$ for any coordinates x on X and η' as above. Choosing an appropriate normalization on Q then recovers the above result.

3.2. The folding case. In the folding case, when we try to solve (34), we will need to allow ϕ to be singular. We will work to construct a solution ϕ with a simple singularity of the form

$$\phi = \theta \pm \frac{2}{3}(-\zeta)^{3/2}.$$

This will translate directly to a solution of our original Eikonal equations.

The analog of the Lagrangians $\Lambda^p_{\eta_0}$ will be played by

$$L^p_{\eta_0} := \{\rho \in P \mid Y(\rho) = (\cdot, \eta_0)\}, \qquad \eta_0 \in \mathbb{R}^{d-1}.$$

Then,

$$\kappa(U \cap L^p_{\eta_0}) = V \cap \{\xi_1^2 - x_1 + \xi_2 = 0\} \cap \{(\xi_2, \dots, \xi_d) = \eta_0\}$$

with V an open neighborhood of $(0,0)$.

$$T_{(x,\xi)} \kappa(U \cap L^p_{\eta_0}) = \operatorname{span}\{\partial_{x_2}, \dots, \partial_{x_d}, H_p\}.$$

In particular, $L^p_{\eta_0}$ is Lagrangian for each fixed η_0. Moreover, $L^p_{\eta_0}$ foliate P.

Let $T \subset P$ be a $d-1$ dimensional submanifold transverse to $L^p_{\eta_0}$ for each η_0. Then for any $f \in C^\infty(T; \mathbb{R})$, we can find $\Phi \in C^\infty(P)$ so that for any η_0,

$$d(\Phi|_{L^p_{\eta_0}}) = \alpha|_{L^p_{\eta_0}}, \qquad \Phi|_T = f.$$

Therefore,

$$p(x, d\Phi|_{L^p_{\eta_0}}) = 0$$

and if we could use (x, η_0) as coordinates on P, we would be done. However, the vanishing of $H_p q$ means that we cannot do this. It will be convenient to choose T contained in the fold surface and $f \equiv 0$.

LEMMA 3.1. *There exist* $\theta, \zeta \in C^\infty(Y(P); \mathbb{R})$ *so that*

$$\Phi = Y^*(\theta \pm \frac{2}{3}(-\zeta)^{3/2}).$$

In addition, $Y^*\zeta$ *is a defining function for the fold and*

$$\zeta = \eta_2, \qquad \text{on } Y(P) \cap (T^*\{x_1 = 0\}). \tag{35}$$

Finally,

$$\partial_{x_1}\zeta \neq 0, \qquad \text{on } Y(P) \cap (T^*\{x_1 = 0\}). \tag{36}$$

PROOF. Let ι be the fold involution for Y and consider

$$\Phi_e := \frac{1}{2}(\Phi + \iota^*\Phi), \qquad \Phi_o := \frac{1}{2}(\Phi - \iota^*\Phi)$$

so that $\Phi = \Phi_e + \Phi_o$, Φ_e is ι even, and Φ_o is ι odd. Observe that ρ is in the fold surface, $\mathcal{F}$ if and only if $\rho = \iota(\rho)$. Therefore, $\Phi_o|_{\mathcal{F}} = 0$. Moreover, since

$$d\Phi|_{L^p_{\eta_0}} = d\alpha|_{L^p_{\eta_0}},$$

we have

$$d\Phi_o|_{L^p_{\eta_0}} = \frac{1}{2}(\alpha|_{L^p_{\eta_0}} - \iota^*\alpha|_{L^p_{\eta_0}})$$

and in particular, $d\Phi_o|_{L^p_{\eta_0}}$ vanishes at $\mathcal{F}$. Therefore, Φ_o vanishes to second (and hence third) order at $\mathcal{F}$.

Now, choose coordinates as in Lemma 2.1. Then

$$Y(x_1, \dots x_{2n-1}) = (x_1, \dots, x_{2n-1}^2)$$

and $\iota(x_1, \dots x_{2n-1}) = (x_1, \dots - x_{2n-1})$. Since Φ_e is even, in x_{2n-1},

$$\Phi_e = \theta(x_1, \dots, x_{2n-2}, x_{2n-1}^2) = \theta(Y(x))$$

for some $\theta \in C^\infty$.

On the other hand,

$$\Phi_o = \tilde{\zeta}(x_1, \dots x_{2n-2}, x_{2n-1}^2)x_{2n-1}^3 = \pm\tilde{\zeta}(Y(x))y_{2n-1}(Y(x))^{3/2}. \tag{37}$$

Setting

$$\zeta(y_1, \dots y_{2n-1}) = \left(\frac{3}{2}\tilde{\zeta}\right)^{2/3}$$

we would have the first claim if $\tilde{\zeta}$ is non-vanishing at $y_{2n-1} = 0$.

To see this, we will prove (35). Note that ξ_2 is independent of the choice of T and κ reducing to normal form. Therefore, we aim to show that Φ_o is independent of these choices and hence that it agrees with the solution in the model case. First, fix κ and let Φ_1, Φ_2 be two solutions associated with different T_1 and T_2. Then,

$$w = \Phi_1 - \Phi_2$$

is constant on each $L^p_{\eta_0}$ and in particular, is a function only of η_0. Since the fold involution ι preserves $L^p_{\eta_0}$, the ι odd part of w is a function of only η_0. Since the ι odd part vanishes on $\mathcal{F}$ and $\mathcal{F}$ intersects every $L^p_{\eta_0}$, it vanishes identically. This implies that Φ_o is independent of the choice of T.

Now,

$$p_n(x, d((\kappa^{-1})^*\Phi)|_{P_n\cap\{\eta=\eta_0\}}) = 0$$

with

$$p_n = \eta_1^2 - y_1 + \eta_2.$$

Parametrizing $P_n \cap \{\eta = \eta_0\}$ by (y', η_1), so that $dx_1 = d(\eta_0 + \eta_1^2) = 2\eta_1 d\eta_1$,

$$\partial_{\eta_1}(\kappa^{-1})^*\Phi(y', \eta_1; \eta_0) = 2\eta_1^2.$$

So, the η_1 odd part of $(\kappa^{-1})^*\Phi$ is given by $\frac{2}{3}\eta_1^3$ which, at $\eta = \eta_0$, $y_1 = 0$ is $\pm\frac{2}{3}(-\eta_2)^{3/2}$.

Moreover, on $P \cap Q$, the fold of Y is that of π^Q. In particular, $\iota(0, x', \xi_1, \xi') = (0, x', -\xi_1, \xi')$. In addition, if $\rho \in P \cap Q$, then

$$\eta_1(\kappa(\rho)) = -\eta_1(\kappa(\iota(\rho))), \qquad (y', \eta')(\kappa(\rho)) = (y', \eta')(\kappa(\iota(\rho))).$$

Therefore, on Q, the ι odd part of Φ is independent of the choice of T and of the reduction to normal form, κ and is given by $Y^*(\pm\frac{2}{3}(-\eta_2)^{3/2})$. This also implies that $\tilde{\zeta}$ from (37) does not vanish near the fold and hence shows that ζ defines the fold set.

Next, observe that

$$Y(P) = \{\eta_2 \le x_1 f(x, \eta)\}$$

with $f(\pi(\rho), 0) \neq 0$ Therefore, since ζ defines the fold set,

$$\zeta = e(x, \eta)(\eta_2 - x_1 f(x, \eta))$$

for some $0 \neq e \in C^\infty$. In particular, $\partial_{x_1}\zeta|_{x_1=\eta_2=0} \neq 0$ completing the proof of (36).

□

We next show that θ is a non-degenerate phase function and can be chosen so that $\theta|_{x_1=0}$ generates κ_∂.

Lemma 3.2. *With θ as above,*

(38) $$d_{x'}\Big(\frac{\partial\theta}{\partial\xi_j}\Big), j = 1, \dots d-1 \text{ are linearly independent on } \zeta \le 0 \text{ near } \rho_0$$

and Φ can be chosen so that $\theta|_{x_1=0}$ generates κ_∂^{-1} on $\eta_2 \le 0$.

Proof. Let (x, ξ) be coordinates near ρ_0 on T^*X and use (y, η) as coordinates near $(0, 0)$ on $T^*\mathbb{R}^d$. Then,

$$\xi' dx' = d_{x'}Y^*(\theta \pm \frac{2}{3}(-\zeta))^{3/2}.$$

Now, $d_{x'}Y^* = Y^* d_{x'}$ since Y does not change the base coordinates. Moreover, at $x_1 = 0$, $\zeta = \xi_2$, so $d_{x'}\zeta|_{x_1=0} = 0$. In particular, at $x_1 = 0$,

$$\xi' dx' = Y^* d_{x'}\theta.$$

That is,

(39) $$\partial_{x'}\theta(x', \eta'(\rho)) = \xi'.$$

But the map $\xi' \to \eta'$ is a diffeomorphism, so (38) follows.

To show that θ can be chosen so that $\theta|_{x_1=0}$ generates κ_∂^{-1}, we need to show that

$$\kappa_\partial(x', \partial_{x'}\theta(x',\eta')) = (\partial_{\eta'}\theta(x',\eta'),\eta'), \qquad x_1 = 0.$$

Suppose $\tilde\theta$ generates κ_∂. Then, on $L^p_{\eta'} \cap Q$,

$$d_{x'}Y^*\tilde\theta = d_{x'}Y^*\theta.$$

In particular, $\tilde\theta$ and θ differ by a normalization T and hence, choosing T appropriately in the solution Φ, we can arrange that $\theta = \tilde\theta$. The restriction to $\eta_2 \le 0$ comes from the fact that $L^p_{\eta'} \cap Q = \emptyset$ if $\eta_2 > 0$. □

Our next task is to extend θ and ζ from $\zeta \le 0$. Notice that $Y(P) = \{\zeta \le 0\}$ is of the form

$$\{\eta_2 \le x_1 f(x,\eta')\}$$

where $f(Y(\rho)) \ne 0$. It will not be possible to solve the eikonal equaitons exactly in $\zeta > 0$. However, we will be able to solve them in formal power series both at $\zeta = 0$ and at $x_1 = 0$. It turns out that due to the behavior of the Airy function at the turning point $x = 0$, this will be enough to construct parametrices.

Lemma 3.3. *There exist θ and $\zeta \in C^\infty$ in a neighborhood of $\pi(\rho) \times 0 \in X \times \mathbb{R}^{d-1}$ so that*

$$\zeta|_{Y(Q)} = \xi_2, \qquad d_{x'}\partial_{\eta_j}\theta \text{ are linearly independent }, \qquad \partial_{x_1}\zeta|_{Y(Q)} \ne 0$$

and (26) *holds in $\zeta \le 0$ and in Taylor series at $x_1 = 0$.*

Proof. At this point, we have solved (26) with smooth functions ζ_0, θ_0 having the above properties and defined in $\zeta_0 \le 0$. We next extend θ_0 and ζ_0 as real so that $\zeta_0(x,\eta')|_{x_1=0} = \eta_2$ continues to hold and $\partial_{x_1}\zeta_0|_{x_1=0} \ne 0$. Then θ_0, ζ_0 solve the eikonal equations (26) modulo infinite order errors. That is,

$$\begin{cases} \langle ad\theta_0, d\theta_0\rangle - \zeta_0\langle ad\zeta_0, d\zeta_0\rangle + \langle b, d\theta_0\rangle + c = e_1 \\ 2\langle ad\theta_0, d\zeta_0\rangle + \langle b, d\zeta_0\rangle = e_2 \end{cases}$$

with e_1 and e_2 vanishing identically in $\zeta_0 \le 0$.

Our aim is to solve the eikonal equations also in formal power series at $x_1 = 0$. That is, we define

$$\theta = \theta_0 + \theta', \qquad \zeta = \zeta_0 + \zeta'$$

where θ', ζ' vanish in $\zeta_0 \le 0$ and have

$$\theta' \sim \sum_{k=1}^\infty x_1^k \theta_k(x',\eta'), \qquad \zeta' \sim \sum_{k=1}^\infty x_1^k \zeta_k(x',\eta').$$

We will solve for θ_k and ζ_k iteratively. At each step it is crucial that the errors and the previous functions vanish on $\zeta_0 \le 0$.

The equations we want to solve in formal power series are

$$\begin{cases} \langle ad\theta', d\theta'\rangle + 2\langle ad\theta', d\theta_0\rangle - \zeta'\langle ad\zeta_0, d\zeta_0\rangle - (\zeta_0+\zeta')(\langle ad\zeta', d\zeta'\rangle + 2\langle ad\zeta', d\zeta_0\rangle) + \langle b, d\theta'\rangle & = -e_1 \\ 2\langle ad\theta', d\zeta'\rangle + 2\langle ad\theta', d\zeta_0\rangle + 2\langle ad\theta_0, d\zeta'\rangle + \langle b, d\zeta'\rangle & = -e_2 \end{cases}$$

Using that $\zeta_0 = \eta_0 + O(x_1)$, we obtain, modulo terms of size x, the equations

$$\begin{cases} [(a_{11}\theta_1 + 2\sum_j a_{1j}\partial_{x_j}\theta_0 + b_1)\theta_1 - a_{11}\zeta_0(\zeta_1 + 2\partial_{x_1}\zeta_0)\zeta_1] & = F_{1,1} \\ [2a_{11}(\zeta_1 + \partial_{x_1}\zeta_0)\theta_1 + (2\sum_j a_{1j}\partial_{x_j}\theta_0 + b_1)\zeta_1] & = F_{2,1} \end{cases}$$

and for $k > 1$, modulo x^k, the equations

$$\begin{cases} kx_1^{k-1}[(2a_{11}\theta_1 + 2\sum_j a_{1j}\partial_{x_j}\theta_0 + b_1)\theta_k - 2a_{11}\zeta_0(\zeta_1 + \partial_{x_1}\zeta_0)\zeta_k] & = F_{1,k} \\ kx_1^{k-1}[2a_{11}(\zeta_1 + \partial_{x_1}\zeta_0)\theta_k + (2a_{11}\theta_1 + 2\sum_j a_{1j}\partial_{x_j}\theta_0 + b_1)\zeta_k] & = F_{2,k} \end{cases}$$

where $F_{i,k}$ vanish to order x_1^{k-1} and to infinite order at $\zeta_0 = 0$ and depend on θ_j, ζ_j $0 \le j \le k-1$.

We start by solving for (ζ_1, θ_1). For this, we write

$$\theta_1 = \zeta(\zeta_1, x', \eta') = \frac{F_{2,1} - (2\sum_j a_{1j}\partial_{x_j}\theta_0 + b_1)\zeta_1}{2a_{11}(\zeta_1 + \partial_{x_1}\zeta_0)}$$

Provided $\zeta_1 = O(\zeta_0^\infty)$, near $\zeta_0 = 0$, we have $\theta_1 = O(\zeta_0^\infty)$.

Now, since $H_{x_1}p = 2\sum_j 2a_{1j}\xi_j + b_1 = 0$ at $x_1 = \zeta_0 = 0$, we have $2\sum_j a_{1j}\partial_{x_j}\theta_0 + b_1 = 0$ there. In particular,

$$\partial_{\zeta_1}[(a_{11}\theta_1 + 2\sum_j a_{1j}\partial_{x_j}\theta_0 + b_1)\theta_1 - a_{11}\zeta_0(\zeta_1 + 2\partial_{x_1}\zeta_0)\zeta_1] = -2a_{11}\zeta_0\partial_{x_1}\zeta_0 + O(\zeta_1) + O(\zeta_0^2)$$

Dividing through by ζ_0 and applying the inverse function theorem then gives a solution ζ_1.

Now, we need to solve for (θ_k, ζ_k). The equations for (θ_k, ζ_k) are linear and the matrix has inverse with norm ζ_0^{-1} and hence since the errors vanish to order ζ_0^∞, we may solve for (θ_k, ζ_k). ☐

4. Solution of the transport equations

Recall that in order to construct a parametrix to our original problem, we arrived at transport equations of the form

$$\begin{cases} 2\langle ad\theta, dg_0\rangle + 2\zeta\langle ad\zeta, dg_1\rangle + \langle b, dg_0\rangle + \langle ad\zeta, d\zeta\rangle g_1 + B_1 g_0 + \zeta B_2 g_1 & = F_1 \\ 2\langle ad\zeta, dg_0\rangle - 2\langle ad\theta, dg_1\rangle - \langle b, dg_1\rangle + B_2 g_0 - B_1 g_1 & = F_2 \end{cases}$$

Writing

$$g^\pm = g_0 \pm (-\zeta)^{1/2} g_1 \qquad G^\pm = B_1 \mp (-\zeta)^{1/2} B_2 \qquad F^\pm = F_1 \mp (-\zeta)^{1/2} F_2$$

then in $\zeta \le 0$, the transport equations are equivalent to

$$2\langle ad_x\phi^\pm, d_x g^\pm\rangle + \langle b, d_x g^\pm\rangle + G^\pm g_x^\pm = F^\pm \tag{40}$$

As before, $g^\pm$, $G^\pm$, and $F^\pm$ pull back to smooth functions on P under Y. this is because they are smooth functions of $(x, \eta', (-\zeta)^{1/2})$. Writing g, G, F for these lifts to P, we then have

$$2\langle ad_x\Phi, g\rangle + \langle b, d_x g\rangle + Gg = F$$

Note now that $2\langle ad_x\Phi, \partial\rangle + \langle b, \partial\rangle = H_p$. To see this, observe that $\pi_X : P \to X$ is a diffeomorphism near ρ. Therefore, we may use x as coordinates on P. In particular, in these coordinates,

$$H_p = \sum_{j=1}^{d} \partial_{\xi_j} p \partial_{x_j}.$$

Now, in coordinates, $\partial_{x_j}\Phi = \xi_j$. Therefore,

$$\partial_{\xi_j} p = \sum_i a_{ij}\xi_j + b_j = \sum_i a_{ij}\partial_{x_j}\Phi + b_j$$

which gives

$$H_p = 2\langle ad_x\Phi, \partial\rangle + \langle b, \partial\rangle.$$

Hence, in $\zeta \leq 0$, we need to solve

$$H_p g + G g = F \tag{41}$$

with specified initial data for g_0 and g_1. In particular, $g_1|_{x_1=0} = 0$ and $g_0(\rho) = 1$. For g, this amounts to $g|_{P\cap Q} \in C^\infty(Q/\mathbb{R}H_q)$ and $g(\rho) = 1$. In fact, we will want to solve these equations with slightly more general boundary conditions that we describe below.

We now work to simplify (41) before proceeding to solve the equations. Clearly, we can solve

$$H_p\tilde{g} + G\tilde{g} = F, \qquad \tilde{g}(\rho) = 0$$

by integrating a smooth, nonvanishing vector field. Subtracting $\tilde{g}$ from g we need only solve the homogeneous problem. Next we remove the order 0 term by solving

$$H_p r = G, \qquad r(\rho) = 0.$$

Then to solve the original problem it is enough to solve the equation

$$H_p u = 0. \tag{42}$$

In particular,

$$g = \tilde{g} + \exp(-r)u$$

has $(H_p + G)g = F$. The boundary conditions or u are, however, more complicated:

$$[\exp(-r)u|_{P\cap Q}]_O = [\tilde{g}|_{P\cap Q}]_O, \qquad u(\rho) = 1.$$

Here,

$$[f]_O = \frac{1}{2\tau}[f - \iota_Q^* f], \qquad [f]_E = \frac{1}{2}[f + \iota_Q^* f]$$

where ι_Q is the involution induced by π^Q and τ is a Q-odd function with $d\tau \neq 0$ on $Y^{-1}(\zeta = 0)$.

More generally, we will want to solve the equations with

$$g_1 = c g_0 + d \quad \text{on } T^*B, \qquad g_0(\rho) = c_0.$$

Note, $g_1 = [g]_O$ and $g_0 = [g]_E$, so $[g]_O = c[g]_E + d$ which implies

$$[\exp(-r)u|_{P\cap Q}]_O = c[\exp(-r)u]_E + f, \qquad u(\rho) = c_0$$

where c and f are given ι_Q even functions.

Next, observe that the existence of a solution u to (42) with given data u_0 on $Q \cap P$ amounts to the ι_P evenness of u_0 where ι_P is the involution induced by π^P.

Therefore, it suffices to find an ι_P even function u_0 on $Q \cap P$ so that

$$[\exp(-r)u_0]_O = c[\exp(-r)u_0]_E + f, \qquad u_0(\rho) = c_0.$$

Moreover, since

$$[\alpha\beta]_E = [\alpha]_E[\beta]_E + \tau^2[\alpha]_O[\beta]_O$$
$$[\alpha\beta]_O = [\alpha]_O[\beta]_E + [\alpha]_E[\beta]_O$$

and $[\exp(-r)]_E(\rho) \neq 0$, it is enough to find an ι_P even function, v with

$$[v]_O = c[v]_E + f, \qquad v(\rho) = c_0.$$

for some given ι_Q even c and f.

LEMMA 4.1. *For any ι_Q even $c, f \in C^\infty(Q \cap P)$, $c_0 \in \mathbb{C}$, there exists a smooth ι_P even function v on $Q \cap P$ so that*

$$[v]_O = c[v]_E + f, \qquad v(\rho) = c_0 \tag{43}$$

in a neighborhood of ρ.

PROOF. For this, we work in the normal form so that

$$\iota_Q(y_2, y'', \eta_1, \eta') = (y_2, y'', -\eta_1, \eta'), \qquad \iota_P(y_2, y'', \eta_1, \eta') = (y_2 - 2\eta_1, y'', -\eta_1, \eta').$$

In this coordinates, then we seek an ι_P even function solving (43). To simplify notation we write $t_1 = \eta_1$ and $t_2 = y_2$, $t' = (y'', \eta')$ so that

$$\iota_Q(t_1, t_2, t'') = (-t_1, t_2, t'), \qquad \iota_P(t_1, t_2, t') = (-t_1, t_2 - 2t_1, t').$$

Since v is ι_P even, $v = v(t_1^2, t_2 - t_1, t')$. We start by solving for v in formal power series at $t_1 = 0$. Assume that $f = t_1^{2p} f_p(t_2, t')$ and $v = t_1^{2p} v_p(t_2 - t_1, t')$. Then,

$$\frac{v_p(t_2 - t_1, t') - v_p(t_2 + t_1, t')}{2t_1} = c(t_1^2, t_2, t')\frac{v_p(t_2 - t_1, t') + v_p(t_2 + t_1, t')}{2} + f_p(t_2, t').$$

Since we are interested in Taylor series at $t_1 = 0$, we send $t_1 \to 0$ and obtain

$$\partial_{t_2} v_p(t_2, t') = -c(0, t_2, t')v_p(t_2, t') + f_p(t_2, t'). \tag{44}$$

In particular, if this is satisfied, then

$$[v]_O = c[v]_E + f_p + O(t^{2p+1}).$$

Moreover, since c, f_p are ι_Q even, the error is actually $O(t^{2p+2})$.

We may clearly solve (44) with $v_p(0, t')$ given. Therefore, there is a formal power series,

$$v_f = \sum_{p=0}^{\infty} v_p(t_2 - t_1, t')t_1^{2p} \tag{45}$$

satisfying (43) in formal power series.

Now, we apply Borel's lemma to sum the series (45) asymptotically to obtain a smooth function v_f that is ι_P even and satisfies

$$[v_f]_O = c[v_f]_E + g + e, \qquad e \in C^\infty,\ \iota_Q^* e = e, \quad \partial_t^\alpha e(t) = O_\alpha(t_1^\infty).$$

Hence, by linearity, we need only solve

$$[v]_O = c[v]_E + e, \qquad e \in C^\infty,\ \iota_Q^* e = e, \quad \partial_t^\alpha e(t) = O_\alpha(t_1^\infty), \quad v|_{t_1=0} = 0 \tag{46}$$

Since we are interested in solving near 0, we may also assume that $\operatorname{supp} e \subset \{t_2 < 1\}$, c is supported near $t_2 = 0$, and $\operatorname{supp} v \subset \{t_2 < 1\}$.

Now, observe that

$$\frac{v - \iota_Q^* v}{2t_1} = c(t)\frac{v + \iota_Q^* v}{2} + e(t)$$

implies

$$v = \frac{1}{1 - t_1 c(t)}[(1 + t_1 c(t))\iota_Q^* v + 2t_1 c(t) e(t)]. \tag{47}$$

Since $\iota_Q^* e = e$, $\iota_Q^* c = c$ $\iota_P^* v = v$, $\iota_P^* t_1 = -t_1$,

$$\begin{aligned}
\iota_P^* v &= \frac{1}{1 + \iota_P^* t_1 \iota_P^* c(t)}[(1 + \iota_P^* t_1 \iota_P^* c(t))\iota_P^* \iota_Q^* v + 2\iota_P^* t_1 \iota_P^* c(t) \iota_P * e(t)] \\
v &= \frac{1 - t_1 \iota_P^* c(t)}{1 - t_1 \iota_P^* c(t)} \iota_P^* \iota_Q^* v - \frac{2t_1 \iota_P^* c(t)}{1 - t_1 \iota_P^* c(t)} \iota_P^* \iota_Q^* e(t) \\
&= \frac{1 - t_1 \beta^* c(t)}{1 - t_1 \beta^* c(t)} \beta^* v + \frac{-2t_1 \beta^* c(t)}{1 - t_1 \beta^* c(t)} \beta^* e(t) \qquad (48)
\end{aligned}$$

where $\beta = \iota_P \circ \iota_Q$ is the induced billiard ball map. Letting

$$a(t) := \frac{1 - t_1 \beta^* c(t)}{1 + t_1 \beta^* c(t)}, \qquad b(t) := \frac{-2t_1 \beta^* c(t)}{1 - t_1 \beta^* c(t)},$$

we then have

$$v = \sum_{m=0}^{M} B_m (\beta^{m+1})^* e + A_M (\beta^{M+1})^* v \tag{49}$$

where

$$B_m = (\beta^m)^* b \cdot \prod_{k=0}^{m-1} (\beta^k)^* a, \qquad A_M = \prod_{k=0}^{M} (\beta^m)^* a.$$

Now, note that

$$\beta(t_1, t_2, t') = (t_1, t_2 + 2t_1, t')$$

and hence, since $\operatorname{supp} v$, $\operatorname{supp} e \subset \{t_2 < 1\}$, for $t_1 > 0$, $t_2 > -1$, and $M > 2t_1^{-1}$, $(\beta^M)^* v = (\beta^M)^* e = 0$. In particular, the the right hand side of (49) is independent of $M > 2t_1^{-1}$ when $t_1 > 0$.

In particular, in $t_1 > 0$, $t_2 > -1$,

$$v = \sum_{m=0}^{2t_1^{-1}} B_m (\beta^{m+1})^* e \tag{50}$$

solves

$$[v]_O = c[v]_E + e, \qquad \text{on } t_1 > 0.$$

We need to show that the sum (50) converges uniformly with all of its derivatives.

We clearly have

$$|1 - b(t)| \leq c|t_1|$$

and hence, for $m \leq M \leq 2t_1^{-1}$,

$$|B_m(t)| \leq (1 + c|t_1|)^M < C < \infty.$$

Therefore, the coefficients on $(\beta^{m+1})^* e$ in (50) are uniformly bounded. Now, e vanishes to all orders at $t_1 = 0$ and is supported in $t_2 < 1$. Therefore, for all $N > 0$, there exists $c_N > 0$ so that

$$|(\beta^m)^* g(t)| \leq C_N |t_1|^{2N} (1 + |t_2 + 1 + mt_1|)^{-N} \leq c_N |t_1|^N (1 + m)^{-N}.$$

In particular, the sum (50) converges rapidly and hence defines a continuous function vanishing to all orders at $t_1 = 0$.

Now, B_m is a product of at most M terms. Thus, when it is differentiated, in (t_2, t'), it produces at most M choose N terms each of which is bounded by $c_N|t_1|$. In particular, differentiation N times produces at most M^N sums of the form (50) except that sum number of the factors in B_m and those involving e are replaced by derivatives. Since the derivatives of g satisfy the same type of bounds as g and the coefficients in each sum are uniformly bounded, we have that andy finite number of derivatves the sum (50) converges uniformly to a function vanishing rapidly at $t_1 = 0$.

This defines a smooth function v in $t_1 \geq 0$, $t_2 > -1$ satisfying (46). The invariance $\iota_P^* v = v$ then defines a smooth function v in $-2 < t_1 < 2$ vanishing in $|t_1| > 1$. On $t_1 \geq 0$, v solves (48). Moreover, $v = \iota_P^* v$. Therefore, v solves (47) and in particular (46). □

The final step in the construction of the amplitude functions is to extend them into $\zeta \geq 0$. As with the eikonal equations, we will only do this in Taylor series at $\zeta = 0$ and $x_1 = 0$. We do this by extending g_0, g_1 arbitrarily as smooth functions and then adding a formal power series at $x_1 = 0$ **[TODO]Flesh out**.

5. Fourier–Airy Operators

CHAPTER 7

Calderón Projectors – Boundary Integral Operators??

APPENDIX A

Notation

- $\mathrm{Diff}^m(M)$ differential operators of order m on a manifold M.